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REGULARITY ON THE INTERIOR FOR THE GRADIENT OF WEAK SOLUTIONS TO NONLINEAR SECOND-ORDER ELLIPTIC SYSTEMS

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Abstract. We consider weak solutions to the Dirichlet problem for nonlinear elliptic systems. Under suitable conditions on the coefficients of the systems we obtain everywhere Hölder regularity on the interior for the gradients of weak solutions. Our sufficient condition for the regularity works even though an excess of the gradient of solution is not very small. More precise partial regularity on the interior can be deduced from our main result. The main result is illustrated through examples at the end of this article.

1. Introduction

In this paper we give conditions guaranteeing that a weak solution to the Dirichlet problem for a nonlinear elliptic system

$$\begin{aligned} -D_\alpha\big(A_i^\alpha(Du)\big) = 0 \quad &\text{in } \Omega,\ i = 1,\dots,N,\\ u = g \quad &\text{on } \partial\Omega \end{aligned} \tag{1.1}$$

belongs to $C^{1,\gamma}_{\rm loc}(\Omega,\mathbb{R}^N)$ space. Here and in the following, summation over repeated indices is understood.

By a weak solution to the Dirichlet problem (1.1), we mean a function u in $W^{1,2}(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega A_i^\alpha(Du) D_\alpha \varphi^i \, dx = 0, \quad \forall\, \varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$$

and $u - g \in W_0^{1,2}(\Omega,\mathbb{R}^N)$.

Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n \geq 3$, the function g belongs to the space $W^{1,2}(\Omega,\mathbb{R}^N)$, the coefficients $(A_i^\alpha)_{i=1,\dots,N,\alpha=1,\dots,n}$ are differentiable, have the linear controlled growth and satisfy the strong uniform ellipticity condition. More precisely, denoting by

$$A_{ij}^{\alpha\beta}(p) = \frac{\partial A_i^\alpha}{\partial p_j^\beta}(p)$$

and assuming that $A_i^\alpha(0) = 0$ we require

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(i) there exists a constant $M > 0$ such that for every $p \in \mathbb{R}^{nN}$

$$|A_i^\alpha(p)| \le M(1 + |p|),$$

(ii) $|A_{ij}^{\alpha\beta}(p)| \le M$,

(iii) the strong ellipticity condition holds; i.e., there exists a constant $\nu > 0$ such that for every p, $\xi \in \mathbb{R}^{nN}$,

$$A_{ij}^{\alpha\beta}(p)\xi_\alpha^i \xi_\beta^j \ge \nu|\xi|^2,$$

(iv) there exists a real function ω defined and continuous on $[0, \infty)$, which is bounded, nondecreasing, increasing on a neighbourhood of zero, $\omega(0) = 0$ and such that for all p, $q \in \mathbb{R}^{nN}$

$$|A_{ij}^{\alpha\beta}(p) - A_{ij}^{\alpha\beta}(q)| \le \omega(|p - q|).$$

We set $\omega_\infty = \lim_{t\to\infty} \omega(t) \le 2M$.

Here it is worth to point out (see [9, pg. 169]) that for uniformly continuous coefficients $A_{ij}^{\alpha\beta}$ there exists the real function ω satisfying the assumption (iv) and, viceversa, (iv) implies the uniform continuity of the coefficients and the absolute continuity of ω on $[0, \infty)$. It is clear that if $\omega(t) = 0$ for $t \in [0, \infty)$, then the system (1.1) is reduced to the system with constant coefficients and in this case the regularity of weak solutions is well understood (see, e.g. [9] and references therein).

The system (1.1) has been extensively studied (see, e.g. [1, 9, 15, 23]). It is well known that the Dirichlet problem has a unique solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$. Moreover, for boundary function $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ it holds

$$\int_\Omega |Du|^2\, dx \le C_D \int_\Omega |Dg|^2\, dx, \tag{1.2}$$

$$\int_\Omega |Du - (Du)_\Omega|^2\, dx \le C_D \int_\Omega |Dg - (Dg)_\Omega|^2\, dx \tag{1.3}$$

where $(Dg)_\Omega = \frac{1}{m(\Omega)} \int_\Omega Dg\, dx$, $m(\Omega) = m_n(\Omega)$ is the n - dimensional Lebesgue measure of Ω and $C_D = n^2N^2(M/\nu)^2$. The estimates (1.2) and (1.3) can be proved by a standard technique (see [10], Remark on pg.113). For reader's convenience the proofs of (1.2) and (1.3) are given in Appendix to this paper.

The first regularity results for $n = 2$ and for nonlinear systems were established by Morrey (see [21]) and they state that the gradient of unique solution to (1.1) is locally Hölder continuous. If $n \ge 3$, it is known that the gradient Du may be discontinuous and unbounded (see [14, 18, 23]).

For $n \ge 3$ and for the nonlinear systems many partial regularity results were obtained, i.e., it was proved that the gradient of any weak solution to (1.1) (or more general system) is locally Hölder continuous up to a singular set of the Hausdorff dimension $n-2$ (see, e.g. [1, 9, 23]). In the last two decades some new methods for proving the partial regularity of weak solutions to the nonlinear systems, based on a generalization of the technique of harmonic approximation, have appeared (see, e.g. [13, 8] and references therein). These methods extend the previous partial regularity results in such a way that they allow to establish the optimal Hölder exponent for the gradients of weak solutions on their regular sets.

In this place, it is worth to mention the papers [24, 25] where the authors through examples showed that (for $n = 3$) the gradient of the unique minimizer of the convex and differentiable functional F (in this case (1.1) is the Euler-Lagrange equation of F) can be discontinuous or unbounded. Thus full regularity cannot be achieved even in this special case. On the other hand, Campanato in [2] proved that the weak solution of the system (1.1) belongs to $W^{2,2+\epsilon}_{loc}(\Omega, \mathbb{R}^N)$ which implies that $Du \in C^{0,\gamma}_{\rm loc}(\Omega, \mathbb{R}^{nN})$ for $n = 2$ and $u \in C^{0,\gamma}_{\rm loc}(\Omega, \mathbb{R}^N)$ for $2 \le n \le 4$, $\gamma \in (0, 1)$. Kristensen and Melcher have recently proved (using a method which avoids employing the Gehring's lemma) in [16] that an analogous result is true under the strong monotonicity and the Lipschitz continuity of the coefficients. Moreover, they have stated the value of the last mentioned ϵ as $\epsilon = \delta\alpha/\beta$ where $\delta > 1/50$ is a universal constant, $0 < \alpha \le \beta$ are the constant of the monotonicity and the Lipschitz continuity constant respectively.

The aim of this paper is to extend the last mentioned results and the results of the paper [7], giving some conditions sufficient for the everywhere interior regularity of the solutions to the systems (1.1) for $n \ge 3$. In the paper [7], the first author with John and Stará gave conditions, expressed in terms of the continuity modulus of the first derivatives of the coefficients of (1.1), that guarantee the local Hölder continuity of the gradients of solutions to (1.1) in Ω. More precisely, they proved that there exists $\nu_0 > 0$ such that for every ellipticity constant $\nu \ge \nu_0$ with the ratio $M/\nu \le P$, where $P > 1$ is a given constant, the gradients of weak solutions to (1.1) are locally Hölder continuous in Ω (see [5] as well). The point of the current paper is to give conditions guaranteeing the same quality of the solutions to (1.1) when the ratio ω_∞/ν is admitted to be arbitrary and no lower bound for the constant of ellipticity ν is needed (we remind that if the constant M is given, then $\omega_\infty \le 2M$).

