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A WAVELET REGULARIZATION METHOD FOR AN INVERSE HEAT CONDUCTION PROBLEM WITH CONVECTION TERM

WEI CHENG, YING-QI ZHANG, CHU-LI FU

Abstract. In this article, we consider an inverse heat conduction problem with convection, which is ill-posed; i.e., the solution does not depend continuously on the given data. A special projection dual least squares method generated by the family of Shannon wavelets is applied to formulate an approximate solution. Also an optimal-order estimate for the error between the approximate solution and exact solution is obtained.

1. Introduction

In many industrial applications it is needed to determine the temperature on the surface of a body, where the surface is inaccessible for measurements [2]. In this case, it is necessary to determine the surface temperature from a measured temperature history at a fixed location inside the body. This is called an inverse heat conduction problem (IHCP) and has been an interesting subject recently. The standard problem is to determine the temperature u in the sideways heat equation

$$\begin{gathered} u_t = u_{xx}, \quad x > 0, t > 0, \\ u(x,0) = 0, \quad x \geq 0, \\ u(1,t) = g(t), \quad t \geq 0, \\ u(x,t) \text{ remains bounded as } x \to \infty, \end{gathered} \tag{1.1}$$

which has been considered by many authors; see [3, 5, 6, 10, 12, 13, 14] and the references therein.

In this article we consider a non-standard inverse heat conduction problem: A heat conduction problem with convection term in a quarter plane which appears in some applied subjects [1, 8, 15, 16],

$$\begin{gathered} u_t + u_x = u_{xx}, \quad x > 0,\ t > 0, \\ u(x,0) = 0, \quad x \geq 0, \\ u(1,t) = g(t), \quad t \geq 0, \\ u(x,t) \text{ remains bounded as } x \to \infty, \end{gathered} \tag{1.2}$$

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where the convection term u_x relates to a fluid going through the body [1]. We want the temperature distribution in the interval $[0,1)$ for problem (1.2). This problem is ill-posed problem in the sense that small perturbations in the data may cause dramatically large errors in the solution. Details can be seen in [8].

Xiong and his colleagues investigated (1.2) by the central difference method in [15, 16]. Regińska [11] solved (1.1) in the interval $[0,1)$ by applying the wavelet dual least squares method, which is based on the family of Meyer wavelets. This regularization method has also been used for solving an unknown source identification problem by Dou and Fu [4]. In this paper, we solve (1.2) in the interval $[0,1)$ by determining the temperature distribution using a wavelet dual least squares method generated by the family of Shannon wavelets.

To the best of our knowledge, so far most theoretical results concerning the error estimates of regularization methods in the literature are of Hölder type; i.e., the approximate solution ν and the exact solution u satisfy

$$\|u(x,\cdot)-\nu(x,\cdot)\| \le 2E^{1-x}\delta^x$$

where E is an a priori bound on $u(0,t)$. However, from the inequality mentioned above we know that when $x \to 0^+$ the accuracy of the regularized solution becomes progressively lower. At $x=0$, it merely implies that the error is bounded by $2E$; i.e., the convergence of the regularized solution at $x=0$ is not proved. In this paper, we apply the wavelet dual least squares method to stabilize the problem (1.2). Taking suitable regularization parameter, we not only obtain the Hölder continuity with $p=0$ in (1.3) for $0<x<1$, but also get a logarithmic Hölder convergence error estimate with $p>0$ for $0 \le x < 1$, especially gain the logarithmic type convergence estimate on the boundary $x=0$. In a sense, this is an improvement of known result in [6], and as our aim here is to obtain only stability estimate.

