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EXISTENCE OF MULTIPLE SOLUTIONS FOR A MIXED BOUNDARY-VALUE PROBLEM

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Abstract. Using three critical points theorems, we prove the existence of at least three solutions for a second-order mixed boundary-value problem.

1. Introduction

In this article, we show the existence of at least three weak solutions for the mixed boundary-value problem

$$\begin{gathered} -(pu')' + qu = \lambda f(x,u) + g(u) \quad \text{in } (0,1), \\ u(0) = 0, \quad u'(1) = 0, \end{gathered} \tag{1.1}$$

where $p, q \in L^\infty([0,1])$ are such that

$$p_0 := \operatorname{ess\,inf}_{x\in[0,1]} p(x) > 0, \quad q_0 := \operatorname{ess\,inf}_{x\in[0,1]} q(x) \geq 0,$$

λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L > 0$; i.e.,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

Motivated by the fact that such problems are used to describe a large class of physical phenomena, many authors looked for existence and multiplicity of solutions for second-order ordinary differential nonlinear equations, with mixed conditions at the ends. For an overview on this subject, we cite the papers [3, 4, 5, 9, 10, 15]. For instance, in [9], Bonanno and Tornatore, using Ricceri's Variational Principle [13], established the existence of infinitely many weak solutions for the mixed boundary-value problem

$$\begin{gathered} -(pu')' + qu = \lambda f(x,u) \quad \text{in } (a,b), \\ u(a) = u'(b) = 0, \end{gathered}$$

where $p, q \in L^\infty([a,b])$ such that

$$p_0 := \operatorname{ess\,inf}_{x\in[a,b]} p(x) > 0, \quad q_0 := \operatorname{ess\,inf}_{x\in[a,b]} q(x) \geq 0,$$

$f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ is a positive real parameter.

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We also refer the reader to [12] which, by means of an abstract critical point result of Ricceri [14], shows the existence of at least three solutions for the two-point boundary-value problem

$$u'' + (\lambda f(t,u) + g(u))h(t,u') = \mu p(t,u)h(t,u') \quad \text{in } (a,b),$$
$$u(a) = u(b) = 0,$$

where λ and μ are positive parameters, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with $g(0) = 0$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ is bounded, continuous, with $m := \inf h > 0$, and $p : [a,b] \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory function.

The goal of the present paper is to establish some new criteria for (1.1) to have at least three weak solutions (Theorems 3.1-3.3). Our analysis is mainly based on three recent critical point theorems that are contained in Theorems 2.1-2.3 below. In fact, employing rather different three critical points theorems, under different assumptions on the nonlinear term f, we obtain the exact collections of λ for whihc (1.1) admits at least three weak solutions in the space $\{u \in W^{1,2}([0,1]) : u(0) = 0\}$.

A special case of our main results is the following theorem.

Theorem 1.1. *Let $p, q \in L^\infty([a,b])$ such that*

$$p_0 := \operatorname{ess\,inf}_{x\in[a,b]} p(x) > 0, \quad q_0 := \operatorname{ess\,inf}_{x\in[a,b]} q(x) \geq 0,$$

$g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$ and $g(0) = 0$ such that $L < p_0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d > 0$ and $F(\xi) \geq 0$ in $[0,d]$ and

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^2} = 0, \quad \limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem

$$-(pu')' + qu = \lambda f(u) + g(u) \quad \text{in } (0,1),$$
$$u(0) = 0, \quad u'(1) = 0,$$

admits at least three weak solutions.

2. Preliminaries

First we here recall for the reader's convenience our main tools to prove the results; in the first one and the second one the coercivity of the functional $\Phi - \lambda\Psi$ is required, while in the third one a suitable sign hypothesis is assumed. The first result has been obtained in [6], the second one in [8] and the third one in [2]. We recall the third as given in [7].

Theorem 2.1 ([6, Theorem 3.1])**.** *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = \Psi(x_0) = 0$ and that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty \quad \textit{for all } \lambda \in [0, +\infty[.$$

Further, assume that there are $r > 0$, $x_1 \in X$ *such that* $r < \Phi(x_1)$ *and*

$$\sup_{x \in \overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(x) < \frac{r}{r + \Phi(x_1)} \Psi(x_1);$$

here $\overline{\Phi^{-1}(]-\infty,r[)}^w$ *denotes the closure of* $\Phi^{-1}(]-\infty,r[)$ *in the weak topology. Then, for each*

$$\lambda \in \Lambda_1 := \Big] \frac{\Phi(x_1)}{\Psi(x_1) - \sup_{x \in \overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(x)} \Big[,$$

the equation

$$\Phi'(u) - \lambda \Psi'(u) = 0 \tag{2.1}$$

has at least three solutions in X *and, moreover, for each* $h > 1$, *there exist an open interval*

$$\Lambda_2 \subseteq \Big[0, \frac{hr}{r\frac{\Psi(x_1)}{\Phi(x_1)} - \sup_{x \in \overline{\Phi^{-1}(-\infty,r[)}^w} \Psi(x)}\Big]$$

and a positive real number σ *such that, for each* $\lambda \in \Lambda_2$, *equation* (2.1) *has at least three solutions in* X *whose norms are less than* σ.