The main results are stated in two theorems. The first of them refers that if ω_∞/ν is small enough, the solutions to (1.1) are regular. This result is not very surprising but, moreover, an upper bound C_{cr} (although probably not optimal) of ω_∞/ν is designed there (see (2.2) below). If $\omega_\infty/\nu > C_{cr}$, then a sufficient condition for regularity of solutions to the system (1.1) is given in Theorem 2.3. A basic advantage of condition (2.4) below is that it admits (for sufficiently big ellipticity constant ν) an arbitrary growth of the continuity modulus $\omega = \omega(t)$ when t is near by zero. Here it is needful to note that Theorem 2.3 works likewise when ν is small but, in this case, the modulus of continuity ω has to grow slowly enough. Many proofs of regularity results for systems like the system (1.1) are based on a certain excess-decay estimate for the excess function $U_r(x)$ (in our case this function is defined by (2.1) below). The key assumption of the excess-decay estimate is that $U_r(x)$ has to be sufficiently small on a ball $B_r(x) \Subset \Omega$. On the other hand, our condition (2.4) does not suppose smallness of the excess function $U_r(x)$ (see Remark 2.4 below). We would like to note that more delicate estimates and careful designing of some parameters in proofs allow us to state these results in a much simpler form than in [7].

Various conditions, guaranteeing the regularity of weak solutions, were studied by Giaquinta and Nečas in [11, 12] (the Liouville's condition for regularity formulated through L^∞-spaces), Daněček in [4] (the Liouville's condition for regularity formulated through BMO-spaces), Chipot and Evans in [3] and Koshelev in [15]. Koshelev's condition, interpreted according to the assumptions (ii) and (iii), is the following : If it is supposed that $nNM|\xi|^2 \ge A^{\alpha\beta}_{ij}(p)\xi^i_\alpha\xi^j_\beta \ge \nu|\xi|^2$ for every p,

$\xi \in \mathbb{R}^{nN}$, $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ and

$$\frac{M}{\nu} < \frac{1}{nN} \frac{\sqrt{1 + \frac{(n-2)^2}{n-1}} + 1}{\sqrt{1 + \frac{(n-2)^2}{n-1}} - 1},$$

then any solution to (1.1) has the locally Hölder continuous gradient in Ω. It is proved in [15] that the above condition is sharp. The same result is proved, by an another method which is based on an estimate of the gradient of solution in a suitable weighted Morrey space, in [18]. Further results concerning the local (and global as well) Hőlder regularity of the solutions and the dispersion of the eigenvalues of the coefficients matrix of elliptic systems can be found in [20, 19]. On the other hand, the last mentioned condition does not cover the linear systems with constant coefficients and the large dispersion of the eigenvalues of $A_{ij}^{\alpha\beta}$, while every linear system with constant coefficients satisfies the conditions (2.2) and (2.4) as well. Chipot and Evans in [3] consider the variational problem and assume that $A_{ij}^{\alpha\beta}(p)$ tend to a constant matrix for p tending to infinity. Thus the modulus of continuity of $A_{ij}^{\alpha\beta}(p)$ approaches zero for sufficiently large p while our assumption requires that its changes are small enough. Herein we would like to note that, as far as we know, the above mentioned condition from the paper [3] was for the first time employed in [4].

The methods of proving main results follow the standard procedures used in the direct proofs of the partial regularity. The novelty is an employment of special complementary Young functions which allows us (through a modification of the Natanson's Lemma - see Lemma 3.7 below) to get some key estimates. As a consequence of our proof of the main result (Theorem 2.3 below) we obtain the partial regularity result concerning the more precise identification of the singular set of the weak solution to (1.1). As it is known (see [9, 23, 13, 8]), the singular set of the weak solution to (1.1) is characterized as follows

$$\Omega_{\text{sing}} = \big\{ x \in \Omega : \liminf_{r \to 0} \fint_{B_r(x)} |Du(y) - (Du)_{x,r}|^2 \, dy > 0 \big\}.$$

Our description of the singular set $\Omega \setminus \Omega_{\mathcal{R}}$, from Theorem 2.6 below, indicates clearly that $\Omega \setminus \Omega_{\mathcal{R}} \subsetneq \Omega_{\text{sing}}$ and the constant which describes $\Omega \setminus \Omega_{\mathcal{R}}$ is computable.

Four examples, illustrating above mentioned results, are given at the end of the paper. The first one presents a system which our results can be applied to. The second and the third of them show typical samples of modulus of continuity that our main result deals with. The fourth one indicates that the regularity of gradient of boundary data, which is considerably weaker than the Campanato's one, does not admit the singularities of the weak solutions to (1.1) in a subdomain.

2. Main results

By $\Omega_0 \Subset \Omega$ we will understand any bounded subdomain Ω_0 which is compactly embedded into Ω (i.e. $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$) and the boundary $\partial\Omega_0$ is smooth. For $x \in \Omega$,

$r > 0$ such that $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\} \subset \Omega$ we set

$$\begin{aligned} U_r(x) &= \frac{1}{m(B_r(x))} \int_{B_r(x)} |Du(y) - (Du)_{x,r}|^2 \, dy \\ &:= \fint_{B_r(x)} |Du(y) - (Du)_{x,r}|^2 \, dy, \\ \phi(x, r) &= \int_{B_r(x)} |Du(y) - (Du)_{x,r}|^2 \, dy, \end{aligned} \tag{2.1}$$

where $(Du)_{x,r} = \fint_{B_r(x)} Du(y)\, dy$ and κ_n is the n - dimensional Lebesgue measure of the unit ball.

Theorem 2.1. *Let $n \le \vartheta < n + 2$, $\Omega_0 \Subset \Omega$ with $\operatorname{dist}(\Omega_0, \partial\Omega) \ge d > 0$ be given. Let u be a weak solution to the Dirichlet problem (1.1) where $g \in W^{1,2}(\Omega)$ and the hypotheses (i), (ii), (iii), (iv) be satisfied with M, ν and the function ω for which*

$$\frac{\omega_\infty}{\nu} \le \frac{1}{\sqrt{8n^2 N^2 (2^{n+5} L)^{\frac{\vartheta}{n+2-\vartheta}}}} := C_{cr} \tag{2.2}$$

where the constant L is given in Lemma 3.10 below. Then

$$\|Du\|_{\mathcal{L}^{2,\vartheta}(\Omega_0, \mathbb{R}^{nN})} \le c d_\vartheta^{-\vartheta} \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})} \tag{2.3}$$

for some $0 < d_\vartheta \le d$. The norm $\|Du\|_{\mathcal{L}^{2,\vartheta}(\Omega_0, \mathbb{R}^{nN})}$ is defined in Definition 3.1 below.

Remark 2.2. The inequality (2.3) implies that $Du \in BMO(\Omega_0, \mathbb{R}^{nN}))$ for $\vartheta = n$ and $Du \in C^{0,(\vartheta-n)/2}(\Omega_0, \mathbb{R}^{nN}))$ for $n < \vartheta < n + 2$.

