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EXISTENCE OF SOLUTIONS TO SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH SMALL PERTURBATIONS OF IMPULSES

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Abstract. In this article we study second-order impulsive differential equations with Dirichlet boundary conditions, depending on two real parameters. We show that an appropriate growth condition of the nonlinear term, under small perturbations of impulsive terms, ensures the existence of three solutions. The approach is based on variational methods.

1. Introduction

Impulsive differential equations are recognized as adequate models to study the evolution of processes that are subject to sudden changes in their states. Processes with such a character arise naturally and often, especially in engineering and physics. In fact, it is known that many biological phenomena involving thresholds, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulse effects. For this reason, the theory of impulsive differential equations has become an important area of investigation in recent years. For an introduction of the basic theory of impulsive differential equations in $\mathbb{R}^n$, see [7] and [2]. Some classical tools have been used to study such problems in the literature, such as the coincidence degree theory of Mawhin, the method of upper and lower solutions with the monotone iterative technique, and some fixed point theorems in cones (see [5, 11, 9]). Recently, some researchers have begun to study the existence of solutions for impulsive boundary value problems by using variational methods (see for instance [10]–[14]).

In this article we consider the nonlinear Dirichlet boundary-value problem

$$\begin{gathered}
-u''(t)+a(t)u'(t)+b(t)u(t)=\lambda g(t,u(t)) \quad t\in[0,T],\ t\neq t_j\\
u(0)=u(T)=0\\
\Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=\mu I_j(u(t_j)),\quad j=1,2,\dots,n
\end{gathered}\tag{1.1}$$

where $\lambda\in]0,+\infty[$, $\mu\in]0,+\infty[$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}$, $a,b\in L^\infty([0,T])$ satisfy the conditions $\operatorname{ess\,inf}_{t\in[0,T]}a(t)\geq 0$, $\operatorname{ess\,inf}_{t\in[0,T]}b(t)\geq 0$, $0=t_0<t_1<t_2<\cdots<$

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$t_n < t_{n+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1, 2, \dots, n$.

By a classical solution of (1.1), we mean a function

$$u \in \left\{w \in C([0,T] : w_{|[t_j,t_{j+1}]} \in H^2([t_j,t_{j+1}])\right\}$$

that satisfies the equation in (1.1) a.e. on $[0,T] \setminus \{t_1,\dots,t_n\}$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j = 1,\dots,n$, exist, satisfy the impulsive conditions $\Delta u'(t_j) = \mu I_j(u(t_j))$m and the boundary condition $u(0) = u(T) = 0$. Clearly, if a, b, g are continuous, then the classical solution $u \in C^2([t_j,t_{j+1}])$, $j = 0, 1, \dots, n$, and satisfies the equation in (1.1) for all $t \in [0,T] \setminus \{t_1,\dots,t_n\}$.

By using variational methods, we show the existence of three solutions for this problem. More precisely, by choosing μ in a suitable way and under a growth condition on the nonlinear term we prove that (1.1) has at least three solutions for every λ lying in a precise interval. In particular, we obtain two main theorems. In the first one (Theorem 3.8) we require on the antiderivative of g both a growth more then quadratic in a suitable interval and a growth less then quadratic at infinity, and at the same time, on the impulse I_j, an asymptotic condition is required. In the second one (Theorem 3.9) we establish the existence of at least three positive solutions uniformly bounded without asymptotic conditions on g and I_j.

As an example, we present a particular case of Theorem 3.9.

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous and non-zero function such that*

$$\lim_{u\to 0^+} \frac{g(u)}{u} = \lim_{u\to+\infty} \frac{g(u)}{u} = 0\,. \tag{1.2}$$

Then, for every

$$\lambda > \frac{(12+T^2)e^{2T}}{2T(e^T-1)(e^{3T/4}-e^{T/4})} \inf_{d>0} \frac{d^2}{\int_0^d g(x)\,dx}$$

and for every negative continuous function $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,\dots,n$, there exists $\delta^* > 0$ such that, for each $\mu \in]0,\delta^*[$, the problem

$$\begin{gathered} -u''(t) + u'(t) + u(t) = \lambda g(u(t)) \quad t \in [0,T],\ t \neq t_j \\ u(0) = u(T) = 0 \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \quad j = 1, 2, \dots, n \end{gathered} \tag{1.3}$$

has at least three non-zero solutions.

We wish to stress that in many papers, as for instance in [16, 17, 8], under assumptions similar to those of our results, the authors ensure the existence of at least only one solution for (1.1) and, moreover, do not give an estimate of λ and μ and an explicit upper bound, uniformly with respect to parameters, of the solutions.

The remainder of the paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, we will state and prove the main results of the paper, as well as give some applications to (1.1).