Theorem 2.2. [8, Theorem 3.6] *Let* X *be a reflexive real Banach space, let* $\Phi : X \to \mathbb{R}$ *be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable whose Gâteaux derivative admits a continuous inverse on* X^*, *and let* $\Psi : X \to \mathbb{R}$ *be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist* $r \in \mathbb{R}$ *and* $u_1 \in X$ *with* $0 < r < \Phi(u_1)$, *such that*

- (A1) $\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}$;
- (A2) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(u_1)}{\Psi(u_1)}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

Theorem 2.3 ([7, Corollary 3.1])**.** *Let* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for each* $\lambda > 0$ *and for every* u_1, u_2 *which are local minima for the functional* $\Phi - \lambda\Psi$ *and such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s \in [0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 *and* $\overline{v} \in X$, *with* $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, *such that*

- (B1) $\frac{\sup_{u \in \Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2\Psi(\overline{v})}{3\Phi(\overline{v})}$;
- (B2) $\frac{\sup_{u \in \Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each λ *in*

$$\Big] \frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\{\frac{r_1}{\sup_{u \in \Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u \in \Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\} \Big[,$$

the functional $\Phi - \lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function and $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

Put

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \quad G(t) := -\int_0^t g(\xi)d\xi$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$. Denote

$$X := \big\{u \in W^{1,2}([0,1]) : u(0) = 0\big\};$$

the usual norm in X is defined by

$$\|u\|_X := \Big(\int_0^1 (u(x))^2 dx + \int_0^1 (u'(x))^2 dx\Big)^{1/2}.$$

For every $u, v \in X$, we define

$$(u,v) := \int_0^1 p(x)u'(x)v'(x)dx + \int_0^1 q(x)u(x)v(x)dx. \tag{2.2}$$

Clearly, (2.2) defines an inner product on X whose corresponding norm is

$$\|u\| := \Big(\int_0^1 p(x)(u'(x))^2 dx + \int_0^1 q(x)(u(x))^2 dx\Big)^{1/2}.$$

Then, it is easy to see that the norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_X$. In the following, we will use $\|\cdot\|$ instead of $\|\cdot\|_X$. Note that X is a separable and reflexive real Banach space.

We say that a function $u \in X$ is a *weak solution* of problem (1.1) if

$$\begin{aligned}&\int_0^1 p(x)u'(x)v'(x)dx + \int_0^1 q(x)u(x)v(x)dx\\ &- \lambda\int_0^1 f(x,u(x))v(x)dx - \int_0^1 g(u(x))v(x)dx = 0\end{aligned}$$

for all $v \in X$.

By standard regularity results, if f is a continuous function, $p \in C^1([0,1])$ and $q \in C^0([0,1])$, then weak solutions of the problem (1.1) belong to $C^2([0,1])$, thus they are classical solutions.

It is well known that $(X, \|\cdot\|)$ is compactly embedded in $(C^0([0,1]), \|\cdot\|_\infty)$ and

$$\|u\|_\infty \le \frac{1}{\sqrt{p_0}}\|u\| \tag{2.3}$$

for all $u \in X$ (see, e.g., [16]).

Also, we use the following notation:

$$\|p\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} p(x), \quad \|q\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} q(x).$$

Suppose that the Lipschitz constant $L > 0$ of the function g satisfies $L < p_0$. Finally, put

$$k := \frac{3p_0}{6\|p\|_\infty + 2\|q\|_\infty}, \quad \tau := \frac{p_0 - L}{p_0 + L}.$$

For other basic notations and definitions, we refer the reader to [11, 17].

3. Main results

Our main results are the following theorems.

Theorem 3.1. *Assume that there exist a function $w \in X$, a positive function $a \in L^1$ and two positive constants r and γ with $\gamma < 2$ such that*

(A1) $\|w\|^2 > \frac{2p_0\, r}{p_0 - L}$;

(A2) $\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx < r\frac{\int_0^1 F(x,w(x))dx}{r+\frac{p_0+L}{2p_0}\|w\|^2}$;

(A3) $F(x,t) \le a(x)(1+|t|^\gamma)$ *for almost every $x \in [0,1]$ and for all $t \in \mathbb{R}$.*

Then, for each λ in

$$\Lambda_1 := \Big]\frac{\frac{p_0+L}{2p_0}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}, \frac{r}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}\Big[,$$

problem (1.1) admits at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2 \subseteq \Big[0, \frac{hr}{\frac{2p_0\, r}{(p_0+L)\|w\|^2}\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}\Big]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the problem (1.1) admits at least three weak solutions in X whose norms are less than σ.

Theorem 3.2. *Assume that there exist a function $w \in X$ and a positive constant r such that*

(B1) $\|w\|^2 > \frac{2p_0\, r}{p_0 - L}$;

(B2) $\frac{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}{r} < \frac{2p_0}{p_0+L}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}$;

(B3) $\frac{2}{p_0-L}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}{r}$.