As we consider (1.2) in $L^2(\mathbb{R})$ with respect to variable t, we extend $u(x,\cdot)$, $g(\cdot) := u(1,\cdot)$, $f(\cdot) := u(0,\cdot)$, and other functions of variable t appearing in the paper to be zero for $t<0$. By a solution of (1.2) we understand a function $u(x,t)$ satisfying (1.2) in the classical sense; and for every fixed $x \in [0,1)$, the functions $u(x,\cdot)$ belongs to $L^2(\mathbb{R})$. Throughout the paper, we assume that for the exact g, the solution u exists and satisfies an a-priori bound

$$\|f(\cdot)\|_p := \|u(0,\cdot)\|_p \le E, \quad p \ge 0, \tag{1.3}$$

where $\|f(\cdot)\|_p$ is defined by

$$\|f(\cdot)\|_p := \Big(\int_{-\infty}^{\infty}(1+\xi^2)^p|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}.$$

Since g is measured by the thermocouple, there will be measurement errors, and we would actually have as data some function $g_\delta \in L^2(\mathbb{R})$, for which

$$\|g_\delta(\cdot)-g(\cdot)\| \le \delta, \tag{1.4}$$

where the constant $\delta>0$ represents a bound on the measurement error, and $\|\cdot\|$ denotes the $L^2(\mathbb{R})$ norm and

$$\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi t}h(t)\,dt$$

is the Fourier transform of function $h(t)$. For the uniqueness of solution, we require that $\|u(x,\cdot)\|$ be bounded [7], which implied that $u(x,\cdot)|_{x\to\infty}$ is bounded. The solution of problem (1.2) is given by its Fourier transform [8, 15]:

$$\hat{u}(x,\xi) = e^{(1-x)\theta(\xi)}\,\hat{g}(\xi), \tag{1.5}$$

where

$$\begin{aligned} \theta(\xi) &= \sqrt{i\xi + 1/4} - 1/2 \\ &= (1/2)\big[\sqrt[4]{1+16\xi^2}(\cos(\beta/2) + i\sin(\beta/2)) - 1\big], \quad \xi \in \mathbb{R}, \end{aligned} \tag{1.6}$$

$$\beta = \arg(1+4i\xi), \quad \tan\beta = 4\xi, \quad -\pi/2 < \beta < \pi/2 \quad \xi \in \mathbb{R}, \tag{1.7}$$

$$\cos(\beta/2) = \frac{\sqrt{\sqrt{1+16\xi^2}+1}}{\sqrt{2}\sqrt[4]{1+16\xi^2}}, \quad \xi \in \mathbb{R}, \tag{1.8}$$

$$\sin(\beta/2) = \sigma\frac{\sqrt{\sqrt{1+16\xi^2}-1}}{\sqrt{2}\sqrt[4]{1+16\xi^2}}, \quad \xi \in \mathbb{R}, \quad \sigma = \operatorname{sign}(\xi). \tag{1.9}$$

It is easy to verify from (1.5) and (1.7) that

$$\hat{f}(\xi) = e^{\theta(\xi)}\hat{g}(\xi), \quad \xi \in \mathbb{R}. \tag{1.10}$$

The following lemma will be used in our proofs.

Lemma 1.1 ([8])**.** *Let $\theta(\xi)$ be given by (1.6), then there holds*

$$e^{-x\sqrt{|\xi|/2}} \le |e^{-x\theta(\xi)}| \le \sqrt{e}e^{-x\sqrt{|\xi|/2}}, \quad 0 \le x \le 1,\ \xi \in \mathbb{R}. \tag{1.11}$$

To formulate problem (1.2) for $x \in [0,1)$ in terms of an operator equation in the space $X = L^2(\mathbb{R})$, we define an operator $K_x : u(x,\cdot) \mapsto g(\cdot)$, i.e.,

$$\forall\, u(x,\cdot) \in X, \quad K_x u(x,t) = g(t), \quad 0 \le x < 1. \tag{1.12}$$

From (1.5), we obtain

$$\widehat{K_x u(x,\xi)} = e^{-(1-x)\theta(\xi)}\hat{u}(x,\xi) = \hat{g}(\xi) \quad 0 \le x < 1. \tag{1.13}$$

