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BLOW-UP AND GENERAL DECAY OF SOLUTIONS FOR A NONLINEAR VISCOELASTIC EQUATION

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ABSTRACT. In this article we investigate a nonlinear viscoelastic equation that admits blow-up and decay. First, we establish blow-up results for this equation, even for vanishing initial energy. Then, we show that the solutions decay under suitable conditions.

1. INTRODUCTION

In this article, we consider the viscoelastic equation

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = u|u|^{p-1}, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 1$, and g is a positive nonincreasing function.

There have been extensive studies on some special cases of this equation and the physical background is also given in these works; see [3, 4, 1, 9, 7, 8, 10, 5, 12, 2, 14] and references therein. For instance, the equation without u_t is studied in [3], the local existence theorem is established, and for certain initial data and suitable conditions on g and p, that this solution is global with energy which decays exponentially or polynomially depending on the rate of the decay of the relaxation function g. In the absence of the viscoelastic term ($g = 0$), for instance, the equation

$$u_{tt} - \Delta u + au_t|u_t|^m = bu|u|^\gamma, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.2}$$

we know that the source term $bu|u|^\gamma (\gamma > 0)$ causes finite-time blow-up of solutions with negative initial energy when $a = 0$, cf. [1]. The interaction between the damping and the source terms was first considered by Levine [7, 8] for the linear damping case ($m = 0$). He showed that solutions with negative initial energy blow up in finite time. Recently, In [14], it is proved that the solution blows up in finite time even for vanishing initial energy. Another case with time dependent damping $b(t)u_t$ is studied in [11]. Georgiev and Todorova [5] extended Levines result to the nonlinear damping case ($m > 0$). In [4], it is showed that the solution blows up in

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finite time even for vanishing initial energy. We mention the work of Liu and Zhou [9], the equation

$$u_{tt} - \Delta u = a^{-k}|u|^{\gamma}, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{1.3}$$

is studied, it is proved that the solutions blow up in finite time with more relaxed initial data and extended index γ.

For the problem (1.1) in $\mathbb{R}^n$, Mohammad Kafinia and Salim Messaoudib in [6] give a finite-time blow-up result under suitable conditions on the initial data and the relaxation function, this work extend the result of [13], established for the wave equation, to the problem (1.1) in $\mathbb{R}^n$. In this paper we improve the result of blow-up in [6], and discuss the phenomenon of decay for the solution of equation (1.1). This is an important breakthrough, since it is only well known that the solution blows up in finite time if the initial energy is negative from all the previous literature.

Now, we list some notation that will be used in our paper. Use $\|\cdot\|_p$ to denote the $L^p(\mathbb{R}^n)$ norm. Throughout this paper, C denotes a generic positive constant (generally large), it may be different from line to line.

The remainder of the paper will be organized as follows. In the next section, we review some preliminaries that will be used in the proof of our main theorems. Then, the blow-up phenomenon will be considered in Section 3. In the last Section, we discuss the decay of the solution to equation (1.1).

2. Preliminaries

In this section we review some preliminaries that will be used in the proof of our main theorems. Throughout this paper,

$$\frac{n+2}{[n-2]_+} = \begin{cases} \infty, & n = 1,2,\\ \frac{n+2}{n-2}, & n \geq 3. \end{cases}$$

The relaxation function g satisfies:

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(\tau)d\tau = l > 0, \quad t \geq 0.$$

(H2) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0.$$

and

$$\Big|\frac{\xi'(t)}{\xi(t)}\Big| \leq k, \xi(t) > 0, \xi'(t) \leq 0, t > 0.$$

Remark 2.1. Since ξ is nonincreasing, then $\xi(t) \leq \xi(0) = M$.

The embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq \frac{2n}{n-2}$, if $n \geq 3$ and $q \geq 2$, if $n = 1,2$; $L^r(\Omega) \hookrightarrow L^q(\Omega)$ for $q < r$; that is to say, there exists a constant C_e, such that

$$\|u\|_q \leq C_e\|\nabla u\|_2, \quad \|u\|_q \leq C_e\|u\|_r. \tag{2.1}$$

These two inequalities will be used frequently in this article.