Then, for each

$$\lambda \in \Big]\frac{p_0+L}{2p_0}\frac{\|w\|^2}{\int_0^1 F(x,w(x))dx}, \frac{r}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}\Big[,$$

problem (1.1) admits at least three weak solutions.

Theorem 3.3. *Suppose that $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies the condition $f(x,t) \ge 0$ for all $x \in [0,1]$ and $t \in \mathbb{R}$. Assume that there exist a function $w \in X$ and two positive constants r_1 and r_2 with $\frac{4p_0\, r_1}{p_0-L} < \|w\|^2 < \frac{p_0\, r_2}{p_0+L}$ such that*

(C1)
$$\frac{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r_1}{p_0-L}}} F(x,t)dx}{r_1} < \frac{4p_0}{3(p_0+L)}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2};$$

(C2)
$$\frac{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r_2}{p_0-L}}} F(x,t)dx}{r_2} < \frac{2p_0}{3(p_0+L)}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}.$$

Then, for each

$$\lambda \in \Big]\frac{3(p_0+L)}{4p_0}\frac{\|w\|^2}{\int_0^1 F(x,w(x))dx},\ \Theta_1\Big[,$$

where

$$\Theta_1 := \min\Big\{\frac{r_1}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}, \frac{\frac{r_2}{2}}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}\Big\},$$

problem (1.1) *admits at least three nonnegative weak solutions* v^1, v^2, v^3 *such that*

$$|v^j(x)| < \sqrt{\frac{2r_2}{p_0-L}}$$

for each $x \in [0,1]$ *and* $j = 1, 2, 3$.

Let us give particular consequences of Theorems 3.1-3.3 for a fixed test function w.

Corollary 3.4. *Assume that there exist a positive function* $a \in L^1$ *and three positive constants* c, d *and* γ *with* $c < \sqrt{2}d$ *and* $\gamma < 2$ *such that Assumption* (A3) *in Theorem 3.1 holds. Furthermore, suppose that*

(A4) $F(x,t) \ge 0$ *for all* $(x,t) \in [0,\frac{1}{2}] \times [0,d]$;

(A5) $\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx < (k\tau c^2)\frac{\int_{1/2}^1 F(x,d)dx}{k\tau c^2+d^2}$.

Then, for each λ *in*

$$\Lambda_1' := \Big]\frac{\frac{p_0+L}{2k}d^2}{\int_{1/2}^1 F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}, \frac{(p_0-L)c^2}{2\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}\Big[,$$

problem (1.1) *admits at least three weak solutions in* X *and, moreover, for each* $h > 1$, *there exist an open interval*

$$\Lambda_2' \subseteq \Big[0, \frac{(p_0-L)hc^2/2}{\frac{2k\tau c^2}{d^2}\int_{1/2}^1 F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}\Big]$$

and a positive real number σ *such that, for each* $\lambda \in \Lambda_2'$, *problem* (1.1) *admits at least three weak solutions in* X *whose norms are less than* σ.

Proof. We claim that all the assumptions of Theorem 3.1 are fulfilled with w given by

$$w(x) := \begin{cases} 2d^2x, & x \in [0,1/2[,\\ d, & x \in [1/2,1]. \end{cases} \tag{3.1}$$

and $r := (p_0-L)c^2/2$. It is easy to verify that $w \in X$ and, in particular, one has

$$2p_0d^2 \le \|w\|^2 \le \frac{p_0d^2}{k}.$$

Hence, taking into account that $c < \sqrt{2}d$, we have

$$\|w\|^2 > \frac{2p_0\, r}{p_0-L}.$$

Thus, (A1) holds. Since $0 \le w(x) \le d$ for each $x \in [0,1]$, the condition (A4) ensures that

$$\int_0^{1/2} F(x,w(x))dx \ge 0,$$

so from (A5),

$$\begin{aligned}
\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx &< (k\tau c^2)\frac{\int_{1/2}^1 F(x,d)dx}{k\tau c^2+d^2} \\
&= \frac{(p_0-L)kc^2}{(p_0-L)kc^2+(p_0+L)d^2}\int_{1/2}^1 F(x,d)dx \\
&= \frac{(p_0-L)c^2}{2}\frac{\int_{1/2}^1 F(x,d)dx}{\frac{(p_0-L)c^2}{2}+\frac{(p_0+L)d^2}{2k}} \\
&\le r\frac{\int_0^1 F(x,w(x))dx}{r+\frac{p_0+L}{2p_0}\|w\|^2},
\end{aligned}$$

and thus (A2) holds. Next notice that

$$\begin{aligned}
&\frac{\frac{p_0+L}{2p_0}\|w\|^2}{\int_0^1 F(x,w(x))dx-\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx} \\
&\le \frac{\frac{p_0+L}{2k}d^2}{\int_{1/2}^1 F(x,d)dx-\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}
\end{aligned}$$

and

$$\frac{r}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}=\frac{(p_0-L)c^2}{2\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.$$

