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EXISTENCE OF ENTIRE SOLUTIONS FOR NON-LOCAL DELAYED LATTICE DIFFERENTIAL EQUATIONS

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Abstract. In this article we study entire solutions for a non-local delayed lattice differential equation with monostable nonlinearity. First, based on a concavity assumption of the birth function, we establish a comparison theorem. Then, applying the comparison theorem, we show the existence and some qualitative features of entire solutions by mixing a finite number of traveling wave fronts with a spatially independent solution.

1. Introduction

The purpose of this article is to study entire solutions to a non-local delayed lattice differential equation which describes the growth of mature population of a single species in a patchy environment (see [6, 9, 8]):

$$u_n'(t) = D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[u_{n-i}(t) - u_n(t)] - du_n(t) + \sum_{i\in\mathbb{Z}} J(i)b\big(u_{n-i}(t-\tau)\big), \quad (1.1)$$

where $n \in \mathbb{Z}$, $t \in \mathbb{R}$, $D > 0$ and $\tau \geq 0$ are given constants, the kernel functions I and J and the birth function b satisfy

- (A1) $I(i) = I(-i) \geq 0$, $J(i) = J(-i) \geq 0$, $\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i) = 1$, $\sum_{i\in\mathbb{Z}} J(i) = 1$, and for every $\lambda \geq 0$, $\sum_{i\in\mathbb{Z}\setminus\{0\}} e^{-\lambda i} I(i) < \infty$, $\sum_{i\in\mathbb{Z}} e^{-\lambda i} J(i) < \infty$;
- (A2) $b \in C^2(\mathbb{R}^+, \mathbb{R}^+)$, $b(0) = b(K) - dK = 0$, $b'(0) > d$, $b(u) > du$ for $u \in (0, K)$, $b'(u) \geq 0$ and $b(u) \leq b'(0)u$ for all $u \in [0, K]$, where $K > 0$ is a constant.

Ma et al [6] proved that there exists a minimal wave speed $c_* > 0$ such that a monotone traveling wave solution (traveling wave front for short) of (1.1) exists if and only if its wave speed is not lower than this minimal wave speed. There is no doubt that the study of traveling wave solutions is important in many applications. It can describe certain dynamical behavior of the studied problem such as (1.1). However, the dynamics of delayed lattice differential equations is so rich that there might be other interesting patterns. Recently, quite a few front-like entire solutions have been found in many problems; see e.g., [1, 2, 3, 5, 4, 7, 8, 11, 10]. Here an entire solution is meant by a classical solution defined for all space and time. It is clear that traveling wave solutions are also entire solutions.

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Recently, Wang et al [8] constructed some types of entire solutions for (1.1) by mixing traveling wave fronts with speeds $c > c_*$ and a spatial independent solution. The basic idea in [8], similar to [2], is to use traveling wave fronts and their exponential decay at $-\infty$ to build subsolution and upper estimates, respectively, and then prove the existence of entire solutions by employing comparison principle. However, the issue of the existence of entire solution of (1.1) connecting traveling wave fronts with minimal wave speed c_* (minimal wave front for short) is still open. Resolving this issue represents a main contribution of our current study.

More precisely, in this paper, we consider the entire solutions of (1.1) connecting the minimal wave front. Since the decay of the minimal wave front at $-\infty$ may not be exponential, the approach in [2, 8] can not be applied directly for (1.1) to construct appropriate upper estimates. To overcome this difficulty, by making a concavity assumption on the birth function b, we establish a comparison theorem (see Lemma 3.1). Applying the comparison theorem, a new upper estimate is obtained and some new types of entire solutions are constructed by mixing any finite number of traveling wave fronts with speeds $c \geq c_*$ and a spatial independent solution (see Theorem 3.4).

We should remark that Wang et al [8] also established the uniqueness of entire solutions and the continuous dependence of entire solutions on parameters, which are not discussed in the present paper, for the spatially discrete Fisher-KPP equation:

$$u_n'(t) = \frac{D}{2}[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] + f(u_n(t)). \tag{1.2}$$

The rest of this article is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish a comparison theorem. Then, we prove the existence and qualitative features of entire solutions of (1.1).