2. Preliminaries

We consider the following problem, which is slightly different form (1.1),

$$\begin{gathered} -(p(t)u'(t))' + q(t)u(t) = \lambda f(t, u(t)) \quad t \in [0,T],\ t \neq t_j \\ u(0) = u(T) = 0 \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \quad j = 1, 2, \dots, n \end{gathered} \tag{2.1}$$

where $p \in C^1([0,T], [0,+\infty[)$, $q \in L^\infty([0,T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} q(t) \geq 0$.

It is easy to see that the solutions of (2.1) are solutions of (1.1) if

$$p(t) = e^{-\int_0^t a(\tau)\,d\tau}, \quad q(t) = b(t)e^{-\int_0^t a(\tau)\,d\tau}, \quad f(t,u) = g(t,u)e^{-\int_0^t a(\tau)\,d\tau}.$$

Let us introduce some notation. In the Sobolev space $H_0^1(0,T)$, consider the inner product

$$(u, v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt\,,$$

which induces the norm

$$\|u\| = \Big(\int_0^T p(t)(u'(t))^2\,dt + \int_0^T q(t)(u(t))^2\,dt\Big)^{1/2}.$$

Let us recall the Poincarè type inequality

$$\Big[\int_0^T u^2(t)\,dt\Big]^{1/2} \leq \frac{T}{\pi}\Big[\int_0^T (u')^2(t)\,dt\Big]^{1/2}. \tag{2.2}$$

Proposition 2.1. *Let $u \in H_0^1(0,T)$. Then*

$$\|u\|_\infty \leq \frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\| \tag{2.3}$$

where $p^* := \min_{t\in[0,T]} p(t)$

Proof. In view of Hölder's inequality one has

$$\|u\|_\infty \leq \frac{\sqrt{T}}{2}\|u'\|_{L^2([0,T])} \leq \frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\|\,.$$

□

Here and in the sequel $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, namely:

(F1) (a) $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$;
(b) $x \to f(t,x)$ is continuous for almost every $t \in [0,T]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,T])$ such that

$$\sup_{|x|\leq\rho} |f(t,x)| \leq l_\rho(t)$$

for almost every $t \in [0,T]$;

Definition 2.2. A function $u \in H_0^1(0,T)$ is said to be a weak solution of (2.1), if u satisfies

$$\begin{aligned} &\int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)dt \\ &- \lambda \int_0^T f(t,u(t))v(t)\,dt + \mu \sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j) = 0, \end{aligned} \tag{2.4}$$

for any $v \in H_0^1(0,T)$.

Lemma 2.3. *$u \in H_0^1(0,T)$ is a weak solution of (2.1) if and only if u is a classical solution of (2.1).*

Proof. Let $u \in H_0^1(0,T)$ be a weak solution of (2.1). Then (2.4) holds for any $v \in H_0^1(0,T)$. Fix $j \in \{0,1,2,\dots,n\}$ and let $\bar{v} \in H_0^1(0,T)$ such that $\bar{v}(t) = 0$ for all $t \in [0,t_j] \cup [t_{j+1},T]$. Thus by (2.4) we obtain

$$\int_{t_j}^{t_{j+1}} [-(p(t)u'(t))'\bar{v}(t) + q(t)u(t)\bar{v}(t)]\,dt - \lambda \int_{t_j}^{t_{j+1}} f(t,u(t))\bar{v}(t)\,dt = 0\,.$$

This implies that

$$-(p(t)u'(t))' + q(t)u(t) = \lambda f(t,u(t))$$

for almost every $t \in [t_j,t_{j+1}]$. Hence, $u \in H^2(t_j,t_{j+1})$ and satisfies the equation

$$-(pu')' + qu = \lambda f(t,u) \quad \text{almost every } t \in [0,T]\,. \tag{2.5}$$

Now multiplying by $v \in H_0^1(0,T)$ and integrating on $[0,T]$, we obtain that

$$-\sum_{j=1}^n \Delta u'(t_j)p(t_j)v(t_j) + \int_0^T [-(p(t)u'(t))' + q(t)u(t) - \lambda f(t,u(t))]\,v(t)\,dt = 0\,.$$

Taking again (2.4) into account, we obtain

$$\sum_{j=1}^n \Delta u'(t_j)p(t_j)v(t_j) = \mu \sum_{j=1}^n I_j(u(t_j))p(t_j)v(t_j)\,.$$

Hence $\Delta u'(t_j) = \mu I_j(u(t_j)$, for every $j = 1,2,\dots,n$, and the impulsive condition in (2.1) is satisfied. □