In addition note

$$\begin{aligned}
&\frac{\frac{p_0+L}{2k}d^2}{\int_{1/2}^1 F(x,d)dx-\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \\
&< \frac{\frac{p_0+L}{2k}d^2}{\Big(\frac{\frac{(p_0-L)c^2}{2}+\frac{(p_0+L)d^2}{2k}}{\frac{(p_0-L)c^2}{2}}-1\Big)\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \\
&= \frac{(p_0-L)c^2}{2\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.
\end{aligned}$$

Finally note that

$$\begin{aligned}
&\frac{hr}{\frac{2p_0\, r}{(p_0+L)\|w\|^2}\int_0^1 F(x,w(x))dx-\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx} \\
&\le \frac{(p_0-L)hc^2/2}{\frac{2k\tau c^2}{d^2}\int_{1/2}^1 F(x,d)dx-\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx},
\end{aligned}$$

and taking into account that $\Lambda_1'\subseteq\Lambda_1$ and $\Lambda_2\subseteq\Lambda_2'$, we have the desired conclusion directly from Theorem 3.1. □

Corollary 3.5. *Assume that there exist two positive constants c and d with $c<d$ such that the assumption* (A4) *in Corollary 3.4 holds. Furthermore, suppose that*

- (B4) $\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx < \frac{k\tau c^2}{d^2}\int_{1/2}^1 F(x,d)dx$;
- (B5) $\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{c^2}$.

Then, for each

$$\lambda \in \Big]\frac{p_0+L}{2k}\frac{d^2}{\int_{1/2}^1 F(x,d)dx}, \frac{(p_0-L)c^2}{2\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}\Big[,$$

problem (1.1) *admits at least three weak solutions.*

Proof. All the assumptions of Theorem 3.2 are fulfilled by choosing w as given in (3.1) and $r := (p_0-L)c^2/2$, and bearing in mind that

$$2p_0d^2 \le \|w\|^2 \le \frac{p_0d^2}{k}.$$

and recalling

$$\int_0^{1/2} F(x,w(x))dx \ge 0.$$

Hence, by applying Theorem 3.2 we have the conclusion. □

Proof of Theorem 1.1. Fix $\lambda > \lambda^* := \frac{(p_0+L)d^2}{kF(d)}$ for some $d>0$. Since

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = 0,$$

there is $\{c_m\}_{m\in\mathbb{N}} \subseteq]0,+\infty[$ such that $\lim_{m\to+\infty} c_m = 0$ and

$$\lim_{m\to+\infty}\frac{\sup_{|\xi|\le c_m} F(\xi)}{c_m} = 0.$$

In fact, one has

$$\lim_{m\to+\infty}\frac{\sup_{|\xi|\le c_m} F(\xi)}{c_m} = \lim_{m\to+\infty}\frac{F(\xi_{c_m})}{\xi_{c_m}^2}.\frac{\xi_{c_m}^2}{c_m} = 0,$$

where $F(\xi_{c_m}) = \sup_{|\xi|\le c_m} F(\xi)$. Hence, there is $\bar{c}>0$ such that

$$\frac{\sup_{|\xi|\le \bar{c}} F(\xi)}{\bar{c}^2} < \min\Big\{\frac{k\tau F(d)}{2d^2};\ \frac{p_0-L}{2\lambda}\Big\}$$

and $\bar{c} < d$. From Corollary 3.5 we have the desired conclusion. □

Corollary 3.6. *Suppose that* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *satisfies the condition* $f(x,t)\ge 0$ *for all* $x\in[0,1]$ *and* $t\in\mathbb{R}$. *Assume that there exist three positive constants* c_1, c_2 *and* d *with* $c_1<d$ *and* $\sqrt{\frac{2}{k\tau}}d < c_2$ *such that*

- (C3) $\int_0^1 \sup_{t\in[-c_1,c_1]} F(x,t)dx < \frac{2k\tau c_1^2}{3d^2}\int_{1/2}^1 F(x,d)dx$;
- (C4) $\int_0^1 \sup_{t\in[-c_2,c_2]} F(x,t)dx < \frac{\tau c_2^2}{3d^2}\int_{1/2}^1 F(x,d)dx$.

Then, for each

$$\lambda \in \Big]\frac{3(p_0+L)}{4k}\frac{d^2}{\int_{1/2}^1 F(x,d)dx}, \Theta_2\Big[,$$

where

$$\Theta_2 := \min\Big\{\frac{(p_0-L)c_1^2}{2\int_0^1 \sup_{t\in[-c_1,c_1]} F(x,t)dx}, \frac{(p_0-L)c_2^2}{4\int_0^1 \sup_{t\in[-c_2,c_2]} F(x,t)dx}\Big\},$$

problem (1.1) *admits at least three nonnegative weak solutions* v^1, v^2, v^3 *such that* $|v^j(x)| < c_2$ *for each* $x\in[0,1]$ *and* $j=1,2,3$.