Denote $\widehat{K_x u(x,\xi)} := \widehat{K_x}\hat{u}(x,\xi)$, and we can see that $\hat{K}_x : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a multiplication operator,

$$\widehat{K_x}\hat{u}(x,\xi) = e^{-(1-x)\theta(\xi)}\hat{u}(x,\xi). \tag{1.14}$$

Lemma 1.2. *Let K_x^* be the adjoint to K_x, then K_x^* corresponds to the following problem where the left-hand side u_t of problem (1.2) is replaced by $-U_t$, says*

$$\begin{gathered} -U_t + U_x = U_{xx}, \quad x >,\ t > 0, \\ U(x,0) = 0, \quad x \ge 0, \\ U(1,t) = g(t), \quad t \ge 0, \\ U(x,t) \textit{remains bounded as } x \to \infty, \end{gathered} \tag{1.15}$$

and

$$\widehat{K_x^*} = e^{-(1-x)\overline{\theta(\xi)}}. \tag{1.16}$$

Proof. By (1.14) and the relations

$$\langle K_x u, v\rangle = \langle \widehat{K_x}\hat{u}, \hat{v}\rangle = \langle \hat{u}, \widehat{K_x}^*\hat{v}\rangle = \langle u, K_x^* v\rangle = \langle \hat{u}, \widehat{K_x^*}\hat{v}\rangle,$$

we have the adjoint operator K_x^* of K_x in frequency domain is

$$\widehat{K_x^*} = \widehat{K_x}^* = e^{-(1-x)\overline{\theta(\xi)}}.$$

On the other hand, Problem (1.15) can be formulated, in frequency space, as follows:

$$\begin{gathered} -i\xi\hat{U} + \hat{U}_x = U_{xx}, \quad x >, \ \xi \in \mathbb{R}, \\ \hat{U}(x,0) = 0, \quad x \geq 0, \\ \hat{U}(1,\xi) = g(\xi), \quad \xi \in \mathbb{R}, \\ \hat{U}(x,\xi) \text{ remains bounded as } x \to \infty. \end{gathered} \tag{1.17}$$

Problem (1.2) can be formulated, in the frequency space as

$$\begin{gathered} i\xi\hat{u} + \hat{u}_x = u_{xx}, \quad x >, \ \xi \in \mathbb{R}, \\ \hat{u}(x,0) = 0, \quad x \geq 0, \\ \hat{u}(1,\xi) = g(\xi), \quad \xi \in \mathbb{R}, \\ \hat{u}(x,\xi) \text{ remains bounded as } x \to \infty \end{gathered} \tag{1.18}$$

Taking the conjugate operator for problem (1.18), we realize that $\hat{U}(x,\xi) = \overline{\hat{u}(x,\xi)}$. Then, with (1.5), we conclude that

$$\hat{U}(x,\xi) = \overline{\hat{u}(x,\xi)} = e^{(1-x)\overline{\theta(\xi)}}\hat{g}(\xi);$$

i.e.,

$$\hat{g}(\xi) = e^{-(1-x)\overline{\theta(\xi)}}\,\hat{U}(x,\xi) = \widehat{K_x^*}\hat{U}(x,\xi) := \widehat{K_x^* U}. \tag{1.19}$$

This completes the proof. □

2. Wavelet dual least squares method

In this section we stabilize the non-standard inverse heat conduction problem (1.2) in the interval $0 \leq x < 1$ under condition (1.3) by a wavelet dual least squares method.