Now, we define the functionals $\Phi, \Psi : H_0^1(0,T) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\|u\|^2 \quad \Psi(u) = \int_0^T F(t,u(t))\,dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)\,dx, \tag{2.6}$$

for each $u \in H_0^1(0,T)$, where $F(t,\xi) = \int_0^\xi f(t,x)\,dx$ for each $(t,\xi) \in [0,T] \times \mathbb{R}$. Using the property of f and the continuity of I_j, $j = 1,2,\dots,n$, we have that $\Phi, \Psi \in C^1(H_0^1(0,T),\mathbb{R})$ and for any $v \in H_0^1(0,T)$, we have

$$\Phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt$$

and

$$\Psi'(u)(v) = \int_0^T f(t,u(t))v(t)\,dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j).$$

So, arguing in a standard way, it is possible to prove that the critical points of the functional $E_{\lambda,\mu}(u) := \Phi(u) - \lambda\Psi(u)$ are the weak solutions of problem (2.1) and so they are classical.

We now state two critical point theorems which are the main tools for the proofs of our results. The following statement comes easily by the results contained in [4] and in [3].

Theorem 2.4 ([4, Theorem 2.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0\,.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\sup_{\Phi(x)\le r} \Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x})$,

(ii) *for each λ in*

$$\Lambda_r := \Big]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\Big[,$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.5 ([3, Theorem 3.2]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0\,.$$

Assume that there exist two positive constants $r_1, r_2 > 0$ and $\bar{x} \in X$, with $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$, such that

(j) $\frac{\sup_{\Phi(x)\le r_1}\Psi(x)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(jj) $\frac{\sup_{\Phi(x)\le r_2}\Psi(x)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(jjj) *for each λ in*

$$\Lambda^*_{r_1,r_2} := \Big]\frac{3}{2}\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min\Big\{\frac{r_1}{\sup_{\Phi(x)\le r_1}\Psi(x)}, \frac{r_2}{2\sup_{\Phi(x)\le r_2}\Psi(x)}\Big\}\Big[$$

and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda\Psi$, and such that $\Psi(x_1) \ge 0$ and $\Psi(x_2) \ge 0$, one has $\inf_{t\in[0,1]}\Psi(tx_1 + (1-t)x_2) \ge 0$.

Then, for each $\lambda \in \Lambda^*_{r_1,r_2}$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

3. Main results

First, we give the following lemma which we will use in the proof of our main result.

Lemma 3.1. *Assume that*

(H1) *there exist constants* $\alpha, \beta > 0$ *and* $\sigma \in [0, 1[$ *such that*
$$|I_j(x)| \leq \alpha + \beta |x|^\sigma \quad \text{for all } x \in \mathbb{R},\ j = 1, 2, \dots, n\,.$$

Then, for any $u \in H_0^1(0,T)$, *one has*
$$\Big|\sum_{j=1}^{n} p(t_j) \int_0^{u(t_j)} I_j(x)\,dx\Big| \leq \sum_{j=1}^{n} p(t_j)\Big(\alpha \|u\|_\infty + \frac{\beta}{\sigma+1}\|u\|_\infty^{\sigma+1}\Big). \tag{3.1}$$

Proof. Thanks to condition (H1), one has
$$\Big|\int_0^{u(t_j)} I_j(x)\,dx\Big| \leq \alpha |u(t_j)| + \frac{\beta}{\sigma+1}|u(t_j)|^{\sigma+1}.$$
Thus, (3.1) is obtained. □

Remark 3.2. It is easy to verify that the condition

(H1') There exist constants $\gamma_j, \beta_j > 0$ and $\sigma_j \in [0, 1[$, $(j = 1, 2, \dots, n)$, such that
$$|I_j(x)| \leq \gamma_j + \beta_j |x|^{\sigma_j} \quad \text{for all } x \in \mathbb{R},\ j = 1, 2, \dots, n\,.$$

is equivalent to (H1). In fact, it is sufficient to put $\beta := \max_{1\leq j\leq n} \beta_j$, $\gamma := \max_{1\leq j\leq n} \gamma_j$, $\alpha = \gamma + \beta$ and $\sigma := \max_{1\leq j\leq n} \sigma_j$.

Now, put
$$\tilde{p} := \sum_{j=1}^{n} p(t_j), \quad k := \frac{6p^*}{12\|p\|_\infty + T^2\|q\|_\infty}, \quad \Gamma_c := \frac{\alpha}{c} + \big(\frac{\beta}{\sigma+1}\big)c^{\sigma-1},$$
where α, β, σ are given by (h_1) and c is a positive constant.