Proof. Following the same way as in the proof of Corollary 3.5, we achieve the stated assertion by applying Theorem 3.3 with w as given in (3.1), $r_1 := (p_0 - L)c_1^2/2$ and $r_2 := (p_0 - L)c_2^2/2$. □

We point out that, applying Theorems 3.1-3.3, we have the relevant results of Corollaries 3.4-3.6 for the following mixed boundary value problem with a complete equation

$$\begin{gathered} -(\bar{p}u')' + \bar{r}u' + \bar{q}u = \lambda f(x,u) + g(u) \quad \text{in } (a,b), \\ u(0) = 0, \quad u'(1) = 0, \end{gathered} \tag{3.2}$$

where $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$ and $g(0) = 0$, $\bar{p} \in C^1([0,1])$, $\bar{q}, \bar{r} \in C^0([0,1])$ and λ is a positive parameter. Moreover, $\bar{p}$ is nonnegative and R is a primitive of $\bar{r}/\bar{p}$.

If fact, since the solutions of problem (3.2) are solutions of the problem

$$\begin{gathered} -(e^{-R}\bar{p}u')' + e^{-R}\bar{q}u = \Big(\lambda f(x,u) + g(u)\Big)e^{-R} \quad \text{in } (0,1), \\ u(0) = 0, \quad u'(1) = 0, \end{gathered}$$

assuming the Lipschitz constant $L > 0$ of the function g satisfies

$$L < \min_{x\in[0,1]} e^{-R(x)}\bar{p}(x),$$

and setting

$$k' := \frac{3\min_{x\in[0,1]} e^{-R(x)}\bar{p}(x)}{6\|e^{-R}\bar{p}\|_\infty + 2\|e^{-R}\bar{q}\|_\infty}, \quad \tau' := \frac{\min_{x\in[0,1]} e^{-R(x)}\bar{p}(x) - L}{\min_{x\in[0,1]} e^{-R(x)}\bar{p}(x) + L},$$

under the assumptions of Corollary 3.4 but with (A5) replaced by the assumption

$$\int_0^1 \sup_{t\in[-c,c]} e^{-R(x)}F(x,t)dx < (k'\tau'c^2)\frac{\int_{1/2}^1 e^{-R(x)}F(x,d)dx}{k'\tau'c^2 + d^2},$$

by the same reasoning as in the proof of Corollary 3.4, using Theorem 3.1, for each λ in

$$\begin{aligned} \Lambda_1'' := \Big]&\frac{\frac{\min_{x\in[0,1]} e^{-R(x)}\bar{p}(x)+L}{2k'}d^2}{\int_{1/2}^1 e^{-R(x)}F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} e^{-R(x)}F(x,t)dx}, \\ &\frac{(\min_{x\in[0,1]} e^{-R(x)}\bar{p}(x) - L)c^2}{2\int_0^1 \sup_{t\in[-c,c]} e^{-R(x)}F(x,t)dx}\Big[, \end{aligned}$$

problem (3.2) admits at least three classical solutions in X; moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2'' \subseteq \Big[0, \frac{(\min_{x\in[0,1]} e^{-R(x)}\bar{p}(x) - L)hc^2/2}{\frac{2k'\tau'c^2}{d^2}\int_{1/2}^1 e^{-R(x)}F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} e^{-R(x)}F(x,t)dx}\Big]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2''$, problem (3.2) admits at least three classical solutions in X whose norms are less than σ. Moreover, under the assumptions of Corollary 3.5, but replacing Assumptions (B4) and (B5) by the assumptions

$$\int_0^1 \sup_{t\in[-c,c]} e^{-R(x)}F(x,t)dx < \frac{k'\tau'c^2}{d^2}\int_{1/2}^1 e^{-R(x)}F(x,d)dx$$

and

$$\limsup_{|t|\to+\infty}\frac{e^{-R(x)}F(x,t)}{t^2}<\frac{\int_0^1\sup_{t\in[-c,c]}e^{-R(x)}F(x,t)dx}{c^2},$$

respectively, by the same reasoning as given in the proof of Corollary 3.5, using Theorem 3.2, for each λ in

$$\Big]\frac{\min_{x\in[0,1]}e^{-R(x)}\bar{p}(x)+L}{2k'}\frac{d^2}{\int_{1/2}^1 e^{-R(x)}F(x,d)dx},\frac{(\min_{x\in[0,1]}e^{-R(x)}\bar{p}(x)-L)c^2}{2\int_0^1\sup_{t\in[-c,c]}e^{-R(x)}F(x,t)dx}\Big[,$$

problem (3.2) admits at least three classical solutions. Also, under the assumptions of Corollary 3.6, but replacing the condition $\sqrt{\frac{2}{k\tau}}d<c_2$, Assumptions (C3) and (C4) by the condition $\sqrt{\frac{2}{k'\tau'}}d<c_2$, the assumptions

$$\int_0^1\sup_{t\in[-c_1,c_1]}e^{-R(x)}F(x,t)dx<\frac{2k'\tau'c_1^2}{3d^2}\int_{1/2}^1 e^{-R(x)}F(x,d)dx$$

and

$$\int_0^1\sup_{t\in[-c_2,c_2]}e^{-R(x)}F(x,t)dx<\frac{\tau'c_2^2}{3d^2}\int_{1/2}^1 e^{-R(x)}F(x,d)dx,$$