2. Preliminaries

In this section, first we state some known results on traveling wave fronts and spatial independent solutions of (1.1). Then, we consider the initial value problem of (1.1) and establish some comparison theorems.

For traveling wave fronts of (1.1), let us substitute $u_n(t) := U(\xi)$, $\xi = n + ct$, into (1.1), then we obtain the corresponding wave equation

$$cU'(\xi) = D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\big[U(\xi - i) - U(\xi)\big] - dU(\xi) + \sum_{i\in\mathbb{Z}} J(i)b(U(\xi - i - c\tau)). \tag{2.1}$$

Obviously, the characteristic function for (2.1) with respect to the trivial equilibrium 0 can be represented by

$$\Delta(c,\lambda) = c\lambda - D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\big[e^{-\lambda i} - 1\big] + d - b'(0)e^{-\lambda c\tau} \sum_{i\in\mathbb{Z}} J(i)e^{-\lambda i} \tag{2.2}$$

for $c \geq 0$ and $\lambda \in \mathbb{C}$,

Properties of $\Delta(c,\lambda)$ and traveling wave solutions of (1.1) were investigated in [6, 8]. For the sake of completeness, we recall them as follows.

Proposition 2.1. *Consider* (1.1) *and* (2.2).

(1) There exist $\lambda_* > 0$ and $c_* > 0$ such that

$$\Delta(c_*, \lambda_*) = 0, \quad \frac{\partial}{\partial\lambda}\Delta(c_*,\lambda)\big|_{\lambda=\lambda_*} = 0.$$

Furthermore, if $c > c_*$, then the equation $\Delta_1(c,\lambda) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) < \lambda_* < \lambda_2(c)$, $\lambda_1'(c) < 0$ and $\frac{\partial}{\partial c}[c\lambda_1(c)] < 0$ for $c > c_*$.

(2) For each $c \geq c_*$, equation (1.1) has a traveling wave front $\phi_c(\xi)$ which satisfies $\phi_c(-\infty) = 0$, $\phi_c(+\infty) = K$ and $\frac{d}{d\xi}\phi_c(\xi) > 0$ for $\xi \in \mathbb{R}$. Moreover, if $c > c_*$, then

$$\lim_{\xi\to-\infty} \phi_c(\xi)e^{-\lambda_1(c)\xi} = 1, \quad \phi_c(\xi) \leq e^{\lambda_1(c)\xi} \text{ for } \xi \in \mathbb{R}.$$

Next, we consider the spatially independent solutions of (1.1); i.e., solutions of the delayed differential equation

$$\Gamma'(t) = -d\Gamma(t) + b(\Gamma(t-\tau)). \tag{2.3}$$

The following result follows from [8, Theorem 4.3].

Proposition 2.2. *There exists a solution $\Gamma(t) : \mathbb{R} \to [0, K]$ of (2.3) which satisfies $\Gamma(-\infty) = 0$ and $\Gamma(+\infty) = K$. Furthermore,*

$$\Gamma'(t) > 0, \quad \lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = 1, \quad \Gamma(t) \leq e^{\lambda^* t} \quad \text{for all } t \in \mathbb{R},$$

where λ^* is the unique positive root of the equation $\lambda + d - b'(0)e^{-\lambda\tau} = 0$.

We now consider the initial value problem of (1.1) with the initial data

$$u_n(s) = \varphi_n(s), \quad n \in \mathbb{Z},\ s \in [-\tau, 0]. \tag{2.4}$$

The definitions of supersolution and subsolution are given as follows.

Definition 2.3. A sequence of differentible functions $v(t) = \{v_n(t)\}_{n\in\mathbb{Z}}$, with $t \in [-\tau, b)$ and $b > 0$, is called a supersolution (resp. subsolution) of (1.1) on $[0, b)$ if $v(t)$ is bounded for $(n,t) \in \mathbb{Z} \times [-\tau, b)$ and

$$v_n'(t) \geq (\text{resp. } \leq) D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[v_{n-i}(t) - v_n(t)] - dv_n(t) + \sum_{i\in\mathbb{Z}} J(i)b\big(v_{n-i}(t-\tau)\big),$$

for $t \in (0, b)$.