Theorem 3.3. *Suppose that* (F1), (H1) *are satisfied. Furthermore, assume that there exist two positive constants* c, d, *with* $c < d$, *such that*

(A1) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in \big([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]\big) \times [0, d]$;

(A2)
$$\frac{\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2} < k \frac{\int_{T/4}^{3T/4} F(t,d)\,dt}{d^2};$$

(A3)
$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} \leq \frac{\pi^2}{4T} \frac{\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2}.$$

Then, for every λ *in*
$$\Lambda := \Big]\frac{2p^*}{kT}\frac{d^2}{\int_{T/4}^{3T/4} F(t,d)\,dt}, \frac{2p^*}{T}\frac{c^2}{\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}\Big[,$$

there exists
$$\delta := \frac{1}{T\tilde{p}} \min\Big\{\frac{2p^*c^2 - \lambda T\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2\Gamma_c}, \frac{k\lambda T\int_{T/4}^{3T/4} F(t,d)\,dt - 2p^*d^2}{d^2\Gamma_{(d/\sqrt{k})}}\Big\}$$

such that, for each $\mu \in [0, \delta[$ the problem (2.1) has at least three distinct classical solutions.

Proof. First, we observe that due to (A2) the interval Λ is non-empty and, consequently, one has $\delta > 0$. Now, fix λ and μ as in the conclusion. Our aim is to apply Theorem 2.4. For this end, take $X = H_0^1(0,T)$ and Φ, Ψ as in (2.6). Put $r = 2c^2p^*/T$. Taking (2.3) into account, for every $u \in X$ such that $\Phi(u) \le r$, one has $\max_{t\in[0,T]} |u(t)| \le c$. Consequently, from Lemma 3.1 it follows that

$$\sup_{\Phi(u)\le r} \Psi(u) \le \int_0^T \max_{|\xi|\le c} F(t,\xi)\,dt + \frac{\mu}{\lambda}\tilde{p}\Big(\alpha c + \frac{\beta}{\sigma+1}c^{\sigma+1}\Big);$$

that is,

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} \le \frac{T}{2p^*}\Big[\frac{\int_0^T \max_{|\xi|\le c} F(t,\xi)\,dt}{c^2} + \frac{\mu}{\lambda}\tilde{p}\Gamma_c\Big].$$

Hence, bearing in mind that $\mu < \delta$, one has

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.2}$$

Put

$$\bar{v}(t) = \begin{cases} \frac{4d}{T}t, & t \in [0, T/4],\\ d, & t \in]T/4, 3T/4],\\ \frac{4d}{T}(T-t), & t \in]3T/4, T]. \end{cases}$$

Clearly $\bar{v} \in X$. Moreover, one has

$$\frac{8p^*}{T}d^2 \le \|\bar{v}\|^2 \le \frac{2d^2(12\|p\|_\infty + T^2\|q\|_\infty)}{3T} = \frac{4d^2p^*}{kT}. \tag{3.3}$$

So, from $c < \sqrt{2}d$ we obtain $r < \Phi(\bar{v})$. Moreover, again from the previous inequality, we have

$$\Phi(\bar{v}) < \frac{2p^*d^2}{kT}.$$

Now, due to Lemma 3.1, (A1), (2.3) and (3.3) one has

$$\begin{aligned}\Psi(\bar{v}) &\ge \int_{T/4}^{3T/4} F(t,d)\,dt - \frac{\mu}{\lambda}\tilde{p}\Big(\alpha\|\bar{v}\|_\infty + \frac{\beta}{\sigma+1}\|\bar{v}\|_\infty^{\sigma+1}\Big)\\ &\ge \int_{T/4}^{3T/4} F(t,d)\,dt - \frac{\mu}{\lambda}\frac{\tilde{p}\,d^2}{k}\Gamma_{(d/\sqrt{k})}.\end{aligned}$$

So, we obtain

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} \ge \frac{kT\int_{T/4}^{3T/4} F(t,d)\,dt - \frac{\mu}{\lambda}\tilde{p}Td^2\Gamma_{(d/\sqrt{k})}}{2p^*d^2}.$$

Since $\mu < \delta$, one has

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} > \frac{1}{\lambda}. \tag{3.4}$$

Therefore, from (3.2) and (3.4), condition (i) of Theorem 2.4 is fulfilled.

Now, to prove the coercivity of the functional $\Phi - \lambda\Psi$, due to (A3), we have

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \big(\frac{\pi^2p^*}{2T^2}\big)\frac{1}{\lambda}.$$

So, we can fix $\varepsilon > 0$ satisfying

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \varepsilon < \big(\frac{\pi^2 p^*}{2T^2}\big)\frac{1}{\lambda}\,.$$

Then, there exists a positive constant h such that

$$F(t,\xi) \le \varepsilon|\xi|^2 + h \quad \forall t \in [0,T],\ \forall \xi \in \mathbb{R}\,.$$