respectively, by the same reasoning as in the proof of Corollary 3.6, using Theorem 3.3, for each

$$\lambda\in\Big]\frac{3(\min_{x\in[0,1]}e^{-R(x)}\bar{p}(x)+L)}{4k'}\frac{d^2}{\int_{1/2}^1 e^{-R(x)}F(x,d)dx},\Theta_3\Big[,$$

where

$$\Theta_3:=\min\Big\{\frac{(\min_{x\in[0,1]}e^{-R(x)}\bar{p}(x)-L)c_1^2}{2\int_0^1\sup_{t\in[-c_1,c_1]}e^{-R(x)}F(x,t)dx},\frac{(\min_{x\in[0,1]}e^{-R(x)}\bar{p}(x)-L)c_2^2}{4\int_0^1\sup_{t\in[-c_2,c_2]}e^{-R(x)}F(x,t)dx}\Big\},$$

problem (3.2) admits at least three nonnegative classical solutions v^1,v^2,v^3 such that $|v^j(x)|<c_2$ for each $x\in[0,1]$ and $j=1,2,3$.

4. Proofs

Proof of Theorem 3.1. Our aim is to apply Theorem 2.1 to our problem. To this end, for each $u\in X$, we let the functionals $\Phi,\Psi:X\to\mathbb{R}$ be defined by

$$\Phi(u):=\frac{1}{2}\|u\|^2+\int_0^1 G(u(x))dx,\quad \Psi(u):=\int_0^1 F(x,u(x))dx,$$

and put

$$I_\lambda(u):=\Phi(u)-\lambda\Psi(u)\quad\forall\, u\in X.$$

The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative at the point $u\in X$ is the functional $\Phi'(u)\in X^*$, given by

$$\Phi'(u)(v)=\int_0^1 p(x)u'(x)v'(x)dx+\int_0^1 q(x)u(x)v(x)dx-\int_0^1 g(u(x))v(x)dx$$

for every $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ is a Lipschitzian operator. Indeed, for any $u, v \in X$, there holds

$$
\begin{aligned}
&\|\Phi'(u) - \Phi'(v)\|_{X^*} \\
&= \sup_{\|w\|\le 1} |(\Phi'(u) - \Phi'(v), w)| \\
&\le \sup_{\|w\|\le 1} |(u - v, w)| + \sup_{\|w\|\le 1} \int_0^1 |g(u(x)) - g(v(x))|\,|w(x)|dx \\
&\le \sup_{\|w\|\le 1} \|u - v\|\,\|w\| + \sup_{\|w\|\le 1} \Big(\int_0^1 |g(u(x)) - g(v(x))|^2 \Big)^{1/2} \Big(\int_0^1 |w(x)|^2 \Big)^{1/2}.
\end{aligned}
$$

Recalling that g is Lipschitz continuous and the embedding $X \hookrightarrow L^2([0,1])$ is compact, the claim is true. In particular, we derive that Φ is continuously differentiable. The inequality (2.3) yields for any $u, v \in X$ the estimate

$$
\begin{aligned}
(\Phi'(u) - \Phi'(v), u - v) &= (u - v, u - v) - \int_0^1 \big(g(u(x)) - g(v(x))\big)\big(u(x) - v(x)\big)dx \\
&\ge \frac{p_0 - L}{p_0}\|u - v\|^2.
\end{aligned}
$$

By the assumption $L < p_0$, it turns out that Φ' is a strongly monotone operator. So, by applying Minty-Browder theorem [17, Theorem 26.A]), $\Phi' : X \to X^*$ admits a Lipschitz continuous inverse. On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gateaux derivative at the point $u \in X$ is given by

$$
\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)dx
$$

for every $v \in X$. Note that the weak solutions of (1.1) are exactly the critical points of I_λ. Also, since g is Lipschitz continuous and satisfies $g(0) = 0$, we have from (2.3) that

$$
\frac{p_0 - L}{2p_0}\|u\|^2 \le \Phi(u) \le \frac{p_0 + L}{2p_0}\|u\|^2, \tag{4.1}
$$

for all $u \in X$, and so Φ is coercive.

Furthermore from (A3) for any fixed $\lambda \in [0, +\infty[$, using (4.1), taking (2.3) into account, we have

$$
\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 + \int_0^1 G(u(x))dx - \lambda \int_0^1 F(x, u(x))dx \\
&\ge \frac{p_0 - L}{2p_0}\|u\|^2 - \lambda \int_0^1 (a(x)(1 + |u(x)|^\gamma)dx \\
&\ge \frac{p_0 - L}{2p_0}\|u\|^2 - \lambda\|a\|_{L^1([0,1])}(1 + \frac{1}{p_0^{\gamma/2}}\|u\|^\gamma),
\end{aligned}
$$

and so

$$
\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty.
$$

Also according to (A1) we achieve $\Phi(w) > r$. From the definition of Φ and by using (4.1) we have