We define the energy corresponding to problem (1.1) as

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.2}$$

here
$$(g \circ v)(t) = \int_0^t g(t-\tau)\|v(t)-v(\tau)\|_2^2 d\tau\,.$$
By a direct calculation we obtain
$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \|u_t\|_2^2 \le \frac{1}{2}(g' \circ \nabla u)(t) \le 0. \tag{2.3}$$
Hence, we can deduce that $E(t) \le E(0)$ for $t \ge 0$.

Remark 2.2. The largest T for which the solution exists on $[0,T) \times \mathbb{R}^n$ is called the lifespan of the solution of (1.1). The supremum of the T's is denoted by T^* . If $T^* = \infty$, we say the solution is global while $T^* < \infty$ we say that solution blows up in finite time.

Lemma 2.3. *If p satisfies $p < \frac{n+2}{[n-2]_+}$, then there exists a positive constant $C > 1$, such that*
$$\|u\|_{p+1}^s \le C\left(\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}\right) \quad \text{with } 2 \le s \le p+1, \tag{2.4}$$
for any u being a solution of (1.1) on $[0,T)$. Consequently,
$$\|u\|_{p+1}^s \le C\left(H(t) + \|u_t\|_2^2 + (g \circ \nabla u)(t) + \|\nabla u\|_2^2\right) \quad \text{with } 2 \le s \le p+1, \tag{2.5}$$
on $[0,T)$ and here $H(t) := -E(t)$.

Proof. If $\|u\|_{p+1} \le 1$, the estimate $\|u\|_{p+1}^s \le \|u\|_{p+1}^2 \le B^2\|\nabla u\|_2^2$ is true.

If $\|u\|_{p+1} > 1$, we have $\|u\|_{p+1}^s \le \|u\|_{p+1}^{p+1}$. Combining the two inequalities we obtain (2.4).

Note that (2.5) follows from (2.4) and the definition of energy corresponding to the solution. □

3. Blow-up phenomenon

Theorem 3.1. *Assume that (H1), (H2) hold, $1 < p < \frac{n+2}{[n-2]_+}$, $\int_0^\infty g(\tau)d\tau < \frac{(p+1)(p-1)}{1+(p+1)(p-1)}$ and $E(0) < 0$. Then the solution blows up in finite time.*

Proof. From the definition of $H(t)$, we have
$$H'(t) = -\frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 + \|u_t\|_2^2 \ge 0,$$
and
$$0 < H(0) \le H(t) \le \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$
Moreover, we define
$$L(t) = H^{1-\alpha}(t) + \epsilon\int_\Omega uu_t\,dx,$$
here ϵ small to be chosen later, $0 < \alpha \le \frac{p-1}{2(p+1)}$.

By differentiating the above equality, we have

$$
\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon \int_\Omega |u_t|^2 dx + \epsilon \int_\Omega u u_{tt} dx \\
&= (1-\alpha)H^{-\alpha}(t)\Big(-\frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2} g(t)\|\nabla u\|_2^2 + \|u_t\|_2^2 \Big) \\
&\quad + \epsilon \|u_t\|_2^2 - \epsilon \|\nabla u\|_2^2 + \epsilon \int_\Omega \nabla u(t) \int_0^t g(t-\tau)\nabla u(\tau)\, d\tau\, dx \\
&\quad - \epsilon \int_\Omega u u_t dx + \epsilon \|u\|_{p+1}^{p+1}.
\end{aligned} \tag{3.1}
$$

Using Young and Schwarz inequality, we obtain

$$
\begin{aligned}
&\int_\Omega \nabla u(t) \int_0^t g(t-\tau)\nabla u(\tau)\, d\tau\, dx \\
&\ge -\delta \|\nabla u\|_2^2 - \frac{1}{4\delta}\Big(\int_0^t g(\tau) d\tau \Big)(g \circ \nabla u)(t) + \Big(\int_0^t g(\tau) d\tau \Big)\|\nabla u\|_2^2 \,,
\end{aligned} \tag{3.2}
$$

$$
\int_\Omega u u_t dx \le \frac{\delta^2}{2}\|u\|_2^2 + \frac{\delta^{-2}}{2}\|u_t\|_2^2 \tag{3.3}
$$