By Definition 2.3, we have the following result, see [8, Lemmas 3.2 and 5.1 and Theorem 3.4].

Proposition 2.4. *(1) For any $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ with $\varphi_n \in C([-\tau, 0], [0, K])$, Equation (1.1) admits a unique solution $u(t;\varphi) = \{u_n(t;\varphi)\}_{n\in\mathbb{Z}}$ on $[0, +\infty)$ satisfies $u_n \in C([-\tau, +\infty), [0, K])$. Moreover, there exists $M > 0$ which is independent of φ such that*

$$|u_n'(t;\varphi)|,\ |u_n''(t;\varphi)| \leq M \quad \text{for all } n \in \mathbb{Z}, t > \tau.$$

(2) Let $\{u_n^+(t)\}_{n\in\mathbb{Z}}$ and $\{u_n^-(t)\}_{n\in\mathbb{Z}}$ be a pair of super- and sub-solutions of (1.1) on $[0,\infty)$ such that $u_n^\pm(t) \geq 0$ and $u_n^-(s) \leq u_n^+(s)$ for $n \in \mathbb{Z}$, $t \in [-\tau, \infty)$ and $s \in [-\tau, 0]$. Then $u_n^+(t) \geq u_n^-(t)$ for $n \in \mathbb{Z}$ and $t \geq 0$.

(3)] Let $u_n^+(t) \in C\big([-\tau, +\infty), [0, +\infty)\big)$ and $u_n^-(t) \in C\big([-\tau, +\infty), (-\infty, K]\big)$ be such that $u_n^+(s) \geq u_n^-(s)$ for all $n \in \mathbb{Z}$ and $s \in [-\tau, 0]$. If

$$\frac{d}{dt}u_n^+(t) \geq D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[u_{n-i}^+(t) - u_n^+(t)] - du_n^+(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)u_{n-i}^+(t-\tau),$$

and

$$\frac{d}{dt}u_n^-(t) \leq D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[u_{n-i}^-(t) - u_n^-(t)] - du_n^-(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)u_{n-i}^-(t-\tau),$$

for $n \in \mathbb{Z}$ and $t > 0$, then $u_n^+(t) \geq u_n^-(t)$ for all $n \in \mathbb{Z}$ and $t \geq 0$.

3. Existence of entire solutions

In this section, we first establish a comparison theorem. Then, applying the comparison theorem, we prove the existence and qualitative features of entire solutions of (1.1). The approach adopted here is inspired by the work of Hamel and Nadirashvili [3].

To obtain the comparison theorem, we need the concavity assumption of the birth function b:

(A3) $b''(u) \le 0$ for $u \in [0,\infty)$.

Lemma 3.1. *Assume* (A1)–(A3). *Let* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$, $\varphi^{(i)} = \{\varphi_n^{(i)}\}_{n\in\mathbb{Z}}$ *with* φ_n *and* $\varphi_n^{(i)}$ *in* $C([-\tau,0],[0,K])$, $i = 1,\dots,m_0$, *be* m_0+1 *given functions with*

$$\varphi_n(s) \le \min\{K,\ \varphi_n^{(1)}(s) + \dots + \varphi_n^{(m_0)}(s)\} \quad \text{for } n \in \mathbb{Z},\ s \in [-\tau,0].$$

Let u *and* $u^{(i)}$ *be the solutions of the Cauchy problems of* (1.1) *with initial data*

$$u_n(s) = \varphi_n(s), \quad n \in \mathbb{Z},\ s \in [-\tau,0], \tag{3.1}$$

$$u_n^{(i)}(s) = \varphi_n^{(i)}(s), \quad n \in \mathbb{Z},\ s \in [-\tau,0], \tag{3.2}$$

respectively. Then

$$0 \le u_n(t) \le \min\{K,\ u_n^{(1)}(t) + \dots + u_n^{(m_0)}(t)\}$$

for all $n \in \mathbb{Z}$ *and* $t \ge 0$.