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \Big\{u \in X : \Phi(u) < r\Big\} \\ &\subseteq \Big\{u \in X : \|u\| < \sqrt{\frac{2p_0 r}{p_0 - L}}\Big\} \\ &\subseteq \Big\{u \in X : |u(x)| < \sqrt{\frac{2r}{p_0 - L}} \quad \text{for all } x \in [0,1]\Big\}.\end{aligned}$$

So, we obtain

$$\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u) \le \int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx.$$

Therefore, from (A2) and (4.1), we have

$$\begin{aligned}\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u) &\le \int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx \\ &< \frac{r}{r + \frac{p_0+L}{2p_0}\|w\|^2} \int_0^1 F(x, w(x))dx \\ &< \frac{r}{r + \Phi(w)} \Psi(w).\end{aligned}$$

Now, we can apply Theorem 2.1. Note for each $x \in [0,1]$,

$$\frac{\Phi(w)}{\Psi(w) - \sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u)} \le \frac{\frac{p_0+L}{2p_0}\|w\|^2}{\int_0^1 F(x, w(x))dx - \int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx}$$

and

$$\frac{r}{\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u)} \ge \frac{r}{\int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx}.$$

Note also that (A2) implies

$$\begin{aligned}&\frac{\frac{p_0+L}{2p_0}\|w\|^2}{\int_0^1 F(x, w(x))dx - \int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx} \\ &< \frac{\frac{p_0+L}{2p_0}\|w\|^2}{(\frac{r + \frac{p_0+L}{2p_0}\|w\|^2}{r} - 1) \int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx} \\ &= \frac{r}{\int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx}.\end{aligned}$$

Also,

$$\begin{aligned}&\frac{hr}{r\frac{\Psi(w)}{\Phi(w)} - \sup_{u \in \overline{\Phi^{-1}(-\infty, r[)}^w} \Psi(u)} \\ &\le \frac{hr}{\frac{2p_0 r}{(p_0+L)\|w\|^2} \int_0^1 F(x, w(x))dx - \int_0^1 \sup_{|t| \le \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx} = \rho.\end{aligned}$$

From (A2) it follows that

$$\begin{aligned}
&\frac{2p_0 r}{(p_0+L)\|w\|^2}\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx\\
&> \Big(\frac{2p_0 r}{(p_0+L)\|w\|^2} - \frac{r}{r+\frac{p_0+L}{2p_0}\|w\|^2}\Big)\int_0^1 F(x,w(x))dx\\
&\ge \Big(\frac{2p_0 r}{(p_0+L)\|w\|^2} - \frac{2p_0 r}{(p_0+L)\|w\|^2}\Big)\int_0^1 F(x,w(x))dx = 0,
\end{aligned}$$

since $\int_0^1 F(x,w(x))dx \ge 0$ (note $F(x,0)=0$ so $\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx \ge 0$ and now apply (A2)). Now with $x_0 = 0$ and $x_1 = w$ from Theorem 2.1 (note $\Psi(0) = 0$) it follows that, for each $\lambda \in \Lambda_1$, the problem (1.1) admits at least three weak solutions and there exist an open interval $\Lambda_2 \subseteq [0,\rho]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2$, the problem (1.1) admits at least three weak solutions whose norms in X are less than σ. Thus, the conclusion is achieved. □

Proof of Theorem 3.2. To apply Theorem 2.2 to our problem, we take the functionals $\Phi,\Psi : X \to \mathbb{R}$ as given in the proof of Theorem 3.1. Let us prove that the functionals Φ and Ψ satisfy the conditions required in Theorem 2.2. The regularity assumptions on Φ and Ψ, as requested in Theorem 2.2 hold. According to (B1) we deduce $\Phi(w) > r$. From the definition of Φ we have

$$\Phi^{-1}(]-\infty,r[) \subseteq \Big\{u \in X : |u(x)| < \sqrt{\frac{2r}{p_0-L}} \quad \text{for all } x \in [0,1]\Big\},$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \le \int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx.$$

Therefore, due to assumption (B2), we have

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} &\le \frac{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}{r}\\
&< \frac{2p_0}{p_0+L}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}\\
&\le \frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Furthermore, from (B3) there exist two constants $\eta,\vartheta \in \mathbb{R}$ with

$$\eta < \frac{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}{r}$$

such that

$$\frac{2}{p_0-L}F(x,t) \le \eta t^2 + \vartheta$$

for all $x \in [0,1]$ and all $t \in \mathbb{R}$. Fix $u \in X$. Then

$$F(x,u(x)) \le \frac{p_0-L}{2}(\eta|u(x)|^2 + \vartheta) \tag{4.2}$$

for all $x \in [0,1]$. Now, to prove the coercivity of the functional $\Phi - \lambda\Psi$, first we assume that $\eta > 0$. So, for any fixed

$$\lambda \in \Big]\frac{p_0 + L}{2p_0}\frac{\|w\|^2}{\int_0^1 F(x,w(x))dx},\ \frac{r}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}\Big[,$$

using (4.2), we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 + \int_0^1 G(u(x))dx - \lambda\int_0^1 F(x,u(x))dx\\ &\ge \frac{p_0 - L}{2p_0}\|u\|^2 - \frac{\lambda(p_0-L)}{2}\Big(\eta\int_0^1 |u(x)|^2dx + \vartheta\Big)\\ &\ge \frac{p_0 - L}{2p_0}\Big(1 - \eta\frac{r}{\int_0^1 \sup_{|t|\le\sqrt{\frac{2r}{p_0-L}}} F(x,t)dx}\Big)\|u\|^2 - \frac{\lambda(p_0-L)}{2}\vartheta,\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$

On the other hand, if $\eta \le 0$, clearly we obtain $\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$. Both cases lead to the coercivity of functional $\Phi - \lambda\Psi$.