Inserting (3.2) and (3.3) into (3.1), we deduce

$$
\begin{aligned}
L'(t) &\ge (1-\alpha)H^{-\alpha}(t)\Big(-\frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2} g(t)\|\nabla u\|_2^2 + \|u_t\|_2^2 \Big) + \epsilon \|u_t\|_2^2 \\
&\quad + \Big(-1 - \delta + \int_0^t g(\tau) d\tau \Big)\epsilon \|\nabla u\|_2^2 - \frac{\epsilon}{4\delta}\Big(\int_0^t g(\tau) d\tau \Big)(g \circ \nabla u)(t) \\
&\quad - \frac{\epsilon \delta^2}{2}\|u\|_2^2 - \frac{\epsilon \delta^{-2}}{2}\|u_t\|_2^2 + \epsilon \|u\|_{p+1}^{p+1}.
\end{aligned}
$$

If we set $\delta^2 = kH^\alpha$, $\delta^{-2} = k^{-1}H^{-\alpha}$, $k > 0$ and we have

$$
H^\alpha(t)\|u\|_2^2 \le C(\frac{1}{p+1})^\alpha \|u\|_{p+1}^{2+\alpha(p+1)}.
$$

Then

$$
\begin{aligned}
L'(t) &\ge \Big(1 - \alpha - \frac{\epsilon}{2k}\Big) H^{-\alpha}(t)\|u_t\|_2^2 + \Big[p + 1 - \frac{kC}{2}\Big(\frac{1}{p+1}\Big)^\alpha\Big]\epsilon H(t) \\
&\quad + \Big[\frac{p-1}{2}\Big(1 - \int_0^t g(\tau) d\tau\Big) - \delta - \frac{kC}{2}\Big(\frac{1}{p+1}\Big)^\alpha\Big]\epsilon \|\nabla u\|_2^2 \\
&\quad + \Big[\frac{p+1}{2} - \frac{1}{4\delta}\int_0^t g(\tau) d\tau - \frac{kC}{2}\Big(\frac{1}{p+1}\Big)^\alpha\Big]\epsilon (g \circ \nabla u)(t) \\
&\quad + \Big[\frac{p+3}{2} - \frac{kC}{2}\Big(\frac{1}{p+1}\Big)^\alpha\Big]\epsilon \|u_t\|_2^2.
\end{aligned}
$$

According to the hypothesis in Theorem 3.1 and take k and δ to be small enough such that

$$
\begin{gathered}
\frac{p-1}{2}\Big(1 - \int_0^t g(\tau) d\tau\Big) - \delta - \frac{kC}{2}\Big(\frac{1}{p+1}\Big)^\alpha > 0, \\
\frac{p+1}{2} - \frac{1}{4\delta}\int_0^t g(\tau) d\tau - \frac{kC}{2}\Big(\frac{1}{p+1}\Big)^\alpha > 0.
\end{gathered}
$$

Choose ϵ (k is fixed) small enough such that

$$1-\alpha-\frac{\epsilon}{2k}\geq 0,\quad L(0)=H^{1-\alpha}(0)+\epsilon\int_\Omega u_0u_1dx>0.$$

Then, we can deduce that

$$L'(t)\geq C[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+(g\circ\nabla u)(t)].$$

Thanks to Hölder and Young inequality, we obtain

$$\begin{aligned}\Big|\int_\Omega uu_tdx\Big|^{1/(1-\alpha)}&\leq\|u\|_2^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}\leq C\|u\|_{p+1}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}\\&\leq C(\|u\|_{p+1}^s+\|u_t\|_2^2)\\&\leq C\left(H(t)+\|u_t\|_2^2+(g\circ\nabla u)(t)+\|\nabla u\|_2^2\right),\end{aligned}\tag{3.4}$$

where $2\leq s=\frac{2}{1-2\alpha}\leq p+1$. Hence,

$$\begin{aligned}L^{1/(1-\alpha)}(t)&=\Big(H^{1-\alpha}(t)+\epsilon\int_\Omega uu_tdx\Big)^{1/(1-\alpha)}\\&\leq 2^{1/(1-\alpha)}\Big(H(t)+\Big|\int_\Omega uu_tdx\Big|^{1/(1-\alpha)}\Big)\\&\leq C\left(H(t)+\|u_t\|_2^2+(g\circ\nabla u)(t)+\|\nabla u\|_2^2\right),\end{aligned}$$