Taking into account Lemma 3.1, Proposition 2.1 and (2.2), it follows that

$$\begin{aligned}
&\Phi(u) - \lambda\Psi(u) \\
&\ge \frac{1}{2}\|u\|^2 - \lambda\varepsilon\|u\|^2_{L^2[0,T]} - \lambda hT - \mu\tilde{p}\Big[\alpha\frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\| + \frac{\beta}{\sigma+1}\Big(\frac{1}{2}\sqrt{\frac{T}{p^*}}\Big)^{\sigma+1}\|u\|^{\sigma+1}\Big] \\
&\ge \Big(\frac{1}{2} - \lambda\varepsilon\frac{T^2}{\pi^2 p^*}\Big)\|u\|^2 - \lambda hT - \mu\tilde{p}\Big[\alpha\frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\| + \frac{\beta}{\sigma+1}\Big(\frac{1}{2}\sqrt{\frac{T}{p^*}}\Big)^{\sigma+1}\|u\|^{\sigma+1}\Big],
\end{aligned}$$

for all $u \in H^1_0(0,T)$. So, the functional $\Phi - \lambda\Psi$ is coercive. Now, the conclusion of Theorem 2.4 can be used. It follows that, for every

$$\lambda \in \Big]\frac{2k}{T}\frac{d^2}{\int_{T/4}^{3T/4} F(t,d)\,dt}, \frac{2}{T}\frac{c^2}{\int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt}\Big[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X, which are the weak solutions of the problem (2.1). This completes the proof. $\square$

Corollary 3.4. *Suppose that* (H1) *holds. Let $h \in L^1([0,T])$ be a nonnegative and non-zero function and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $h_0 := \int_{T/4}^{3T/4} h(t)dt$ and $G(\xi) = \int_0^\xi g(x)dx$ for all $\xi \in \mathbb{R}$, and assume that there exist two positive constants c, d, with $c < d$, such that*

(A1') $G(\xi) \ge 0$ *for all* $\xi \in [0,d]$*;*

(A2')

$$\frac{\max_{|\xi|\le c} G(\xi)}{c^2} < \frac{1}{2}\frac{h_0}{\|h\|_1}\frac{G(d)}{d^2};$$

(A3') $\limsup_{|\xi|\to+\infty} G(\xi)/\xi^2 \le 0$.

Then, for every λ in

$$\Lambda := \Big]\frac{4}{Th_0}\frac{d^2}{G(d)}, \frac{2}{T\|h\|_1}\frac{c^2}{\max_{|\xi|\le c} G(\xi)}\Big[,$$

there exists

$$\delta := \frac{1}{Tn}\min\Big\{\frac{2c^2 - \lambda T\|h\|_1 \max_{|\xi|\le c} G(\xi)}{c^2\Gamma_c}, \frac{\frac{\lambda T h_0}{2}G(d) - 2d^2}{d^2\Gamma_{(\sqrt{2}d)}}\Big\}$$

such that, for each $\mu \in [0,\delta[$ the problem

$$\begin{gathered}
-u''(t) = \lambda h(t)g(u(t)) \quad t \in [0,T],\ t \ne t_j \\
u(0) = u(T) = 0 \\
\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \quad j = 1,2,\dots,n
\end{gathered} \tag{3.5}$$

has at least three classical solutions.

The proof of the above corollary follows from Theorem 3.3 by choosing $f(t,x) = h(t)g(x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$ and taking into account that $k = 1/2$.

Remark 3.5. Clearly, if g is nonnegative then assumption (A1') is verified and (A2') becomes

$$\frac{G(c)}{c^2} < \frac{1}{2}\frac{h_0}{\|h\|_1}\frac{G(d)}{d^2}.$$

Now, we state a result without asymptotic conditions on I_j. The following lemma will be crucial in our arguments.

Lemma 3.6. *Suppose that* (F1) *is satisfied. Moreover, assume that $f(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$ and $I_j(x) \leq 0$ for all $x \in \mathbb{R}$, $j = 1, \dots, n$. If u is a classical solution of* (2.1)*, then $u(t) \geq 0$ for all $t \in [0,T]$.*

Proof. If u is a classical solution of(2.1), then

$$\int_0^T (p(t)u'(t))'v(t)\,dt - \int_0^T q(t)u(t)v(t)\,dt + \lambda \int_0^T f(t,u(t))v(t)\,dt = 0$$

for all $v \in X$. Let $v(t) = \max\{-u(t), 0\}$ for all $t \in [0,T]$; clearly $v \in X$ and we have