2.1. **Dual least squares method.** For an operator equation $Ku = g$, $K : X = L^2(\mathbb{R}) \mapsto X = L^2(\mathbb{R})$, a general projection method is generated by two subspace families $\{V_j\}$ and $\{Y_j\}$ of X and the approximate solution $u_j \in V_j$ is defined to be the solution of the problem

$$\langle Ku_j, y\rangle = \langle g, y\rangle, \quad \forall\, y \in Y_j, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in X. If $V_j \subset R(K^*)$ and subspaces Y_j are chosen in such a way that

$$K^* Y_j = V_j.$$

Then we obtain a special case of projection method known as the dual least squares method. If $\{\psi_\lambda\}_{\lambda\in\tilde{I}_j}$ is an orthogonal basis of V_j and y_λ is the solution of the equation

$$K^* y_\lambda = k_\lambda \psi_\lambda, \quad \|y_\lambda\| = 1, \tag{2.2}$$

the approximate solution is explicitly given by the expression

$$u_j = \sum_{\lambda\in\tilde{I}_j} \langle g, y_\lambda\rangle \frac{1}{k_\lambda}\psi_\lambda. \tag{2.3}$$

2.2. **Shannon wavelets.** The Shannon scaling function is $\phi = \frac{\sin(\pi t)}{\pi t}$ and its Fourier transform is

$$\hat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le \pi, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

The corresponding wavelet function ψ is given by its Fourier transform

$$\hat{\psi}(\xi) = \begin{cases} e^{-i\frac{\xi}{2}}, & \pi \le |\xi| \le 2\pi, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Let us list some notation: $\phi_{j,k}(t) := 2^{j/2}\phi(2^j t - k)$, $\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k)$, $j, k \in \mathbb{Z}$, $\Psi_{-1,k} := \phi_{0,k}$ and $\Psi_{l,k} := \psi_{l,k}$ for $l \ge 0$, the index set

$$\begin{gathered} \tilde{I} = \{\{j,k\} : j,\, k \in \mathbb{Z}\} \subset \mathbb{Z}^2, \\ \tilde{I}_J = \{\{j,k\} : j = -1, 0, \dots, J-1;\, k \in \mathbb{Z}\} \subset \mathbb{Z}^2. \end{gathered} \tag{2.6}$$

Due to the equality $V_J = V_{J-1} \oplus W_{J-1} = V_{J-2} \oplus W_{J-2} \oplus W_{J-1} = \dots = V_0 \oplus W_1 \oplus \dots \oplus W_{J-1}$, we can define the subspaces

$$V_J = \overline{\operatorname{span}\{\Psi_\lambda\}_{\lambda \in \tilde{I}_J}}. \tag{2.7}$$

We define an orthogonal projection $P_J : L^2(\mathbb{R}) \to V_J$:

$$P_J \varphi = \sum_{\lambda \in \tilde{I}_J} \langle \varphi, \Psi_\lambda \rangle \Psi_\lambda, \quad \forall \varphi \in L^2(\mathbb{R}), \tag{2.8}$$

according to (2.3) we easily conclude $u_J = P_J u$. From the point of view of an application to the problem (1.2), the important property of Shannon wavelets is the compactness of their support in the frequency space. Indeed, since

$$\hat{\psi}_{j,k}(\xi) = 2^{-j/2} e^{-i2^{-j}k\xi} \hat{\psi}(2^{-j}\xi),\ \hat{\phi}_{j,k}(\xi) = 2^{-j/2} e^{-i2^{-j}k\xi} \hat{\phi}(2^{-j}\xi), \tag{2.9}$$

it follows that for any $k \in \mathbb{Z}$.

$$\operatorname{supp}(\hat{\psi}_{j,k}) = \{\xi : \pi 2^j \le |\xi| \le \pi 2^{j+1}\}, \quad \operatorname{supp}(\hat{\phi}_{j,k}) = \{\xi : |\xi| \le \pi 2^j\}. \tag{2.10}$$

From (2.8), P_J can be seen as a low pass filter. The frequencies with greater than $\pi 2^{J+1}$ are filtered away.