which implies that $L'(t)\geq\lambda L^{1/(1-\alpha)}(t)$, where λ is a constant depending on C, p, α and ϵ. Therefore

$$L(t)\geq(L^{\frac{-\alpha}{1-\alpha}}(0)+\frac{-\alpha}{1-\alpha}\lambda t)^{-\frac{1-\alpha}{\alpha}}.$$

So $L(t)$ approaches infinite as t tends to $(1-\alpha)/\big(\alpha\lambda L^{\frac{\alpha}{1-\alpha}}(0)\big)$. This completes the proof. □

To obtain another blow-up result we first give the following lemma.

Lemma 3.2. *Assume that* (H1), (H2) *hold, additionally, assume that*

$$\|u_0\|_{p+1}>\lambda_0\equiv B_0^{\frac{-2}{p-1}},\quad E(0)<E_0=\Big(\frac12-\frac{1}{p+1}\Big)B_0^{\frac{-2(p+1)}{p-1}}.$$

Then

$$\|u\|_{p+1}>\lambda_0,\quad\|\nabla u\|_2>B_0^{\frac{-(p+1)}{p-1}},\quad\textit{for all } t\geq 0,$$

where $B_0=\frac{B}{l^{1/2}}$ *for* $\|u\|_{p+1}\leq B\|\nabla u\|_2$.

Proof. From (2.2) and the hypothesis, we know that

$$\begin{aligned}E(t)&=\frac12\|u_t\|_2^2+\frac12\Big(1-\int_0^tg(\tau)d\tau\Big)\|\nabla u\|_2^2+\frac12(g\circ\nabla u)(t)-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\\&\geq\frac12\Big(1-\int_0^tg(\tau)d\tau\Big)\|\nabla u\|_2^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\\&\geq\frac{l}{2}\|\nabla u\|_2^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\geq\frac{1}{2B_0^2}\|u\|_{p+1}^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}.\end{aligned}$$

Set $h(\xi)=\frac{1}{2B_0^2}\xi^2-\frac{1}{p+1}\xi^{p+1}$, $\xi\geq 0$. Then $h(\xi)$ satisfies

- $h(\xi)$ is strictly increasing on $[0,\lambda_0)$;
- $h(\xi)$ takes its maximum value $(\frac12-\frac{1}{p+1})B_0^{\frac{-2(p+1)}{p-1}}$ at λ_0;

- $h(\xi)$ is strictly decreasing on (λ_0, ∞).

Since $E_0 > E(0) \geq E(t) \geq h(\|u\|_{p+1})$ for all $t \geq 0$, there is no time t^* such that $\|u(\cdot,t^*)\|_{p+1} = \lambda_0$. By the continuity of the $\|u(\cdot,t)\|_{p+1} - norm$ with respect to the time variable, one has

$$\|u(\cdot,t)\|_{p+1} > \lambda_0 = B_0^{\frac{-2}{p-1}} \quad \text{for all } t \geq 0,$$

and consequently,

$$\|\nabla u(\cdot,t)\|_2 \geq \frac{1}{l^{1/2}B_0}\|u(\cdot,t)\|_{p+1} > \frac{1}{l^{1/2}} B_0^{\frac{-(p+1)}{p-1}} > B_0^{\frac{-(p+1)}{p-1}}.$$

This completes the proof. □

Theorem 3.3. *Suppose that*(H1), (H2) *hold,* $1 < p < \frac{n+2}{[n-2]_+}$,

$$\int_0^\infty g(\tau)d\tau < \frac{(p+1)(p-1)}{1+(p+1)(p-1)},$$

$\|u_0\|_{p+1} > \lambda_0$ *and* $E(0) \leq E_0$. *Then the solution of* (1.1) *blows up in finite time.*