Proof. Set $Q_n(t) = u_n^{(1)}(t)+\dots+u_n^{(m_0)}(t)$. By Proposition 2.4, we have $0 \le u_n(t) \le K$ for all $n \in \mathbb{Z}$ and $t \ge 0$. Thus, it suffices to show that $u_n(t) \le Q_n(t)$ for all $n \in \mathbb{Z}$ and $t \ge 0$. First, we show that for any $v_i \in (0,K]$, $i = 1,\dots,m_0$,

$$b(v_1 + \dots + v_{m_0}) \le b(v_1) + \dots + b(v_{m_0}). \tag{3.3}$$

For $m_0 = 1$, (3.3) holds obviously. For $m_0 = 2$, using the concavity of the function b, we have

$$\frac{b(v_1+v_2)-b(v_1)}{v_2} \le \frac{b(v_1)}{v_1}, \quad \frac{b(v_1+v_2)-b(v_2)}{v_1} \le \frac{b(v_2)}{v_2},$$

which imply that

$$v_1 b(v_1+v_2) \le (v_1+v_2)b(v_1), \quad v_2 b(v_1+v_2) \le (v_1+v_2)b(v_2).$$

Thus, we have $b(v_1+v_2) \le b(v_1)+b(v_2)$. Using mathematical induction, we can show that (3.3) holds. It then follows that

$$\begin{aligned}
Q_n'(t) &= \sum_{k=1}^{m_0} \frac{d}{dt} u_n^{(k)}(t) \\
&= D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[Q_{n-i}(t) - Q_n(t)] - dQ_n(t) + \sum_{i\in\mathbb{Z}} J(i) \sum_{k=1}^{m_0} b\big(u_{n-i}^{(k)}(t-\tau)\big) \\
&\ge D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[Q_{n-i}(t) - Q_n(t)] - dQ_n(t) + \sum_{i\in\mathbb{Z}} J(i) b\big(Q_{n-i}(t-\tau)\big)
\end{aligned}$$

for all $n \in \mathbb{Z}$ and $t > 0$; that is, the function $Q(t) = \{Q_n(t)\}_{n\in\mathbb{Z}}$ is a supersolution of (1.1) on $[0,\infty)$. By our assumption, $u_n(s) \le Q_n(s)$ for $n \in \mathbb{Z}$ and $s \in [-\tau,0]$. Therefore, from the assertion (2) of Proposition 2.4, we have $u_n(t) \le Q_n(t)$ for all $n \in \mathbb{Z}$ and $t \ge 0$. This completes the proof. □

In the sequel, we assume that $\phi_c(\xi)$ and $\Gamma(t)$ are the traveling wave front and spatially independent solution of (1.1) decided in Propositions 2.1 and 2.2, respectively. For any $k \in \mathbb{N}$, $l, m \in \mathbb{N} \cup \{0\}$, $\theta_1, \dots, \theta_l, \theta_1', \dots, \theta_m', \theta \in \mathbb{R}$, $c_1, \dots, c_l, c_1', \dots, c_m' \geq c_*$ and $\chi \in \{0, 1\}$ with $l + m + \chi \geq 2$, we denote

$$\varphi_n^{(k)}(s) := \max\big\{ \max_{1 \leq i \leq l} \phi_{c_i}(n + c_i s + \theta_i), \max_{1 \leq j \leq m} \phi_{c_j}(-n + c_j' s + \theta_j'), \chi\Gamma(s + \theta)\big\},$$

$$\underline{u}_n(t) := \max\big\{ \max_{1 \leq i \leq l} \phi_{c_i}(n + c_i t + \theta_i), \max_{1 \leq j \leq m} \phi_{c_j}(-n + c_j' t + \theta_j'), \chi\Gamma(t + \theta)\big\},$$

where $n \in \mathbb{Z}$, $s \in [-k - \tau, -k]$ and $t > -k$. Let $U^{(k)}(t) = \{U_n^{(k)}(t)\}_{n \in \mathbb{Z}}$ be the unique solution of (1.1) with the initial data:

$$U_n^{(k)}(s) = \varphi_n^{(k)}(s), \; n \in \mathbb{Z}, s \in [-k - \tau, -k]. \tag{3.4}$$

By Proposition 2.4, we have $\underline{u}_n(t) \leq U_n^{(k)}(t) \leq K$ for all $n \in \mathbb{Z}$ and $t \geq -k$.