Theorem 2.1. *If $u(x,t)$ is the solution of* (1.2) *satisfying the condition $\|u(0,\cdot)\|_p \le E$, then for any fixed $x \in [0,1)$,*

$$\|u(x,\cdot) - P_J u(x,\cdot)\| \le \sqrt{e}\,(2^{J+1})^{-p}\, e^{-x\sqrt{\frac{1}{2}\pi 2^J}} E. \tag{2.11}$$

Proof. From (2.8), we have

$$\begin{gathered} u(x,\cdot) = \sum_{\lambda} \langle u(x,\cdot), \Psi_\lambda \rangle \Psi_\lambda, \\ P_J u(x,\cdot) = \sum_{\lambda \in \tilde{I}_J} \langle u(x,\cdot), \Psi_\lambda \rangle \Psi_\lambda. \end{gathered}$$

Due to Parseval relation and (1.5) (1.10) (1.11) (1.3), we obtain

$$\begin{aligned} &\|u(x,\cdot) - P_J u(x,\cdot)\| \\ &= \|\hat{u}(x,\cdot) - \widehat{P_J u}(x,\cdot)\| \\ &= \|\sum_{\lambda \in \tilde{I}} \langle \hat{u}, \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda - \sum_{\lambda \in \tilde{I}_J} \langle \hat{u}, \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda\| \end{aligned}$$

$$
\begin{aligned}
&= \Big\| \sum_{\lambda \in \tilde{I}_{j \geq J+1}} \langle \hat{u}, \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda \Big\| \\
&= \Big\| \sum_{\lambda \in \tilde{I}_{j \geq J+1}} \langle e^{(1-x)\theta(\xi)} \, \hat{g}(\cdot), \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda \Big\| \\
&= \Big\| \sum_{\lambda \in \tilde{I}_{j \geq J+1}} \langle e^{-x\theta(\xi)} \, \hat{f}(\cdot), \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda \Big\| \\
&\leq \sup_{\pi 2^J \leq |\xi| \leq \pi 2^{J+1}} \big[|\xi|^{-p} |e^{-x\theta(\xi)}| \big] \Big\| \sum_{\lambda \in \tilde{I}_{j \geq J+1}} \langle (1+(\cdot)^2)^{p/2} \hat{f}(\cdot), \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda \Big\| \\
&\leq \sup_{\pi 2^J \leq |\xi| \leq \pi 2^{J+1}} \sqrt{e} |\xi|^{-p} e^{-x\sqrt{|\xi|/2}} E \\
&\leq \sqrt{e} (2^{J+1})^{-p} e^{-x\sqrt{\pi 2^J/2}} E.
\end{aligned}
$$

The proof is complete. □

2.3. **Subspaces Y_j.** In this section, we present some properties of the subspaces Y_j. According to $K^* Y_j = V_j$, the subspaces Y_j are spanned by ρ_λ, $\lambda \in \tilde{I}_J$, where

$$
K^* \rho_\lambda = \Psi_\lambda, \quad k_\lambda = \|\rho_\lambda\|^{-1}, \quad y_\lambda = \frac{\rho_\lambda}{\|\rho_\lambda\|} = k_\lambda \rho_\lambda. \tag{2.12}
$$

The value ρ_λ can be determined by solving the parabolic equation (see Lemma 1.2)

$$
\begin{gathered}
-U_t + U_x = U_{xx}, \quad x >, t > 0, \\
U(x,0) = 0, \quad x \geq 0, \\
U(1,t) = \Psi_{j,k}(t), \quad t \geq 0, \\
U(x,t) \text{ remains bounded as } x \to \infty.
\end{gathered} \tag{2.13}
$$

Because $\operatorname{supp} \hat{\psi}_{j,k}$ is compact, the solution exists for any $t \in (0, \infty)$. Similarly the solution of the adjoint equation is unique. So for a given Ψ_λ, ρ_λ can be uniquely determined according to (2.13), and

$$
\hat{\rho}_\lambda = e^{(1-x)\overline{\theta(\xi)}} \hat{\Psi}_\lambda(\xi) \; \Leftrightarrow \; \hat{y}_\lambda = e^{(1-x)\overline{\theta(\xi)}} k_\lambda \hat{\Psi}_\lambda(\xi), \; \lambda = \{j,k\}. \tag{2.14}
$$