$$\begin{aligned}
0 &= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (p(t)u'(t))'v(t)\,dt - \int_0^T q(t)u(t)v(t)\,dt + \lambda \int_0^T f(t,u(t))v(t)\,dt \\
&= \sum_{j=0}^{n} p(t)u'(t)v(t)\big|_{t_j}^{t_{j+1}} - \int_0^T p(t)u'(t)v'(t)\,dt - \int_0^T q(t)u(t)v(t)\,dt \\
&\quad + \lambda \int_0^T f(t,u(t))v(t)\,dt \\
&= -\sum_{j=1}^{n} \Delta u'(t_j)p(t_j)v(t_j) - \int_0^T p(t)u'(t)v'(t)\,dt - \int_0^T q(t)u(t)v(t)\,dt \\
&\quad + \lambda \int_0^T f(t,u(t))v(t)\,dt \\
&= -\mu \sum_{j=1}^{n} I_j(u(t_j))p(t_j)v(t_j) + \int_0^T p(t)(v'(t))^2\,dt + \int_0^T q(t)(v(t))^2\,dt \\
&\quad + \lambda \int_0^T f(t,u(t))v(t)\,dt \\
&\geq \|v\|^2 .
\end{aligned}$$

So $v(t) = 0$ for $t \in [0,T]$. □

Put

$$\Im_c := \sum_{j=1}^{n} \min_{|\xi| \leq c} \int_0^{\xi} I_j(x)\,dx, \quad \text{for all } c > 0 .$$

Our other main result is as follows.

Theorem 3.7. *Suppose that* (F1) *is satisfied. Furthermore, assume that there exist three positive constants c_1, c_2, d, with $c_1 < d < \sqrt{\frac{k}{2}}c_2$, such that*

(B1) $f(t,\xi) \geq 0$ *for all* $(t,\xi) \in [0,T] \times [0,c_2]$*;*

(B2)
$$\frac{\int_0^T F(t,c_1)\,dt}{c_1^2} < \frac{2}{3}k\frac{\int_{T/4}^{3T/4} F(t,d)\,dt}{d^2};$$

(B3)
$$\frac{\int_0^T F(t,c_2)\,dt}{c_2^2} < \frac{k}{3}\frac{\int_{T/4}^{3T/4} F(t,d)\,dt}{d^2}.$$

Let
$$\Lambda' := \Big]\frac{3p^*}{kT}\frac{d^2}{\int_{T/4}^{3T/4} F(t,d)\,dt}, \frac{p^*}{T}\min\Big\{\frac{2c_1^2}{\int_0^T F(t,c_1)\,dt}, \frac{c_2^2}{\int_0^T F(t,c_2)\,dt}\Big\}\Big[.$$

Then, for every $\lambda \in \Lambda'$ *and for every negative continuous function* I_j*,* $j = 1,\dots,n$*, there exists*
$$\delta^* := \frac{1}{T\|p\|_\infty}\min\Big\{\frac{\lambda T\int_0^T F(t,c_1)\,dt - 2p^*c_1^2}{\Im_{c_1}}, \frac{\lambda T\int_0^T F(t,c_2)\,dt - p^*c_2^2}{\Im_{c_2}}\Big\}$$
such that, for each $\mu \in]0,\delta^*[$ *the problem* (2.1) *has at least three classical solutions* u_i*,* $i = 1,2,3$*, such that* $0 < \|u_i\|_\infty \leq c_2$*.*

Proof. Without loss of generality, we can assume $f(t,x) \geq 0$ for all $(t,x) \in [0,T]\times\mathbb{R}$. Fix λ, I_j and μ as in the conclusion and take X, Φ and Ψ as in the proof of Theorem 3.3. Put $\bar{v}$ as in Theorem 3.3, $r_1 = \frac{2p^*c_1^2}{T}$ and $r_2 = \frac{2p^*c_2^2}{T}$. Therefore, one has $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$ and since $\mu < \delta^*$, one has
$$\begin{aligned}
\frac{1}{r_1}\sup_{\Phi(u)<r_1}\Psi(u) &\leq \frac{T}{2p^*c_1^2}\Big(\int_0^T F(t,c_1)\,dt - \frac{\mu}{\lambda}\|p\|_\infty\Im_{c_1}\Big)\\
&< \frac{1}{\lambda} < \frac{T}{3k}\frac{\int_{T/4}^{3T/4} F(t,d)\,dt}{d^2}\\
&\leq \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})},
\end{aligned}$$
and
$$\begin{aligned}
\frac{2}{r_2}\sup_{\Phi(u)<r_2}\Psi(u) &\leq \frac{T}{p^*c_2^2}\Big(\int_0^T F(t,c_2)\,dt - \frac{\mu}{\lambda}\|p\|_\infty\Im_{c_2}\Big)\\
&< \frac{1}{\lambda} < \frac{T}{3k}\frac{\int_{T/4}^{3T/4} F(t,d)\,dt}{d^2}\\
&\leq \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}.
\end{aligned}$$
Therefore, conditions (j) and (jj) of Theorem 2.5 are satisfied. Finally, let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (2.1). Hence, owing to Lemma 3.6, we obtain $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for all $t \in [0,T]$. So, one has $\Psi(su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$. From Theorem 2.5 the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are weak solutions of (2.1). This complete the proof. $\square$

Let $A(t)$ a primitive of $a(t)$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function and put

$$G(t,\xi) = \int_0^\xi g(t,x)\,dx, \quad \tilde{k} := \frac{6e^{-A(T)}}{12 + T^2\|be^{-A}\|_\infty}\,.$$

By Theorems 3.3 and 3.7, we obtain the following results for problem (1.1).