For the rest of this article, we always suppose that $\omega_\infty/\nu > C_{cr}$.

Theorem 2.3. *Let $\Omega_0 \Subset \Omega$ with $\operatorname{dist}(\Omega_0, \partial\Omega) \ge 2d > 0$ and $n \le \vartheta < n + 2$ be given. Let u be a weak solution to the Dirichlet problem (1.1) where $g \in W^{1,2}(\Omega)$ and the hypotheses (i), (ii), (iii), (iv) be satisfied with M, ν and the function ω. Then the condition*

$$\frac{1}{5} \mathcal{M} c_0 \sqrt{U_{2d}(x)} \le 1, \quad \forall\, x \in \Omega_0 \tag{2.4}$$

where $0 < c_0 \le 1$ and

$$\mathcal{M} = \sup_{t_0 < t < \infty} \frac{\frac{\omega^2(t)}{\varepsilon} \, \mathrm{e}^{(\frac{\omega^2(t)}{2\sqrt{\mu}\varepsilon})^{2/(2\mu-1)}} - \mathrm{e}^{(\frac{1}{2\sqrt{\mu}})^{2/(2\mu-1)}}}{t - t_0}$$

implies that $Du \in C^{0,(\vartheta-n)/2}(\Omega_0, \mathbb{R}^{nN})$ in the case $\vartheta > n$ and $Du \in BMO(\Omega_0, \mathbb{R}^{nN})$ for $\vartheta = n$. Here $t_0 > 0$, $\omega(t_0) = \sqrt{\varepsilon}$, $\varepsilon > 0$ is specified in (4.8) where the constant $\epsilon_0 = \frac{1}{4(2^{n+5}L)^{\vartheta/(n+2-\vartheta)}}$ (L is the constant from Lemma 3.10) and $\mu \ge 2$.

Remark 2.4. As it is visible from the condition (2.4), an appropriate choice of the constant c_0 guarantees the regularity even if the excess U_{2d} is not assumed to be very small in Ω_0. Moreover, the term $(U_{2d}(x))^{1/2}$ in (2.4) can be replaced with $\|Du\|_{L^2(\Omega, \mathbb{R}^{nN})}/(2d)^{n/2}$ or, in the case of the Dirichlet problem (1.1), with $C_D^{1/2} \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}/(2d)^{n/2}$ where C_D is from (1.2). See Example 5.2 and 5.3 for additional information.

Remark 2.5. It can be seen (according to the assumption (iii)) that $\mathcal{M}$ is finite. On the parameter μ we only quote that its main goal is to damp the exponential growth. A structure of the Young functions in (3.1) and the estimates (4.3) - (4.5) below indicate a role of μ. It is visible from these estimates that it is possible to find a value of the parameter μ which is optimal in some measure.

The next theorem is a straightforward consequence of Theorem 2.3. It presents the well-known partial regularity result but unlike the other partial regularity results this theorem describes the so-called singular set a little bit more precisely.

Theorem 2.6. *Let $n < \vartheta < n+2$ be given and $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (1.1). *Let the hypotheses (i), (ii), (iii), (iv) be satisfied with M, ν and the function ω. Then there exists an open set $\Omega_{\mathcal{R}} \subset \Omega$ such that $u \in C^{1,(\vartheta-n)/2}(\Omega_{\mathcal{R}}, \mathbb{R}^N)$, and $\mathcal{H}^{n-2}(\Omega \setminus \Omega_{\mathcal{R}}) = 0$, where $\mathcal{H}^{n-2}$ is the $(n-2)$ - dimensional Hausdorff measure. Moreover,*

$$\Omega \setminus \Omega_{\mathcal{R}} = \big\{ x \in \Omega : \liminf_{r \to 0} ⨍_{B_r(x)} |Du(y) - (Du)_{x,r}|^2 \, dy \geq (\frac{5}{\mathcal{M} c_0})^2 \big\} \tag{2.5}$$

where the constants $\mathcal{M}$ and c_0 are defined in Theorem 2.3.

3. Preliminaries

Besides the spaces $C_0^\infty(\Omega, \mathbb{R}^N)$, the Hölder spaces $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and the Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W_0^{k,p}(\Omega, \mathbb{R}^N)$, we use the Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ (see Definition 3.1 below). By $X_{loc}(\Omega, \mathbb{R}^N)$ we denote the space of functions which belong to $X(\widetilde{\Omega}, \mathbb{R}^N)$ for every subdomain $\widetilde{\Omega} \Subset \Omega$ with a smooth boundary.

Definition 3.1 ([17])**.** Let $\lambda \in [0, n+q]$, $q \in [1, \infty)$. The Campanato space $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is the subspace of such functions $u \in L^q(\Omega, \mathbb{R}^N)$ for which

$$[u]^q_{\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)} = \sup_{r>0, x \in \Omega} \frac{1}{r^\lambda} \int_{\Omega_r(x)} |u(y) - u_{x,r}|^q \, dy < \infty$$

where $u_{x,r} = ⨍_{\Omega_r(x)} u(y)\, dy$ and $\Omega_r(x) = \Omega \cap B_r(x)$. The norm in the space $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is defined by $\|u\|_{\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)} = \|u\|_{L^q(\Omega, \mathbb{R}^N)} + [u]_{\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)}$.

Proposition 3.2 ([1, 9, 17])**.** *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary, for $q \in [1, \infty)$ and $0 < \lambda < \mu < \infty$ the relation $\mathcal{L}^{q,\mu}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ holds and $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $C^{0,(\lambda-n)/q}(\overline{\Omega}, \mathbb{R}^N)$, for $n < \lambda \leq n+q$.*

Now, let Φ, Ψ be a pair of the complementary Young functions

$$\Phi(u) = u \ln_+^\mu(au), \quad \Psi(u) \leq \overline{\Psi}(u) = \frac{1}{a} u \mathrm{e}^{(\frac{u}{2\sqrt{\mu}})^{2/(2\mu-1)}} \quad \text{for } u \geq 0 \tag{3.1}$$

where $a > 0$ and $\mu \geq 2$ are constants,

$$\ln_+(au) = \begin{cases} 0 & \text{for } 0 \leq u < 1/a, \\ \ln(au) & \text{for } u \geq 1/a. \end{cases} \tag{3.2}$$

Then the Young inequality for Φ, Ψ reads

$$uv \leq \Phi(u) + \Psi(v), \quad u, v \geq 0. \tag{3.3}$$

Lemma 3.3 ([26, pg.37]). *Let $\phi : [0,\infty) \to [0,\infty)$ be a nondecreasing function which is absolutely continuous on every closed interval of finite length, $\phi(0) = 0$. If $w \geq 0$ is measurable and $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ then*

$$\int_{\mathbb{R}^n} \phi \circ w \, dy = \int_0^\infty m\big(E(t)\big)\phi'(t)\, dt.$$

The next Lemma will be employed in the proof of Theorem 2.3.