Proof. Set $G(t) = E_0 + H(t)$, then

$$G'(t) = -\frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 + \|u_t\|_2^2 \geq 0,$$

from which we obtain

$$\begin{aligned} 0 < G(t) = E_0 + H(t) &= \big(\frac{1}{2} - \frac{1}{p+1}\big)B_0^{\frac{-2(p+1)}{p-1}} + H(t) \\ &< \big(\frac{1}{2} - \frac{1}{p+1}\big)\|\nabla u\|_2^2 + H(t) < C(\|\nabla u\|_2^2 + H(t)) \end{aligned}$$

and

$$\begin{aligned} &0 < G(t) \\ &= E_0 - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\leq E_0 - \frac{1}{2}\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\leq \big(\frac{1}{2} - \frac{1}{p+1}\big)B_0^{\frac{-2(p+1)}{p-1}} - \frac{l}{2}\big(\frac{1}{l^{1/2}}\big)^2 B_0^{\frac{-2(p+1)}{p-1}} + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\leq \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned}$$

Let

$$Q(t) = G^{1-\alpha}(t) + \epsilon\int_\Omega uu_t dx,$$

with ϵ small to be chosen later, $0 < \alpha \leq \frac{p-1}{2(p+1)}$.

By the same process as in the proof of Theorem 3.1, deduce that

$$Q'(t) \geq C[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t)].$$

Thanks to (3.4), we obtain

$$Q^{1/(1-\alpha)}(t) = \Big(G^{1-\alpha}(t) + \epsilon\int_\Omega uu_t dx\Big)^{1/(1-\alpha)}$$

$$
\begin{aligned}
&\le 2^{1/(1-\alpha)}\Big(G(t)+\Big|\int_\Omega uu_t dx\Big|^{1/(1-\alpha)}\Big)\\
&\le C\left(H(t)+\|u_t\|_2^2+(g\circ\nabla u)(t)+\|\nabla u\|_2^2\right),
\end{aligned}
$$

which implies that $Q'(t)\ge\lambda Q^{1/(1-\alpha)}(t)$, where λ is a constant depending on C, p, α and ϵ. Therefore

$$
Q(t)\ge (Q^{\frac{-\alpha}{1-\alpha}}(0)+\frac{-\alpha}{1-\alpha}\lambda t)^{-\frac{1-\alpha}{\alpha}}.
$$

So $Q(t)$ approaches infinite as t tends to $\frac{1-\alpha}{\alpha\lambda Q^{\frac{\alpha}{1-\alpha}}(0)}$. This completes the proof. □

4. Decay solutions

The purpose of this section is to give a decay result of the solution. Set

$$
I(t)=\Big(1-\int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2+(g\circ\nabla u)(t)-\|u\|_{p+1}^{p+1}.
$$

As in [10], to give our decay result, we first prove the following lemmas.

Lemma 4.1. *Suppose that* (H1), (H2) *hold,* $p<\frac{n+2}{[n-2]_+}$*, and* $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *such that*

$$
\beta=\frac{C_e^{p+1}}{l}\Big(\frac{2(p+1)E(0)}{(p-1)l}\Big)^{(p-1)/2}<1, I(u_0)>0, \tag{4.1}
$$

then $I(u(t))>0$*, for all* $t>0$*. Here* C_e *is given in* (2.1).

Proof. Since $I(u_0)>0$, there exists $T_m<T$, such that

$$
I(u(t))>0,\quad \forall t\in[0,T_m],
$$

which gives

$$
\begin{aligned}
&\frac12\Big(1-\int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2+\frac12(g\circ\nabla u)(t)-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\\
&=\frac{p-1}{2(p+1)}\Big[\Big(1-\int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\Big]+\frac{1}{p+1}I(t)\\
&\ge\frac{p-1}{2(p+1)}\Big[\Big(1-\int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\Big].
\end{aligned}
$$

So we have

$$
l\|\nabla u\|_2^2\le\Big(1-\int_0^t g(\tau)d\tau\Big)\|\nabla u\|_2^2\le\frac{2(p+1)}{p-1}E(t)\le\frac{2(p+1)}{p-1}E(0). \tag{4.2}
$$