Theorem 3.8. *Suppose that* (H1) *holds. Furthermore, assume that there exist two positive constants c, d, with $c < d$, such that*

(I1) $G(t,\xi) \geq 0$ *for all* $(t,\xi) \in \big([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]\big) \times [0, d]$;

(I2)
$$\frac{\int_0^T e^{-A(t)} \max_{|\xi|\leq c} G(t,\xi)\,dt}{c^2} < \tilde{k}\frac{\int_{T/4}^{3T/4} e^{-A(t)}G(t,d)\,dt}{d^2};$$

(I3) $\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} e^{-A(t)}G(t,\xi)}{\xi^2} \leq \frac{\pi^2}{4T}\frac{\int_0^T e^{-A(t)} \max_{|\xi|\leq c} G(t,\xi)\,dt}{c^2}$.

Let

$$\Lambda := \Big]\frac{2}{\tilde{k}Te^{\|a\|_1}}\frac{d^2}{\int_{T/4}^{3T/4} e^{-A(t)}G(t,d)\,dt}, \frac{2}{Te^{\|a\|_1}}\frac{c^2}{\int_0^T e^{-A(t)} \max_{|\xi|\leq c} G(t,\xi)\,dt}\Big[.$$

Then, for every $\lambda \in \Lambda$, there exists

$$\begin{aligned}\delta := \frac{1}{\tilde{e}T}\min\Big\{&\frac{2c^2e^{-\|a\|_1} - \lambda T\int_0^T e^{-A(t)} \max_{|\xi|\leq c} G(t,\xi)\,dt}{c^2\Gamma_c}\,,\\ &\frac{\tilde{k}\lambda T\int_{T/4}^{3T/4} e^{-A(t)}G(t,d)\,dt - 2e^{-\|a\|_1}d^2}{d^2\Gamma_{(d/\sqrt{k})}}\Big\}\end{aligned}$$

such that, for each $\mu \in [0, \delta[$ the problem (1.1) *has at least three distinct classical solutions.*

The above theorem follows immediately from Theorem 3.3 taking into account Section 2.

Theorem 3.9. *Assume that there exist three positive constants c_1, c_2, d, with $c_1 < d < \sqrt{\frac{\tilde{k}}{2}}c_2$, such that*

(J1) $g(t,\xi) \geq 0$ *for all* $(t,\xi) \in [0, T] \times [0, c_2]$;

(J2)
$$\frac{\int_0^T e^{-A(t)}G(t,c_1)\,dt}{c_1^2} < \frac{2}{3}\tilde{k}\frac{\int_{T/4}^{3T/4} e^{-A(t)}G(t,d)\,dt}{d^2};$$

(J3)
$$\frac{\int_0^T e^{-A(t)}G(t,c_2)\,dt}{c_2^2} < \frac{\tilde{k}}{3}\frac{\int_{T/4}^{3T/4} e^{-A(t)}G(t,d)\,dt}{d^2}.$$

Then, for every λ in

$$\begin{aligned}\Lambda' := \Big]&\frac{3e^{-\|a\|_1}}{\tilde{k}T}\frac{d^2}{\int_{T/4}^{3T/4} e^{-A(t)}G((t,d)\,dt},\\ &\frac{e^{-\|a\|_1}}{T}\min\Big\{\frac{2c_1^2}{\int_0^T e^{-A(t)}G(t,c_1)\,dt}, \frac{c_2^2}{\int_0^T e^{-A(t)}G(t,c_2)\,dt}\Big\}\Big[,\end{aligned}$$

and for every negative continuous function I_j, $j = 1, \dots, n$, there exists

$$\delta^* := \frac{1}{T} \min\Big\{ \frac{\lambda T \int_0^T e^{-A(t)} G(t, c_1)\, dt - 2e^{-\|a\|_1} c_1^2}{\Im_{c_1}}, \frac{\lambda T \int_0^T e^{-A(t)} G(t, c_2)\, dt - e^{-\|a\|_1} c_2^2}{\Im_{c_2}} \Big\}$$

such that, for each $\mu \in]0, \delta^*[$, problem (1.1) *has at least three classical solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \le c_2$.*

The above theorem follows from Theorem 3.7 when taking into account Section 2.