Lemma 3.4 ([5, pg.388]). *Let $v \in L^2_{\rm loc}(\Omega, \mathbb{R}^N)$, $N \geq 1$, $B_r(x) \Subset \Omega$, $b > 0$ and $s \in (1, +\infty)$. Then*

$$\int_{B_r(x)} \ln_+^s(b|v|^2)\, dy \leq s\big(\frac{s-1}{e}\big)^{s-1} b \int_{B_r(x)} |v|^2\, dy.$$

The following Lemma is a small modification of [1, Lemma 1.IV].

Lemma 3.5. *Let A, $R_0 \leq R_1$ be positive numbers, $n \leq \vartheta < n+2$, η a nonnegative and nondecreasing function on $(0,\infty)$. Then there exist ϵ_0, c positive so that for any nonnegative, nondecreasing function ϕ defined on $[0, 2R_1]$ and satisfying with $(B_1 + B_2\eta(U_{2R_0})) \in [0, \epsilon_0]$ the inequality*

$$\phi(\sigma) \leq \big\{A(\frac{\sigma}{R})^{n+2} + \frac{1}{2}\big(1 + A(\frac{\sigma}{R})^{n+2}\big)[B_1 + B_2\eta(U_{2R})]\big\}\phi(2R) \tag{3.4}$$

for all σ, R such that $0 < \sigma < R \leq R_0$, it holds

$$\phi(\sigma) \leq c\sigma^\vartheta \phi(2R_0), \quad \forall \sigma\, :\, 0 < \sigma \leq R_0. \tag{3.5}$$

Remark 3.6. Note that we can take

$$\epsilon_0 = \frac{1}{2(2^{n+3}A)^{\frac{\vartheta}{n+2-\vartheta}}}, \quad c = \Big(\frac{(2^{n+3}A)^{\frac{1}{n+2-\vartheta}}}{2R_0}\Big)^\vartheta.$$

Proof. I. Without loss of generality we can suppose that $A \geq 1$. Choose $\tau \in (0,1)$ so that $2^{n+3}A\tau^{n+2-\vartheta} = 1$, i.e. $\tau = (\frac{1}{2^{n+3}A})^{1/(n+2-\vartheta)}$, $\epsilon_0 = \tau^\vartheta/2$.

II. We will prove by induction that

$$\phi(2\tau^k R_0) \leq \tau^{k\vartheta}\phi(2R_0), \quad U_{2\tau^k R_0} \leq U_{2R_0}. \tag{3.6}$$

Let $k = 1$. Putting $\sigma = 2\tau R_0$, $R = R_0$ in (3.4) we obtain thanks to the assumptions on τ, B_1, $B_2\eta$, ϵ_0, that

$$\begin{aligned}
&\phi(2\tau R_0)\\
&\leq \big\{2^{n+2}A\tau^{n+2} + \frac{1}{2}(1 + A(2\tau)^{n+2})[B_1 + B_2\eta(U_{2R_0})]\big\}\phi(2R_0)\\
&\leq \tau^\vartheta\big\{2^{n+2}A\tau^{n+2-\vartheta} + \frac{1}{2}(1 + 2^{n+2}A\tau^{n+2-\vartheta})[B_1 + B_2\eta(U_{2R_0})]\tau^{-\vartheta}\big\}\phi(2R_0)\\
&\leq \tau^\vartheta\Big(2^{n+2}A\tau^{n+2-\vartheta} + \frac{3}{4}\epsilon_0\tau^{-\vartheta}\Big)\phi(2R_0)\\
&\leq \tau^\vartheta\phi(2R_0).
\end{aligned}$$

Therefore,

$$U_{2\tau R_0} \leq U_{2R_0}.$$

Suppose (3.6) is valid for $j = 1, \dots, k$ and put $\sigma = 2\tau^{k+1}R_0$, $R = \tau^k R_0$ into (3.4). We obtain

$$\phi(2\tau^{k+1}R_0) \leq \big\{2^{n+2}A\tau^{n+2} + \frac{1}{2}(1 + A(2\tau)^{n+2})[B_1 + B_2\eta(U_{2\tau^k R_0})]\big\}\phi(2\tau^k R_0).$$

Using now (3.6) for $j = k$, choice of τ, assumptions on B_1, $B_2\eta$, ϵ_0 and estimates of $\phi(2\tau^k R_0)$ we have

$$\begin{aligned}\phi(2\tau^{k+1}R_0) &\le \Big(2^{n+2}A\tau^{n+2-\vartheta} + \frac{3}{4}\epsilon_0\tau^{-\vartheta}\Big)\tau^{\vartheta}\phi(2\tau^k R_0)\\ &\le \tau^{\vartheta}\phi(2\tau^k R_0) = \tau^{\vartheta(k+1)}\phi(2R_0).\end{aligned}$$

As $\vartheta \ge n$ it immediately implies the estimate $U_{2\tau^{k+1}R_0} \le U_{2R_0}$ and we have (3.6).

III. Now let σ be an arbitrary positive number less than R_0. Then there is an integer k such that $2\tau^{k+1}R_0 \le \sigma < 2\tau^k R_0$. Using the monotonicity of ϕ, this inequality and (3.6) we obtain

$$\phi(\sigma) \le \phi(2\tau^k R_0) \le \tau^{k\vartheta}\phi(2R_0) \le \sigma^{\vartheta}\frac{1}{(2\tau R_0)^{\vartheta}}\phi(2R_0).$$

If we set $c = (2\tau R_0)^{-\vartheta}$ in this estimate, the proof is complete. □

In the proof of Theorem 2.3 we will use a modification of the Natanson's Lemma [22, pg.262]. It reads as follows.

Lemma 3.7. *Let $f : [a,\infty) \to \mathbb{R}$ be a nonnegative function which is integrable on $[a,b]$ for all $a < b < \infty$ and*

$$\mathcal{N} = \sup_{0<h<\infty} \frac{1}{h}\int_a^{a+h} f(t)\,dt < \infty$$

is satisfied. Let $g : [a,\infty) \to \mathbb{R}$ be an arbitrary nonnegative, non-increasing and integrable function. Then

$$\int_a^{\infty} f(t)g(t)\,dt$$

exists and

$$\int_a^{\infty} f(t)g(t)\,dt \le \mathcal{N}\int_a^{\infty} g(t)\,dt\,.$$

Remark 3.8. The foregoing estimate is optimal because if we put $f(t) = 1$, $t \in [a,\infty)$ then an equality will be achieved.