By using (H1), (4.1) and (4.2), we obtain

$$
\|u\|_{p+1}^{p+1}\le C_e^{p+1}\|\nabla u\|_2^{p+1}\le\beta l\|\nabla u\|_2^2<\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2
$$

Hence,

$$
I(t)=\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)-\|u\|_{p+1}^{p+1}>0,\quad\forall t\in[0,T_m].
$$

By repeating this process, and using that

$$
\lim_{t\to T_m}\frac{C_e^{p+1}}{l}\Big(\frac{2(p+1)E(u,u_t)}{(p-1)l}\Big)^{(p-1)/2}\le\beta<1,
$$

we show that T_m is extended to T. □

To establish the decay rate, we use the functional

$$F(t) = E(t) + \epsilon_1 \Psi(t) + \epsilon_2 \Phi(t), \tag{4.3}$$

where ϵ_1 and ϵ_2 are positive constants and

$$\Psi(t) = \xi(t) \int_\Omega u u_t dx, \quad \Phi(t) = -\xi(t) \int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau\, dx.$$

This functional, for $\xi(t) = 1$, was first introduced in [3] and [2]. Now, let us consider some useful properties of this functional.

Lemma 4.2. *Assume that $u(x,t)$ is the solution of* (1.1) *and that* (4.1) *holds. Then there exists $k_1 < 1$ and $k_2 > 1$ such that*

$$k_1 E(t) \le F(t) \le k_2 E(t). \tag{4.4}$$

Proof. Using Young, Schwarz and Poincaré inequality, we obtain

$$\int_\Omega u u_t dx \le \frac{C_*^2}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u_t\|_2^2, \tag{4.5}$$

$$\int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau\, dx \le \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2}(1-l) C_*^2 (g \circ \nabla u)(t). \tag{4.6}$$

Using (4.5) and (4.6), we have

$$\begin{aligned} k_2 E(t) - F(t) \ge & \Big[\Big(\frac{k_2 - 1}{2} - \frac{k_2 - 1}{p+1} \Big) l - \frac{\epsilon_1 C_*^2 M}{2} \Big] \|\nabla u\|_2^2 \\ & + \frac{1}{2} \{ k_2 - 1 - (\epsilon_1 + \epsilon_2) M \} \|u_t\|_2^2 + \frac{k_2 - 1}{p+1} I(t) \\ & + \Big[\frac{k_2 - 1}{2} - \frac{k_2 - 1}{p+1} - \frac{\epsilon_2 (1-l) C_*^2 M}{2} \Big] (g \circ \nabla u)(t). \end{aligned}$$

Similarly,

$$\begin{aligned} F(t) - k_1 E(t) \ge & \Big[\Big(\frac{1 - k_1}{2} - \frac{1 - k_1}{p+1} \Big) l - \frac{\epsilon_1 C_*^2 M}{2} \Big] \|\nabla u\|_2^2 \\ & + \frac{1}{2} [1 - k_1 - (\epsilon_1 + \epsilon_2) M] \|u_t\|_2^2 + \frac{1 - k_1}{p+1} I(t) \\ & + \Big[\frac{1 - k_1}{2} - \frac{1 - k_1}{p+1} - \frac{\epsilon_2 (1-l) C_*^2 M}{2} \Big] (g \circ \nabla u)(t). \end{aligned}$$

By choosing ϵ_1 and ϵ_2 small enough, such that $k_2 E(t) - F(t) \ge 0$ and $F(t) - k_1 E(t) \ge 0$, we complete the proof. □

Lemma 4.3. *Let* (H1) and (H2) *hold, and $p \le \frac{n+2}{[n-2]_+}$. Assume that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and u is the solution of* (1.1)*. Then*

$$\begin{aligned} \Psi'(t) \le & \Big(1 + \frac{(1-k)(1+k) C_*^2}{l} \Big) \xi(t) \|u_t\|_2^2 + \frac{1-l}{2l} \xi(t) (g \circ \nabla u)(t) \\ & - \frac{l}{4} \xi(t) \|\nabla u\|_2^2 + \xi(t) \|u\|_{p+1}^{p+1} \end{aligned} \tag{4.7}$$