Applying the comparison lemma 3.1, we obtain the following result which provides appropriate upper estimate of $U^{(k)}(t)$.

Lemma 3.2. *Assume* (A1)–(A3). *The function $U^{(k)}(t) = \{U_n^{(k)}(t)\}_{n \in \mathbb{Z}}$ satisfies*

$$U_n^{(k)}(t) \leq \overline{U}_n(t) := \min\big\{K, \Pi(n, t)\big\}$$

for any $n \in \mathbb{Z}$ and $t \geq -k$, where

$$\Pi(n, t) = \sum_{i=1}^{l} \phi_{c_i}(n + c_i t + \theta_i) + \sum_{j=1}^{m} \phi_{c_j}(-n + c_j' t + \theta_j') + \chi\Gamma(t + \theta).$$

Before stating our main results in this subsection, we give the following definition and notation.

Definition 3.3. Let $m_0 \in \mathbb{N}$ and $p, p_0 \in \mathbb{R}^{m_0}$. We say that a sequence of functions $\Psi_p(t) = \{\Psi_{n;p}(t)\}_{n \in \mathbb{Z}}$ converges to a function $\Psi_{p_0}(t) = \{\Psi_{n;p_0}(t)\}_{n \in \mathbb{Z}}$ in the sense of topology $\mathcal{T}$ if, for any compact set $S \subset \mathbb{Z} \times \mathbb{R}$, the functions $\Psi_{n;p}(t)$ and $\Psi_{n;p}'(t)$ converge uniformly in S to $\Psi_{n;p_0}(t)$ and $\Psi_{n;p_0}'(t)$ respectively as p tends to p_0.

For any $N_1 \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, denote the regions $T_{N_1,\gamma}^i$ $(i = 1, \dots, l)$ and $\tilde{T}_{N_1,\gamma}^j$ $(j = 1, \dots, m)$, by

$$T_{N_1,\gamma}^i := \{n \in \mathbb{Z} | n \geq N_1\} \times [\gamma, +\infty), \quad i = 1, \dots, l, \; T_\gamma := \mathbb{R}^N \times (-\infty, \gamma],$$

$$\tilde{T}_{N_1,\gamma}^j := \{n \in \mathbb{Z} || n \leq N_1\} \times [\gamma, +\infty), \quad j = 1, \dots, m, \; \tilde{T}_\gamma := \mathbb{Z} \times [\gamma, +\infty).$$

Following the priori estimate of Proposition 2.4 and the upper estimate of Lemma 3.2, we can obtain the following result.

Theorem 3.4. *Assume* (A1)–(A3). *For any $l, m \in \mathbb{N} \cup \{0\}$, $\theta_1, \dots, \theta_l, \theta_1', \dots, \theta_m'$, $\theta \in \mathbb{R}$, $c_1, \dots, c_l, c_1', \dots, c_m' \geq c_*$ and $\chi \in \{0, 1\}$ with $l + m + \chi \geq 2$, there exists an entire solution $U_p(t) = \big\{U_{n;p}(t)\big\}_{n \in \mathbb{Z}}$ of* (1.1) *such that*

$$\underline{u}_n(t) \leq U_{n;p}(t) \leq \overline{U}_n(t) \quad \text{for all } (n, t) \in \mathbb{Z} \times \mathbb{R}, \tag{3.5}$$

where $p := p_{l,m,\chi} = \big(c_1, \theta_1, \dots, c_l, \theta_l, c_1', \theta_1', \dots, c_m', \theta_m', \chi\theta\big)$. Furthermore, the following properties hold.