Proof. For $a < b < \infty$ we put

$$\mathcal{N}_b = \sup_{0<h\le b-a} \frac{1}{h}\int_a^{a+h} f(t)\,dt < \infty\,.$$

The integral $\int_a^b f(t)g(t)\,dt$ exists because $f(t)g(t) \le g(a)f(t)$, for almost all $t \ge a$. If we put $F(t) = \int_a^t f(s)\,ds$ and use the integration by parts and the fact that $F(t) \le (t-a)\mathcal{N}_b$, we obtain

$$\begin{aligned}\int_a^b f(t)g(t)\,dt &= \int_a^b F'(t)g(t)\,dt = F(b)g(b) + \int_a^b F(t)(-g'(t))\,dt\\ &\le \mathcal{N}_b\Big[(b-a)g(b) + \int_a^b (t-a)(-g'(t))\,dt\Big] = \mathcal{N}_b\int_a^b g(t)\,dt\,.\end{aligned}$$

For an increasing sequence $\{b_k\}_{k=1}^{\infty}$ such that $b_k > a$ and $\lim_{k\to\infty} b_k = \infty$ put

$$f_k(t) = \begin{cases} f(t) & \text{for } a \le t \le b_k\\ 0 & \text{for } b_k < t < \infty\end{cases} \quad\text{and}\quad g_k(t) = \begin{cases} g(t) & \text{for } a \le t \le b_k\\ 0 & \text{for } b_k < t < \infty\,.\end{cases}$$

It is clear that if $k \to \infty$ then $f_k g_k \to fg$ a.e. in $[a,\infty)$ and

$$\int_a^\infty f_k(t)g_k(t)\,dt = \int_a^{b_k} f(t)g(t)\,dt \le \mathcal{N}_{b_k}\int_a^{b_k} g(t)\,dt \le \mathcal{N}\int_a^\infty g(t)\,dt.$$

Now the Fatou's Lemma implies the result. $\square$

In the proof of the next proposition we employ the following form of the Caccioppoli's inequality, which is possible to derive by the difference quotient method (see [9], pg.43-46). For the weak solution to the system (1.1) it holds

$$\int_{B_\sigma(x)} |D^2u|^2\,dy \le \frac{C_{Cacc}}{(\varrho-\sigma)^2}\int_{B_\varrho(x)} |Du-(Du)_{x,\varrho}|^2\,dy \tag{3.7}$$

where $x\in\Omega,\ 0<\sigma<\varrho\le \operatorname{dist}(x,\partial\Omega))$, $C_{Cacc}=16n^2N^2(M/\nu)^2$.

Proposition 3.9. *Let $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to the system (1.1). Then for every ball $B_{2R}(x)$, $x\in\Omega$ and arbitrary constants $b>0$, $\mu\ge 2$, c_1, $c_2\in\mathbb{R}$ we have*

$$\begin{aligned}
&\int_{B_R(x)} |Du(y)-(Du)_{B_R(x)}|^2 \ln_+^\mu(b|Du(y)-c_1|^2)\,dy\\
&\le C_P^2 C_{Cacc}\Big(C_{q\mu} b \fint_{B_R(x)} |Du(y)-c_1|^2\,dy\Big)^{1-1/q}\int_{B_{2R}(x)} |Du(y)-c_2|^2\,dy
\end{aligned}$$

where $1<q\le n/(n-2)$, $C_{q\mu}=\frac{q\mu\kappa_n}{q-1}\big(\frac{(\mu-1)q+1}{(q-1)\mathrm{e}}\big)^{\frac{(\mu-1)q+1}{q-1}}$ and $C_P(n,q)$ is the Sobolev - Poincarè constant.

Proof. Let $x\in\Omega$ and $0\le R\le \operatorname{dist}(x,\partial\Omega)/4$. We denote $B_R=B_R(x)$ for simplicity. By means of the Hölder inequality with $q\le n/(n-2)$, the Sobolev - Poincarè's and the Caccioppoli's inequalities we obtain

$$\begin{aligned}
&\int_{B_R} |Du-(Du)_{B_R}|^2\ln_+^\mu(b|Du-c_1|^2)\,dy\\
&\le \Big(\int_{B_R}|Du-(Du)_{B_R}|^{2q}\,dy\Big)^{1/q}\Big(\int_{B_R}\ln_+^{q\mu/(q-1)}(b|Du-c_1|^2)\,dy\Big)^{1-1/q}\\
&\le C_P^2 R^{n(-1+1/q)+2}\int_{B_R}|D^2u|^2\Big(\int_{B_R}\ln_+^{q\mu/(q-1)}(b|Du-c_1|^2)\,dy\Big)^{1-1/q}\\
&\le C_P^2 C_{Cacc}\Big(\fint_{B_R}\ln_+^{q\mu/(q-1)}(b|Du-c_1|^2)\,dy\Big)^{1-1/q}\int_{B_{2R}}|Du-c_2|^2\,dy
\end{aligned}$$

and finally, we obtain the result by means of Lemma 3.4. $\square$

The next Lemma 3.10 is well known; see, e.g. [1, 9, 23].

Lemma 3.10. *Let $v\in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to the linear system with constant coefficients of the type (1.1) satisfying (ii) and (iii). Then there exists a constant $L=c_L(n,N)(M/\nu)^{2(n+1)}$ such that for every $x\in\Omega$ and $0<\sigma\le R\le \operatorname{dist}(x,\partial\Omega)$ the estimate*

$$\int_{B_\sigma(x)}|Dv(y)-(Dv)_{x,\sigma}|^2\,dy \le L\big(\frac{\sigma}{R}\big)^{n+2}\int_{B_R(x)}|Dv(y)-(Dv)_{x,R}|^2\,dy$$

holds.

4. Proofs of theorems

Proof of Theorem 2.1. At first we recall that we set $\phi(r) = \phi(x_0, r) = \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2\,dx$ for $B_r(x_0) \subset \Omega$. Now let x_0 be any fixed point of $\overline{\Omega}_0 \subset \Omega$, with $\operatorname{dist}(\Omega_0, \partial\Omega) \ge d > 0$ and let $0 < R \le d$. Where no confusion can raise, we will use the notation B_R, $\phi(R)$ and $(Du)_R$ instead of $B_R(x_0)$, $\phi(x_0, R)$ and $(Du)_{x_0,R}$. Denoting by $A^{\alpha\beta}_{ij,0} = A^{\alpha\beta}_{ij}((Du)_R)$,

$$\widetilde{A}^{\alpha\beta}_{ij} = \int_0^1 A^{\alpha\beta}_{ij}((Du)_R + t(Du - (Du)_R))\,dt$$

we can rewrite the system (1.1) as

$$-D_\alpha\Big(A^{\alpha\beta}_{ij,0} D_\beta u^j\Big) = -D_\alpha\Big(\big(A^{\alpha\beta}_{ij,0} - \widetilde{A}^{\alpha\beta}_{ij}\big)\big(D_\beta u^j - (D_\beta u^j)_R\big)\Big).$$

Split u as $v + w$ where v is the solution to the Dirichlet problem

$$\begin{gathered} -D_\alpha\Big(A^{\alpha\beta}_{ij,0} D_\beta v^j\Big) = 0 \quad \text{in } B_R \\ v - u \in W^{1,2}_0(B_R, \mathbb{R}^N) \end{gathered}$$

and $w \in W^{1,2}_0(B_R, \mathbb{R}^N)$ is the weak solution of the system

$$-D_\alpha\Big(A^{\alpha\beta}_{ij,0} D_\beta w^j\Big) = -D_\alpha\Big(\big(A^{\alpha\beta}_{ij,0} - \widetilde{A}^{\alpha\beta}_{ij}\big)\big(D_\beta u^j - (D_\beta u^j)_R\big)\Big).$$

For every $0 < \sigma \le R$ it follows from Lemma 3.10 that

$$\int_{B_\sigma} |Dv - (Dv)_\sigma|^2\,dx \le L(\frac{\sigma}{R})^{n+2} \int_{B_R} |Dv - (Dv)_R|^2\,dx$$

hence

$$\begin{aligned} &\int_{B_\sigma} |Du - (Du)_\sigma|^2\,dx \\ &\le 2L(\frac{\sigma}{R})^{n+2} \int_{B_R} |Dv - (Dv)_R|^2\,dx + 2\int_{B_R} |Dw|^2\,dx \\ &\le 4L(\frac{\sigma}{R})^{n+2} \int_{B_R} |Du - (Du)_R|^2\,dx + 2(1 + 2L(\frac{\sigma}{R})^{n+2}) \int_{B_R} |Dw|^2\,dx. \end{aligned}$$