Proof. By a direct computation, we have

$$\begin{aligned}\Psi'(t) &= \xi(t)\Big(\|u_t\|_2^2 + \|u\|_{p+1}^{p+1} - \|\nabla u\|_2^2 + \int_\Omega \nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\,dx \\ &\quad - \int_\Omega uu_t dx\Big) + \xi'(t)\int_\Omega uu_t dx \\ &:= \xi(t)\Big(\|u_t\|_2^2 + \|u\|_{p+1}^{p+1} - \|\nabla u\|_2^2 + A_1 - A_2\Big) + \xi'(t)A_2.\end{aligned}\tag{4.8}$$

By Young, Schwarz and Poincaré inequality, we have

$$A_1 \le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\big(1+\frac{1}{\eta}\big)(1-l)(g\circ\nabla u)(t) + \frac{1}{2}(1+\eta)(1-l)^2\|\nabla u\|_2^2, \tag{4.9}$$

$$A_2 \le \alpha C_*^2\|\nabla u\|_2^2 + \frac{1}{4\alpha}\|u_t\|_2^2. \tag{4.10}$$

Combining (4.8) and (4.9) with (4.10) yields

$$\begin{aligned}\Psi'(t) &\le \big(1+\frac{1-k}{4\alpha}\big)\xi(t)\|u_t\|_2^2 + \frac{1}{2}\big(1+\frac{1}{\eta}\big)(1-l)\xi(t)(g\circ\nabla u)(t) \\ &\quad - \Big[\frac{1}{2} - \frac{(1+\eta)(1-l)^2}{2} - (1+k)\alpha C_*^2\Big]\xi(t)\|\nabla u\|_2^2 + \xi(t)\|u\|_{p+1}^{p+1}.\end{aligned}$$

We choose $\eta = l/(1-l)$ and $\alpha = l/(4(1+k)C_*^2)$; then (4.7) is true. □

Lemma 4.4. *Let* (H1) *and* (H2) *hold,* $p \le \frac{n+2}{[n-2]_+}$, $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *and* u *is the solution of* (1.1). *Then*

$$\begin{aligned}\Phi'(t) &\le \delta\Big[1 + 2(1-l)^2 + C_e^{2p}\Big(\frac{2(p+1)E(0)}{l(p-1)}\Big)^{p-1}\Big]\xi(t)\|\nabla u\|_2^2 \\ &\quad - \frac{g(0)C_*^2}{4\delta}\xi(t)(g'\circ\nabla u)(t) + \Big[\big(2\delta+\frac{1}{2\delta}\big)(1-l) + \frac{(2+k)(1-l)C_*^2}{4\delta}\Big] \\ &\quad \times \xi(t)(g\circ\nabla u)(t) + \Big[\delta(2+k) - \int_0^t g(\tau)d\tau\Big]\xi(t)\|u_t\|_2^2.\end{aligned}\tag{4.11}$$

Proof. Straightforward computations show that

$$\begin{aligned}\Phi'(t) &= \xi(t)\int_\Omega \nabla u\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\,dx \\ &\quad - \xi(t)\int_\Omega\Big(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx \\ &\quad + \xi(t)\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\ &\quad - \xi(t)\int_\Omega u|u|^{p-1}\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\ &\quad - \xi(t)\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\,dx - \xi(t)\Big(\int_0^t g(\tau)d\tau\Big)\|u_t\|_2^2 \\ &\quad - \xi'(t)\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\ &:= \xi(t)\Big[I_1+I_2+I_3+I_4+I_5 - \Big(\int_0^t g(\tau)d\tau\Big)\|u_t\|_2^2\Big] - \xi'(t)I_3.\end{aligned}\tag{4.12}$$

By Young and Poincaré inequality, we have

$$I_1 \le \delta\|\nabla u\|_2^2 + \frac{1-l}{4\delta}(g\circ\nabla u)(t), \tag{4.13}$$

$$I_2 \le \Big(2\delta + \frac{1}{4\delta}\Big)(1-l)(g\circ\nabla u)(t) + 2\delta(1-l)^2\|\nabla u\|_2^2, \tag{4.14}$$

$$I_3 \le \delta\|u_t\|_2^2 + \frac{C_*^2(1-l)}{4\delta}(g\circ\nabla u)(t), \tag{4.15}$$

$$I_4 \le \delta C_e^{2p}\Big(\frac{2(p+1)E(0)}{l(p-1)}\Big)^{p-1}\|\nabla u\|_2^2 + \frac{(1-l)C_*^2}{4\delta}(g\circ\nabla u)(t), \tag{4.16}$$

$$I_5 \le \delta\|u_t\|_2^2 - \frac{g(0)C_*^2}{4\delta}(g'\circ\nabla u)(t). \tag{4.17}$$

Combining (4.12)-(4.17), we have the required estimate (4.11). $\square$

We are ready to give our decay result.