So, the assumptions (A1) and (A2) in Theorem 2.2 are satisfied. Hence, by using Theorem 2.2, the problem (1.1) admits at least three distinct weak solutions in X. □

Proof of Theorem 3.3. Let Φ and Ψ be as in the proof of Theorem 3.1. Let us apply Theorem 2.3 to our functionals. Obviously, Φ and Ψ satisfy the condition (1) of Theorem 2.3.

Now, we show that the functional $\Phi-\lambda\Psi$ satisfies the assumption (2) of Theorem 2.3. Let u^* and u^{**} be two local minima for $\Phi - \lambda\Psi$. Then u^* and u^{**} are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1.1), and in particular they are nonnegative. Indeed, by the similar reasoning as given in [10, Theorem 3.1], let u_0 be a weak solution of the problem (1.1). Arguing by a contradiction, assume that the set $A = \{x \in]0,1] : u_0(x) < 0\}$ is nonempty and of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in [0,1]$. Clearly, $\bar{v} \in X$ and, taking into account that u_0 is a weak solution and by choosing $v = \bar{v}$, one has

$$\begin{aligned}&\int_0^1 p(x)u_0'(x)\bar{v}'(x)dx + \int_0^1 q(x)u_0(x)\bar{v}(x)dx\\ &- \lambda\int_0^1 f(x,u_0(x))\bar{v}(x)dx - \int_0^1 g(u_0(x))\bar{v}(x)dx = 0\,.\end{aligned}$$

Thus, from our sign assumptions on the data, we have

$$\int_A p(x)|u_0'(x)|^2dx + \int_A q(x)|u_0(x)|^2dx - \int_A g(u_0(x))u_0(x)dx \le 0\,.$$

On the other hand,

$$\begin{aligned}&\frac{p_0 - L(m(A))^2}{p_0}\|u_0\|^2_{W^{1,2}(A)}\\ &\le \int_A p(x)|u_0'(x)|^2dx + \int_A q(x)|u_0(x)|^2dx - \int_A g(u_0(x))u_0(x)dx,\end{aligned}$$

where $m(A)$ is the Lebesgue measure of the set A. Hence, $u_0 \equiv 0$ on A which is absurd. Then, $u^*(x) \geq 0$ and $u^{**}(x) \geq 0$ for every $x \in [0,1]$. Thus, it follows that $su^* + (1-s)u^{**} \geq 0$ for all $s \in [0,1]$, and that

$$f(x, su^* + (1-s)u^{**}) \geq 0,$$

and consequently, $\Psi(su^* + (1-s)u^{**}) \geq 0$, for every $s \in [0,1]$.

Moreover, from the condition $\frac{4p_0\, r_1}{p_0 - L} < \|w\|^2 < \frac{p_0\, r_2}{p_0 + L}$, we observe $2r_1 < \Phi(w) < \frac{r_2}{2}$. From the definition of Φ we have

$$\Phi^{-1}(]-\infty, r[) \subseteq \Big\{u \in X : |u(x)| < \sqrt{\frac{2r}{p_0 - L}} \quad \text{for all } x \in [0,1]\Big\},$$

and it follows that

$$\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) \leq \int_0^1 \sup_{|t| \leq \sqrt{\frac{2r}{p_0 - L}}} F(x,t)dx.$$

Therefore, due to the assumption (C1), we infer that

$$\begin{aligned} \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{r_1} &\leq \frac{\int_0^1 \sup_{|t| \leq \sqrt{\frac{2r_1}{p_0 - L}}} F(x,t)dx}{r_1} \\ &< \frac{4p_0}{3(p_0 + L)} \frac{\int_0^1 F(x, w(x))dx}{\|w\|^2} \\ &\leq \frac{2}{3} \frac{\Psi(w)}{\Phi(w)}. \end{aligned}$$

As above, from assumption (C2), we deduce that

$$\begin{aligned} \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u)}{r_2} &\leq \frac{\int_0^1 \sup_{|t| \leq \sqrt{\frac{2r_2}{p_0 - L}}} F(x,t)dx}{r_2} \\ &< \frac{2p_0}{3(p_0 + L)} \frac{\int_0^1 F(x, w(x))dx}{\|w\|^2} \\ &\leq \frac{1}{3} \frac{\Psi(w)}{\Phi(w)}. \end{aligned}$$

So, the assumptions (B1) and (B2) in Theorem 2.3 are satisfied. Hence, by using Theorem 2.3, the problem (1.1) admits at least three distinct weak solutions in X. This completes the proof. $\square$

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