Now $w \in W^{1,2}_0(B_R, \mathbb{R}^N)$ satisfies

$$\begin{aligned} &\int_{B_R} A^{\alpha\beta}_{ij,0} D_\beta w^j D_\alpha \varphi^i\,dx \\ &\le \int_{B_R} |A^{\alpha\beta}_{ij,0} - \widetilde{A}^{\alpha\beta}_{ij}||D_\beta u^j - (D_\beta u^j)_R||D_\alpha \varphi^i|\,dx \\ &\le nN\Big(\int_{B_R} \omega^2(|Du - (Du)_R|)|Du - (Du)_R|^2\,dx\Big)^{1/2}\Big(\int_{B_R} |D\varphi|^2\,dx\Big)^{1/2} \end{aligned}$$

for any $\varphi \in W^{1,2}_0(B_R, \mathbb{R}^N)$. Choosing $\varphi = w$, we obtain

$$\nu^2 \int_{B_R} |Dw|^2\,dx \le n^2 N^2 \int_{B_R} \omega^2(|Du - (Du)_R|)|Du - (Du)_R|^2\,dx.$$

Now

$$\begin{aligned} \phi(\sigma) &= \int_{B_\sigma} |Du-(Du)_\sigma|^2\,dx \\ &\le 4L(\frac{\sigma}{R})^{n+2}\int_{B_R}|Du-(Du)_R|^2\,dx \\ &\quad + \frac{2n^2N^2(1+2L(\frac{\sigma}{R})^{n+2})}{\nu^2}\int_{B_R}\omega^2(|Du-(Du)_R|)|Du-(Du)_R|^2\,dx. \end{aligned} \tag{4.1}$$

As ω is bounded by ω_∞, we can deduce from (4.1) that

$$\phi(\sigma) \le \big[4L(\frac{\sigma}{R})^{n+2} + \frac{1}{2}(1+4L(\frac{\sigma}{R})^{n+2})4n^2N^2(\frac{\omega_\infty}{\nu})^2\big]\phi(R)$$

for any $0<\sigma<R<d$. Following Lemma 3.5 we put $A=4L$, $B_2=0$ and $B_1=4n^2N^2(\frac{\omega_\infty}{\nu})^2$. Now the assumptions of Lemma 3.5 will be fulfilled if

$$4n^2N^2(\frac{\omega_\infty}{\nu})^2 \le \epsilon_0\,.$$

Using (2.2) we can conclude (taking into account (1.2), (1.3) as well) that the result follows in a standard way. □

Proof of Theorem 2.3. We recall again that we set $\phi(r)=\phi(x_0,r)=\int_{B_r(x_0)}|Du-(Du)_{x_0,r}|^2\,dx$ and $U_r=U_r(x_0)=⨍_{B_r(x_0)}|Du(x)-(Du)_{x_0,r}|^2\,dx$ for $B_r(x_0)\subset\Omega$. Let x_0 be any fixed point of $\overline{\Omega}_0\subset\Omega$, $\operatorname{dist}(\Omega_0,\partial\Omega)\ge 2d>0$, $B_{2R}(x_0)\subset\Omega$. Following the first part of the proof of Theorem 2.1 step by step, we obtain the estimate (4.1).

To estimate the last integral in (4.1) we use the Young inequality (3.3) (here complementary functions are defined through (3.1)) and for any $0<\varepsilon<\omega_\infty^2$ we obtain

$$\begin{aligned} &\int_{B_R}\omega^2(|Du-(Du)_R|)|Du-(Du)_R|^2\,dx \\ &\le \varepsilon\int_{B_R}|Du-(Du)_R|^2\ln_+^\mu\big(a\varepsilon|Du-(Du)_R|^2\big)\,dx + \int_{B_R}\overline{\Psi}(\frac{\omega_R^2}{\varepsilon})\,dx \\ &= \varepsilon I_1 + I_2 \end{aligned} \tag{4.2}$$

where $\omega_R^2(x)=\omega^2(|Du(x)-(Du)_R|)$.

The term I_1 can be estimated by means of Proposition 3.9 and we obtain

$$I_1 \le C_P^2 C_{Cacc} C_{q\mu}^{1-1/q}(2^n a\varepsilon U_{2R})^{1-1/q}\phi(2R) = K(a\varepsilon U_{2R})^{1-1/q}\phi(2R) \tag{4.3}$$

where $1<q\le n/(n-2)$ and $K=C_P^2C_{Cacc}(2^nC_{q\mu})^{1-1/q}$.

Applying Lemma 3.3 to the second integral I_2 we have

$$I_2 = \int_{B_R}\overline{\Psi}(\frac{\omega_R^2}{\varepsilon})\,dx = \frac{1}{a}\int_0^\infty \frac{d}{dt}\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon})m_R(t)\,dt := \frac{1}{a}\widetilde{I}_2 \tag{4.4}$$

where

$$\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) = \frac{\omega^2(t)}{\varepsilon}e^{(\frac{\omega^2(t)}{2\sqrt{\mu}\,\varepsilon})^{2/(2\mu-1)}} \quad \text{for } t>0$$

and $m_R(t)=m(\{y\in B_R(x_0):|Du-(Du)_R|>t\})$.

Using the estimate $m_R(t) \le \kappa_n R^n$, κ_n is the Lebesgue measure of the unit ball, we have (we use Lemma 3.7)

$$\begin{aligned}
\widetilde{I}_2 &\le \int_0^{t_0} \frac{d}{dt}\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) m_R(t)\,dt + \int_{t_0}^{\infty} \frac{d}{dt}\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) m_R(t)\,dt \\
&\le \kappa_n R^n \int_0^{t_0} \frac{d}{dt}\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon})\,dt + \sup_{t_0<t<\infty}\Big(\frac{1}{t-t_0}\int_{t_0}^{t}\frac{d}{ds}\widetilde{\Psi}(\frac{\omega^2(s)}{\varepsilon})\,ds\Big)\int_{t_0}^{\infty} m_R(s)\,ds \\
&\le \kappa_n \widetilde{\Psi}\big(\frac{\omega^2(t_0)}{\varepsilon}\big) R^n + \sup_{t_0<t<\infty}\Big[\frac{\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) - \widetilde{\Psi}\big(\frac{\omega^2(t_0)}{\varepsilon}\big)}{t-t_0}\Big]\int_{B_R}|Du-(Du)_R|\,dx \\
&\le \frac{\kappa_n}{2^n U_{2R}}\widetilde{\Psi}\big(\frac{\omega^2(t_0)}{\varepsilon}\big)\phi(2R) + \frac{\mathcal{M}}{2^{n/2}}\kappa_n^{1/2}(2R)^{n/2}\phi^{1/2}(2R) \\
&< \Big[\frac{\widetilde{\Psi}(\frac{\omega^2(t_0)}{\varepsilon})}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\Big]\phi(2R)
\end{aligned} \tag{4.5}$$

where

$$\mathcal{M} = \sup_{t_0<t<\infty} \frac{\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) - \widetilde{\Psi}\big(\frac{\omega^2(t_0)}{\varepsilon}\big)}{t-t_0}. \tag{4.6}$$

If for some $R>0$ the average $U_R = 0$ then it is clear that x_0 is the regular point. So in the next we can suppose U_R is positive for all $R>0$.