Theorem 4.5. *Suppose that* (H1), (H2) *and* (4.1) *hold,* $p \le \frac{n+2}{[n-2]_+}$, $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$. *Then there exists positive constants* α *and* λ *such that the solution of* (1.1) *satisfies*

$$E(t) \le \alpha e^{-\lambda\int_{t_0}^t \xi(\tau)d\tau}, \quad t \ge t_0.$$

Proof. Since g is positive, continuous and $g(0) > 0$, then for any $t_0 > 0$, we have

$$\int_0^t g(\tau)d\tau \ge \int_0^{t_0} g(\tau)d\tau = g_0 > 0, \quad \forall t \ge t_0. \tag{4.18}$$

Combining (2.3), (4.3), (4.7), (4.11) and (4.18), for $t \ge t_0$, we have

$$\begin{aligned}
F'(t) &\le -\Big\{\epsilon_2[g_0 - \delta(2+k)] - \epsilon_1\Big(1 + \frac{(1-k)(1+k)C_*^2}{l}\Big)\Big\}\xi(t)\|u_t\|_2^2 \\
&\quad - \Big\{\frac{\epsilon_1 l}{4} - \epsilon_2\delta\Big[1 + 2(1-l)^2 + C_e^{2p}\Big(\frac{2(p+1)E(0)}{l(p-1)}\Big)^{p-1}\Big]\Big\}\xi(t)\|\nabla u\|_2^2 \\
&\quad + \Big\{\frac{\epsilon_1(1-l)}{2l} + \epsilon_2\Big[\Big(2\delta + \frac{1}{2\delta}\Big)(1-l) + \frac{(2+k)(1-l)C_*^2}{4\delta}\Big]\Big\}\xi(t)(g\circ\nabla u)(t) \\
&\quad + \Big(\frac12 - \frac{\epsilon_2 g(0)C_*^2 M}{4\delta}\Big)(g'\circ\nabla u)(t) + \epsilon_1\xi(t)\|u\|_{p+1}^{p+1} \\
&:= -J_1\xi(t)\|u_t\|_2^2 - J_2\xi(t)\|\nabla u\|_2^2 + J_3\xi(t)(g\circ\nabla u)(t) \\
&\quad + J_4(g'\circ\nabla u)(t) + \epsilon_1\xi(t)\|u\|_{p+1}^{p+1}.
\end{aligned} \tag{4.19}$$

We choose suitable constants ϵ_1 and ϵ_2 satisfying

$$\frac{\epsilon_1\big(1 + \frac{(1-k)(1+k)C_*^2}{l}\big)}{g_0 - \delta(2+k)} < \epsilon_2 < \frac{\epsilon_1 l}{4\delta\big[1 + 2(1-l)^2 + C_e^{2p}\big(\frac{2(p+1)E(0)}{l(p-1)}\big)^{p-1}\big]}$$

and δ, ϵ_1 small enough, such that

$$g_0 - (2+k)\delta > \frac12 g_0, \quad J_1 > 0, \quad J_2 > 0, \quad k_3 := J_4 - J_3 > 0,$$

which imply

$$J_4(g'\circ\nabla u)(t) + J_3\xi(t)(g\circ\nabla u)(t) \le -k_3\xi(t)(g\circ\nabla u)(t).$$

Applying (4.4) and (4.19) yields

$$F'(t) \le -\gamma\xi(t)E(t) \le \frac{-\gamma}{k_2}\xi(t)F(t).$$

Therefore, after integrating the above inequality and using (4.4) again, we obtain the desire result. □

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