The approximate solution for noisy data g_δ is explicitly given by

$$
P_J u^\delta(x,t) = u_J^\delta = \sum_{\lambda \in \tilde{I}_J} \langle u^\delta, \, \Psi_\lambda \rangle \Psi_\lambda = \sum_{\lambda \in \tilde{I}_J} \langle g_\delta, \, y_\lambda \rangle \frac{1}{k_\lambda} \Psi_\lambda. \tag{2.15}
$$

We call it the wavelet dual least squares approximation solution of problem (1.2) in the interval $0 \leq x < 1$.

3. Error estimates

In this section we estimating the error $\|P_J u^\delta - P_J u\|$.

Theorem 3.1. *If g_δ is noisy data satisfying $\|g(\cdot) - g_\delta(\cdot)\| \leq \delta$, then for any fixed $x \in [0,1)$,*

$$
\|P_J u^\delta - P_J u\| \leq c_4 e^{(r-r_1)\sqrt{\pi 2^J/2}} \delta. \tag{3.1}
$$

Proof. From (2.14), we obtain $\hat{y}_\lambda = e^{(1-x)\overline{\theta(\xi)}}\, k_\lambda \hat{\Psi}_\lambda$. Note that $P_J u^\delta$ given by (2.15), $P_J u$ given by (2.3) and (1.11), for $0 \le x < 1$, we have

$$\begin{aligned}
\|P_J u^\delta(x,\cdot) - P_J u(x,\cdot)\| &= \|\sum_{\lambda\in\tilde{I}_J} \langle g_\delta - g,\, y_\lambda\rangle \frac{1}{k_\lambda}\Psi_\lambda\| \\
&= \|\sum_{\lambda\in\tilde{I}_J} \langle \hat{g}_\delta - \hat{g},\, \hat{y}_\lambda\rangle \frac{1}{k_\lambda}\hat{\Psi}_\lambda\| \\
&= \|\sum_{\lambda\in\tilde{I}_J} \langle \hat{g}_\delta - \hat{g},\, e^{(1-x)\overline{\theta(\xi)}}\, k_\lambda \hat{\Psi}_\lambda\rangle \frac{1}{k_\lambda}\hat{\Psi}_\lambda\| \\
&\le \sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J} |e^{(1-x)\overline{\theta(\xi)}}| \cdot \Big\|\sum_{\lambda\in\tilde{I}_J} \langle \hat{g}_\delta - \hat{g},\, \hat{\Psi}_\lambda\rangle \hat{\Psi}_\lambda\Big\| \\
&\le \sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J} \left|e^{(1-x)\theta(\xi)}\right| \cdot \|\hat{P}_J(\hat{g}_\delta - \hat{g})\| \\
&\le \sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J} \left|e^{(1-x)\theta(\xi)}\right| \cdot \delta \\
&\le \sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J} e^{(1-x)\sqrt{|\xi|/2}}\delta \\
&\le e^{(1-x)\sqrt{\pi 2^J/2}}\delta.
\end{aligned}$$

This completes the proof. □

We now give the following result which is the main conclusion of this article.