Inserting (4.2)–(4.5) into (4.1) yields

$$\begin{aligned}
\phi(\sigma) &\le 4L(\frac{\sigma}{R})^{n+2}\phi(R) + 2n^2N^2(1+2L(\frac{\sigma}{R})^{n+2}) \\
&\quad\times\Big[\frac{\varepsilon K}{\nu^2}(2^n a\varepsilon U_{2R})^{1-1/q} + \frac{1}{a\nu^2}\Big(\frac{\widetilde{\Psi}\big(\frac{\omega^2(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\Big)\Big]\phi(2R).
\end{aligned} \tag{4.7}$$

In (4.7) we can choose

$$a = \frac{16\mathrm{e}n^2N^2}{\epsilon_0\nu^2 c_0\, U_{2R}} \quad \text{for } U_{2R} > 0$$

where $0 < c_0 \le 1$ be an arbitrary constant and

$$\varepsilon = \epsilon_0^{\alpha}\nu^{\beta} \tag{4.8}$$

where α, $\beta \in \mathbb{R}$ are constants, $\epsilon_0 = \frac{1}{4(2^{n+5}L)^{\vartheta/(n+2-\vartheta)}}$ (we remind that $\omega^2(t_0) = \varepsilon$).

Then for $U_{2R} > 0$, we obtain

$$\begin{aligned}
\phi(\sigma) &\le 4L(\frac{\sigma}{R})^{n+2}\phi(R) + \frac{1}{2}\Big(1+2L(\frac{\sigma}{R})^{n+2}\Big) \\
&\quad\times\big[KK_1\epsilon_0^{\alpha+(\alpha-1)(1-1/q)}\nu^{(\beta-2)(2-1/q)} + \frac{\epsilon_0}{4\mathrm{e}^2}\big(\mathrm{e} + \mathcal{M}\sqrt{U_{2R}}\big)\big]\phi(2R) \\
&= 4L\big(\frac{\sigma}{R}\big)^{n+2}\phi(R) + \frac{1}{2}\Big(1+2L(\frac{\sigma}{R})^{n+2}\Big) \\
&\quad\times\big[KK_1(\epsilon_0^{\alpha-1}\nu^{\beta-2})^{2-1/q} + \frac{c_0}{4} + \frac{\mathcal{M}}{4\mathrm{e}}c_0\sqrt{U_{2R}}\big]\epsilon_0\phi(2R)
\end{aligned} \tag{4.9}$$

where $K_1 = 4n^2N^2(2^{n+4}\mathrm{e}n^2N^2/c_0)^{1-1/q}$.

The constants α and β can be always chosen in such a way that

$$KK_1(\epsilon_0^{\alpha-1}\nu^{\beta-2})^{2-1/q} \le \frac{1}{4}$$

and finally we have

$$\phi(\sigma) \le 4L(\frac{\sigma}{R})^{n+2}\phi(R) + \frac{1}{2}\big(1 + 2L(\frac{\sigma}{R})^{n+2}\big)\big(\frac{1}{2} + \frac{1}{10}\mathcal{M}c_0\sqrt{U_{2R}}\big)\epsilon_0\phi(2R). \quad (4.10)$$

We can put

$$B_1 = \frac{1}{2}\epsilon_0, \quad B_2 = \frac{1}{10}\mathcal{M}\epsilon_0$$

and if we take into account assumption (2.4) of Theorem 2.3 we can use Lemma 3.5. □

Proof of Theorem 2.6. Let $x_0 \in \Omega_{\mathcal{R}}$ and $R_1 > 0$ be chosen in such a way that $B_{2R_1}(x_0) \subset \Omega$ and let $0 < R < R_1$. Using the same procedure as in the proof of Theorem 2.3 gives us the estimates (4.10). As $x_0 \in \Omega_{\mathcal{R}}$, it is clear that there exists $0 < R_0 < R_1$ such that $U_{2R_0}(x_0) < 25/(\mathcal{M}c_0)^2$ and so (2.4) is satisfied and we can use Lemma 3.5 in the same way as at the end of the proof of Theorem 2.3. The claim then follows in a standard way (see, e.g. [5, Chapter VI]. □

5. Illustrating examples and comments

Example 5.1 ([6])**.** A class of systems where the above results can be applied is the class of the perturbed linear elliptic systems. Suppose $\mathcal{L} = (L_{ij}^{\alpha\beta})_{i,j,\alpha,\beta=1}^n$ is symmetric positive definite constant matrix such that

$$\lambda|\xi|^2 \le L_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j$$

and put

$$A_i^\alpha(p) = L_{ij}^{\alpha\beta}p_\beta^j + m(\sin\sqrt{|p_\alpha^i|} - \sqrt{|p_\alpha^i|}\cos\sqrt{|p_\alpha^i|})$$

where $0 < m \le \lambda$. The modulus of continuity ω from (iv) has the form

$$\omega(t) = \begin{cases} \frac{1}{2}m\sqrt{t} & \text{for } 0 \le t \le 4, \\ m & \text{for } t > 4. \end{cases}$$

If m is chosen in a suitable way (with respect to λ) then our results can guarantee the interior regularity of the gradient of weak solution to the Dirichlet problem (1.1).

Example 5.2. To illustrate some parameters from the proof of Theorem 2.3 we can consider the following modulus of continuity

$$\omega(t) = \begin{cases} \omega_0(t) = \frac{(1+s)^s\sqrt{\varepsilon}}{(1+\ln\frac{t_0 e^s}{t})^s} & \text{for } 0 < t \le t_0,\ s > 0, \\ \omega_1(t) = \sqrt{\varepsilon}\,kt^\gamma, & \text{for } t_0 < t \le t_1,\ 0 < \gamma \le 1,\ k > 0 \\ \omega_\infty & \text{for } t > t_1 \end{cases}$$

where $\varepsilon > 0$ is from (4.8), $\omega_0(t_0) = \omega_1(t_0) = \sqrt{\varepsilon} < \omega_\infty$.