1. *$0 < U_{n;p}(t) < K$ and $\frac{d}{dt} U_{n;p}(t) > 0$ for any $(n, t) \in \mathbb{Z} \times \mathbb{R}$.*
2. *$\lim_{t \to +\infty} \sup_{n \in \mathbb{Z}} \big|U_{n;p}(t) - K\big| = 0$ and $\lim_{t \to -\infty} \sup_{|n| \leq N_0} U_{n;p}(t) = 0$ for any $N_0 \in \mathbb{N}$.*

(3) *If $b'(u) \le b'(0)$ for $u \in [0,K]$, then for any $\gamma \in \mathbb{R}$, $U_{n;p_{l,m,1}}(t)$ converges to $U_{n;p_{l,m,0}}(t)$ as $\theta \to -\infty$ in $\mathcal{T}$, and uniformly on $(n,t) \in T_\gamma$.*

(4) *For any $N_1 \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, $U_p(t)$ converges to K in the sense of topology $\mathcal{T}$ as $\theta_i \to +\infty$ and uniformly on $(n,t) \in T^i_{N_1,\gamma}$; $U_p(t)$ converges to K in the sense of topology $\mathcal{T}$ as $\theta'_j \to +\infty$ and uniformly on $(n,t) \in \tilde{T}^j_{N_1,\gamma}$; and $U_p(t)$ converges to K in the sense of topology $\mathcal{T}$ as $\theta \to +\infty$ and uniformly on $(n,t) \in \tilde{T}_\gamma$.*

Proof. By Proposition 2.4(2) and Lemma 3.2, we have

$$\underline{u}_n(t) \le U_n^{(k)}(t) \le U_n^{(k+1)}(t) \le \overline{U}_n(t) \quad \text{for all } n \in \mathbb{Z} \text{ and } t \ge -k. \tag{3.6}$$

Using the priori estimate of Proposition 2.4 and the diagonal extraction process, there exists a subsequence $U^{(k_l)}(t) = \{U_n^{(k_l)}(t)\}_{l\in\mathbb{N}}$ of $U^{(k)}(t)$ such that $U^{(k_l)}(t)$ converges to a function $U_p(t) = \{U_{n;p}(t)\}_{n\in\mathbb{Z}}$ in the sense of topology $\mathcal{T}$. Since $U_n^{(k)}(t) \le U_n^{(k+1)}(t)$ for any $t > -k$, we have

$$\lim_{k\to+\infty} U_n^{(k)}(t) = U_{n;p}(t) \quad \text{for any } (n,t) \in \mathbb{Z}\times\mathbb{R}.$$

The limit function is unique, whence all of the functions $U^{(k)}(t)$ converge to the function $U_p(t)$ in the sense of topology $\mathcal{T}$ as $k \to +\infty$. Clearly, $U_p(t)$ is an entire solution of (1.1). Also, (3.5) follows from (3.6). The proof of assertion of part (1) is similar to that of Wang et al [8, Theorem 1.1] and is omitted. The assertion of part (2) is a direct consequence of (3.5).

(3) For $\chi = 0$, we denote $\varphi^{(k)}(s) = \{\varphi_n^{(k)}(s)\}_{n\in\mathbb{Z}}$, by $\varphi^{(k)}_{p_{l,m,0}}(s) = \{\varphi^{(k)}_{n;p_{l,m,0}}(s)\}_{n\in\mathbb{Z}}$, and $U^{(k)}(t) = \{U_n^{(k)}(t)\}_{n\in\mathbb{Z}}$ by $U^{(k)}_{p_{l,m,0}}(t) = \{U^{(k)}_{n;p_{l,m,0}}(t)\}_{n\in\mathbb{Z}}$. Similarly, for $\chi = 1$, we denote $\varphi^{(k)}(s)$ by $\varphi^{(k)}_{p_{l,m,1}}(s)$, and $U^{(k)}(t)$ by $U^{(k)}_{p_{l,m,1}}(t)$. Let

$$W^{(k)}(t) = \{W_n^{(k)}(t)\}_{n\in\mathbb{Z}} := U^{(k)}_{p_{l,m,1}}(t) - U^{(k)}_{p_{l,m,0}}(t), \quad t \ge -k-\tau.$$