Theorem 3.2. *Let u be the exact solution of* (1.2) *and let $P_J u^\delta$ be given by* (2.15). *Let the measured data $g_\delta(t)$, satisfy the condition* (1.4) *at $x = 1$, and the a priori condition* (1.3) *hold. If we select the regularization parameter*

$$J = \log_2\Big[\frac{2}{\pi}\Big(\ln\big(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p}\big)\Big)^2\Big], \tag{3.2}$$

then for any fixed $x \in [0,1)$,

$$\|u(x,\cdot) - P_J u^\delta(x,\cdot)\| \le E^{1-x}\delta^x\big(\ln\frac{E}{\delta}\big)^{-2p(1-x)}\big(\sqrt{e}+1+o(1)\big) \quad \text{as } \delta\to 0. \tag{3.3}$$

Proof. Combining Theorem 3.1 with Theorem 2.1, and noting the choice (3.2) of J, we have

$$\begin{aligned}
&|u(x,\cdot) - P_J u^\delta(x,\cdot)\| \\
&\le \sqrt{e}(2^{J+1})^{-p}\, e^{-x\sqrt{\frac{1}{2}\pi 2^J}} E + e^{(1-x)\sqrt{\pi 2^J/2}}\delta \\
&\le E\sqrt{e}\Big(\ln\big(\frac{E}{\delta}\big(\ln\frac{E}{\delta}\big)^{-2p}\big)\Big)^{-2p}\Big(\frac{E}{\delta}\big(\ln\frac{E}{\delta}\big)^{-2p}\Big)^{-x} \\
&+ \delta\Big(\frac{E}{\delta}\big(\ln\frac{E}{\delta}\big)^{-2p}\Big)^{1-x} \\
&\le E^{1-x}\delta^x\big(\ln\frac{E}{\delta}\big)^{-2p(1-x)}\Big\{\frac{\sqrt{e}(\ln\frac{E}{\delta})^{2p}}{\Big(\ln\big(\frac{E}{\delta}\big(\ln\frac{E}{\delta}\big)^{-2p}\big)\Big)^{2p}} + 1\Big\}.
\end{aligned}$$

Note that

$$\frac{\ln \frac{E}{\delta}}{\ln \Big(\frac{E}{\delta} \big(\ln \frac{E}{\delta} \big)^{-2p} \Big)} = \frac{\ln \frac{E}{\delta}}{\ln \frac{E}{\delta} - 2p \ln \big(\ln \frac{E}{\delta} \big)} \to 1 \quad \text{as } \delta \to 0;$$

therefore, for $\delta \to 0$,

$$\|u(x,\cdot) - P_J u^\delta(x,\cdot)\| \le E^{1-x} \delta^x \big(\ln \frac{E}{\delta} \big)^{-2p(1-x)} (\sqrt{e} + 1 + o(1)).$$

The proof is complete. □

Remark 3.3. (i) When $p = 0$ and $0 < x < 1$, estimate (3.3) is a Hölder stability estimate given by

$$\|u(x,\cdot) - P_J u^\delta(x,\cdot)\| \le (\sqrt{e} + 1) E^{1-x} \delta^x. \tag{3.4}$$

(ii) When $p > 0$ and $0 \le x < 1$, estimate (3.3) is a logarithmical Hölder stability estimate.

(iii) When $p > 0$ and $x = 0$, estimate (3.3) becomes

$$\|u(0,\cdot) - P_J u^\delta(0,\cdot)\| u^\delta(x,\cdot)\| \le E \big(\ln \frac{E}{\delta} \big)^{-2p} (\sqrt{e} + 1 + o(1)) \to 0 \quad \text{as } \delta \to 0. \tag{3.5}$$

We can see this estimate is a logarithmical stability estimate similar to the convergence estimate in [9].

Remark 3.4. In general, the a-priori bound E is unknown in practice. In this case, with

$$J = \log_2 \Big[\frac{2}{\pi} \Big(\ln \big(\frac{1}{\delta} (\ln \frac{1}{\delta})^{-2p} \big) \Big)^2 \Big], \tag{3.6}$$

we have

$$\|u(x,\cdot) - P_J u^\delta(x,\cdot)\| \le \delta^x \big(\ln \frac{1}{\delta} \big)^{-2p\,(1-x)} (\sqrt{e}\, E + 1 + o(1)) textas \delta \to 0,$$

where E is only a bounded positive constant and it is not necessary known, exactly.

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