For $\mathcal{M}$ from (4.6) (see (4.4) and (4.7) as well) where ω is the above function we obtain the estimate

$$\begin{aligned} \mathcal{M} &= \sup_{t_0<t<t_1} \frac{\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) - \widetilde{\Psi}(\frac{\omega^2(t_0)}{\varepsilon})}{t - t_0} \\ &= k^2 \sup_{t_0<t<t_1} \frac{t^{2\gamma}e^{(\frac{k^2t^{2\gamma}}{2\sqrt{\mu}})^{\frac{2}{2\mu-1}}} - t_0^{2\gamma}e^{(\frac{1}{2\sqrt{\mu}})^{\frac{2}{2\mu-1}}}}{t - t_0}. \end{aligned}$$

Example 5.3. As an another typical sample of the function $\omega = \omega(t)$ considered in Theorem 2.3, we can take modulus of continuity

$$\omega(t) = \begin{cases} \omega_0(t) = \frac{(1+s)^s\sqrt{\varepsilon}}{(1+\ln\frac{t_0 e^s}{t})^s} & \text{for } 0 < t \le t_0,\ s > 0, \\ \omega_1(t) = \sqrt{\varepsilon \ln(1+\theta(t))}, & \text{for } \quad t_0 < t \le t_1, \\ \omega_\infty & \text{for } t > t_1 \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is from (4.8), $\omega_0(t_0) = \omega_1(t_0) = \sqrt{\varepsilon} < \omega_\infty$, $\theta(t)$ is a suitable increasing function such that $\lim_{t\to t_0^+}\theta(t) = \mathrm{e} - 1$. For $\mathcal{M}$ defined by (4.6), where ω is the above function, we obtain

$$\begin{aligned} \mathcal{M} &= \sup_{t_0<t<t_1} \frac{\widetilde{\Psi}(\frac{\omega^2(t)}{\varepsilon}) - \widetilde{\Psi}\big(\frac{\omega^2(t_0)}{\varepsilon}\big)}{t - t_0} \\ &= \sup_{t_0<t<t_1} \frac{(1+\theta(t))^{\frac{1}{2\sqrt{\mu}}[\frac{1}{2\sqrt{\mu}}\ln(1+\theta(t))]^{-1+\frac{2}{2\mu-1}}} \ln(1+\theta(t)) - \mathrm{e}^{(\frac{1}{2\sqrt{\mu}})^{\frac{2}{2\mu-1}}}}{t-t_0}. \end{aligned}$$

If we choose $\mu = 2$, $t_0 \ge 1$ and $\theta(t) = \Theta(\mathrm{e}^2 + t)^{\ln^{1/3}(1+t)}$ ($\Theta > 0$ is a constant), we can see that $\mathcal{M} \le 1$ for $t_0 < t < t_1$. In this case the condition ((2.4) takes the form

$$\frac{1}{5}c_0\sqrt{U_{2d}(x)} \le 1\,, \quad \forall\, x \in \Omega_0\,.$$

Example 5.4. In $\Omega = B_R(0) \subset \mathbb{R}^n$ (the fact, that the ball B_R is centered at zero, has no importance for next considerations) we consider the Dirichlet problem (1.1) for $g \in W^{1,2}_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ and, moreover, we assume that for $0 \le \lambda \le n+2$ the estimate $R^{-\lambda}\int_{B_R(0)} |Dg - (Dg)_{0,R}|^2\,dy \le c(\lambda)$, with $c(\lambda) > 0$ holds. Then, choosing $\Omega_0 = B_r(0)$, $0 < r < R$ and $d = (R-r)/2$, the condition (2.4) will have the form

$$\frac{1}{5}\mathcal{M}c_0\Big(C_D\, c(\lambda)\kappa_n^{-1}(1-\frac{r}{R})^{-n}\, R^{\lambda-n}\Big)^{1/2} \le 1, \quad \forall\, x \in B_r(0) \tag{5.2}$$

where the constant C_D is from the estimate (1.3). If the function ω is defined by (5.1) then the condition (2.4) will have the form

$$\frac{1}{5}c_0\Big(C_D\, c(\lambda)\kappa_n^{-1}(1-\frac{r}{R})^{-n}\, R^{\lambda-n}\Big)^{1/2} \le 1, \quad \forall\, x \in B_r(0). \tag{5.3}$$

The last two conditions show that a suitable choice of R and λ gives regularity of solution in $\Omega_0 = B_r(0)$.

6. Appendix

Proof of the estimate (1.2). Denote by $A^{\alpha\beta}_{ij}(p) = \partial A^\alpha_i(p)/\partial p^\beta_j$ and put

$$\widetilde{A}^{\alpha\beta}_{ij} = \int_0^1 A^{\alpha\beta}_{ij}(tDu)\,dt.$$

Then we have

$$\begin{aligned} 0 &= -D_\alpha(A^\alpha_i(Du)) = -D_\alpha[A^\alpha_i(Du) - A^\alpha_i(0)] \\ &= -D_\alpha\Big(\int_0^1 \frac{d}{dt}A^\alpha_i(tDu)\,dt\Big) = -D_\alpha\Big(\int_0^1 A^{\alpha\beta}_{ij}(tDu)D_\beta u^j\,dt\Big) \\ &= -D_\alpha(\widetilde{A}^{\alpha\beta}_{ij}D_\beta u^j). \end{aligned}$$

Now the definition of the weak solution to (1.1) has the form

$$0 = \int_\Omega \widetilde{A}_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i \, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

Setting $\varphi = u - g$ into the previous equality and using (ii), (iii) we obtain

$$\nu \int_\Omega |Du|^2 \, dx \le M \sum_{i,\alpha} \sum_{j,\beta} \int_\Omega |D_\beta u^j||D_\alpha g^i| \, dx.$$

The estimate

$$\sum_{k=1}^{nN} |c_k| \le \Big(nN \sum_{k=1}^{nN} |c_k|^2 \Big)^{1/2}, \quad c_k \in \mathbb{R}$$

leads to

$$\nu \int_\Omega |Du|^2 \, dx \le nNM \Big(\int_\Omega |Du|^2 \, dx \Big)^{1/2} \Big(\int_\Omega |Dg|^2 \, dx \Big)^{1/2}.$$

The estimate (1.2) follows from the above inequality. □

Proof of the estimate(1.3). Denote by $A_{ij}^{\alpha\beta}(p) = \partial A_i^\alpha(p)/\partial p_j^\beta$ and put

$$\widetilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}((Dg)_\Omega + t(Du - (Dg)_\Omega)) \, dt.$$

The same procedure as above gives

$$0 = -D_\alpha(A_i^\alpha(Du)) = -D_\alpha[A_i^\alpha(Du) - A_i^\alpha((Dg)_\Omega)] = -D_\alpha(\widetilde{A}_{ij}^{\alpha\beta}(D_\beta u^j - (D_\beta g^j)_\Omega)).$$

Now the definition of weak solution to (1.1) is

$$0 = \int_\Omega \widetilde{A}_{ij}^{\alpha\beta}(D_\beta u^j - (D_\beta g^j)_\Omega) D_\alpha \varphi^i \, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

Setting $\varphi^i = [(u^i - (D_\alpha g^i)_\Omega x_\alpha) - (g^i - (D_\alpha g^i)_\Omega x_\alpha)]$ we have

$$0 = \int_\Omega \widetilde{A}_{ij}^{\alpha\beta}(D_\beta u^j - (D_\beta g^j)_\Omega)[(D_\alpha u^i - (D_\alpha g^i)_\Omega) - (D_\alpha g^i - (D_\alpha g^i)_\Omega)] \, dx$$

and finally (as in the proof of the estimate (1.2)) we obtain

$$\int_\Omega |Du - (Dg)_\Omega|^2 \, dx \le n^2 N^2 (\frac{M}{\nu})^2 \int_\Omega |Dg - (Dg)_\Omega|^2 \, dx.$$

Now the estimate

$$\int_\Omega |Du - (Du)_\Omega|^2 \, dx \le \int_\Omega |Du - c|^2 \, dx, \quad \forall c \in \mathbb{R}$$

gives the result. □

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