We want to point out that the function $a(t)$ can be taken of any sign, provided changes are made to the constant $\tilde{k}$. When $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, the assumptions in Theorem 3.9 take a simpler form:

Theorem 3.10. *Put*

$$\theta := \frac{\int_{T/4}^{3T/4} e^{-A(t)}\, dt}{\|e^{-A}\|_1}, \quad k^* := \frac{2}{3} \tilde{k} \theta, \quad L := \frac{e^{-\|a\|_1}}{T \|e^{-A}\|_1}.$$

Assume that there exist three positive constants c_1, c_2, d, with $c_1 < d < \frac{1}{2}\sqrt{\frac{3k^*}{\theta}} c_2$, such that

(J2") $G(c_1)/c_1^2 < k^* G(d)/d^2$;
(J3") $G(c_2)/c_2^2 < \frac{k^*}{2} G(d)/d^2$.

Then, for every λ in

$$\Lambda'' := \Big] \frac{2L}{k^*} \frac{d^2}{G(d)}, L \min\Big\{ \frac{2c_1^2}{G(c_1)}, \frac{c_2^2}{G(c_2)} \Big\} \Big[,$$

and for every negative continuous function I_j, $j = 1, \dots, n$, there exists

$$\delta^* := \|e^{-A}\|_1 \min\Big\{ \frac{\lambda G(c_1) - 2Lc_1^2}{\Im_{c_1}}, \frac{\lambda G(c_2) - Lc_2^2}{\Im_{c_2}} \Big\}$$

such that, for each $\mu \in]0, \delta^*[$, problem (1.1) *has at least three classical solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \le c_2$.*

Now using Theorem 3.10, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix $\lambda > \lambda_1$, put $G(\xi) = \int_0^\xi g(x)\, dx$ for all $\xi \in \mathbb{R}$, and let $d > 0$ such that $G(d) > 0$ and

$$\lambda > \frac{(12 + T^2) e^{2T}}{2T(e^T - 1)(e^{3T/4} - e^{T/4})} \frac{d^2}{G(d)}.$$

From (1.2) there is $c_1 > 0$ such that $c_1 < d$ and $G(c_1)/c_1^2 < 2/(T(e^T - 1)\lambda)$, and there is $c_2 > 0$ such that

$$d < \sqrt{\frac{3e^{-T}}{12 + T^2}} c_2, \quad \frac{G(c_2)}{c_2^2} < \frac{1}{T(e^T - 1)\lambda}.$$

Therefore, Theorem 3.10 ensures the conclusion. □

Finally, we give two applications of the results above.

Example 3.11. The problem

$$\begin{gathered} -u''(t) + (\frac{t}{\pi} - 1)^2 u(t) = \lambda u^2(3-4u)\sin t \quad \text{a.e. in } [0,\pi] \\ u(0) = u(\pi) = 0 \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \mu(1 - \sqrt[3]{u(t_1)}) \end{gathered} \tag{3.6}$$

admits at least three non-trivial solutions for each $\lambda \in [7,20]$ and for each $0 < \mu < \frac{1}{38\pi}(1 - \frac{63\lambda\pi}{4096})$.

Indeed, it is sufficient to apply Theorem 3.3 by choosing, for instance, $c = 1/64$ and $d = 1/2$.

We remark that although [17, Theorem 3.2] can be applied, it guarantees the existence of at least one solution, only. Our results go further than [1, Theorem 1], we have precise values of the parameter λ for which the problem admits solutions.

Example 3.12. Let $g : (t,x) \in (0,1] \times \mathbb{R} \to \mathbb{R}$, be defined as

$$g(t,x) = \begin{cases} 10^{-4} e^{t^2} / \sqrt[4]{t} & \text{if } x \le 10^{-2} \\ x^2 e^{t^2} / \sqrt[4]{t} & \text{if } 10^{-2} < x \le 1 \\ e^{t^2} / (x^2 \sqrt[4]{t}) & \text{if } x > 1. \end{cases}$$

By Theorem 3.9, for each $\lambda \in [33, 55]$ and each $\mu \in]0, 3.4 \times 10^{-4}[$ the problem

$$\begin{gathered} -u''(t) + 2tu'(t) + (1-t)u(t) = \lambda g(t, u(t)) \quad \text{a.e. in } [0,1] \\ u(0) = u(1) = 0 \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \mu(-1 - |u(t_1)|^3) \end{gathered}$$

admits at least three non-trivial solutions u_i, such that $0 < |u_i(t)| < 10^2$ for all $t \in [0,1]$, $i = 1,2,3$.

It suffices to choose, for instance, $c_1 = 10^{-2}$, $c_2 = 10^2$, $d = 1$.

We observe that in Example 3.11 we do not have the negativity of the impulsive term, so we cannot apply Theorem 3.9. On the other hand, in Example 3.12 the function is negative, but it does not have the sublinear growth.

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