Then $0 \le W_n^{(k)}(t) \le K$ for all $(n,t) \in \mathbb{Z}\times[-k,+\infty)$. Moreover, by the assumption $b'(u) \le b'(0)$ for $u \in [0,K]$, it is easy to verify that

$$\begin{aligned}
&\frac{d}{dt}W_n^{(k)}(t)\\
&= D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[W_{n-i}^{(k)}(t) - W_n^{(k)}(t)] - dW_n^{(k)}(t)\\
&\quad + \sum_{i\in\mathbb{Z}} J(i)\big[b\big(U^{(k)}_{n-i;p_{l,m,1}}(t-\tau)\big) - b\big(U^{(k)}_{n-i;p_{l,m,0}}(t-\tau)\big)\big]\\
&\le D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[W_{n-i}^{(k)}(t) - W_n^{(k)}(t)] - dW_n^{(k)}(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)W_{n-i}^{(k)}(t-\tau)
\end{aligned}$$

for $n \in \mathbb{Z}$, $t > -k$. Let us define the function

$$\widehat{W}(t) = \{\widehat{W}_n(t)\}_{n\in\mathbb{Z}} = \{e^{\lambda^*(t+\theta)}\}_{n\in\mathbb{Z}}.$$

By Proposition 2.2, we have

$$W_n^{(k)}(s) = \varphi^{(k)}_{n;p_{l,m,1}}(s) - \varphi^{(k)}_{n;p_{l,m,0}}(s) \le \Gamma(s+\theta) \le e^{\lambda^*(s+\theta)} = \widehat{W}_n(s)$$

for $n \in \mathbb{Z}$, $s \in [-k-\tau, -k]$. Moreover, it is easy to see that $\widehat{W}(t)$ satisfies the linear equation

$$\frac{d}{dt}\widehat{W}_n(t) = D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)[\widehat{W}_{n-i}(t) - \widehat{W}_n(t)] - d\widehat{W}_n(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\widehat{W}_{n-i}(t-\tau).$$

It then follows from the statement (3) of Proposition 2.4 that

$$0 \le W_n^{(k)}(t) \le \widehat{W}_n(t) = e^{\lambda^*(t+\theta)} \text{ for all } (n,t) \in \mathbb{Z}\times[-k,+\infty).$$

Since $\lim_{k\to+\infty} U^{(k)}_{n;p_{l,m,i}}(t) = U_{n;p_{l,m,i}}(t)$, $i = 0, 1$, we get

$$0 \le U_{n;p_{l,m,1}}(t) - U_{n;p_{l,m,0}}(t) \le e^{\lambda^*(t+\theta)}$$

for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$, which implies that $U_{p_{l,m,1}}(t)$ converges to $U_{p_{l,m,0}}(t)$ as $\theta \to -\infty$ uniformly on $(n,t) \in T_\gamma$ for any $\gamma \in \mathbb{R}$. For any sequence θ^ℓ with $\theta^\ell \to -\infty$ as $\ell \to +\infty$, the functions $U_{p^\ell_{l,m,1}}(t)$ (here $p^\ell_{l,m,1} := (c_1, \theta_1, \dots, c_l, \theta_l, c'_1, \theta'_1, \dots, c'_m, \theta'_m, \theta^\ell)$) converge to a solution of (1.1) (up to extraction of some subsequence) in the sense of topology $\mathcal{T}$, which turns out to be $U_{p_{l,m,0}}(t)$. The limit does not depend on the sequence θ^ℓ, whence all of the functions $U_{p_{l,m,1}}(t)$ converge to $U_{p_{l,m,0}}(t)$ in the sense of topology $\mathcal{T}$ as $\theta \to -\infty$.

The proof of part (4) is similar to that of part (3), and omitted. This completes the proof. □

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