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GROUND STATES FOR THE FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. In this article, we show the existence of ground state solutions for the nonlinear Schrödinger equation with fractional Laplacian

$$(-\Delta)^\alpha u + V(x)u = \lambda |u|^p u \quad \text{in } \mathbb{R}^N \text{ for } \alpha \in (0,1).$$

We use the concentration compactness principle in fractional Sobolev spaces H^α for $\alpha \in (0,1)$. Our results generalize the corresponding results in the case $\alpha = 1$.

1. Introduction

This article is devoted to the study of existence of ground state solutions for the fractional nonlinear Schrödinger equation

$$\begin{gathered} (-\Delta)^\alpha u + V(x)u = \lambda |u|^p u \quad \text{in } \mathbb{R}^N, \\ u \in H^\alpha(\mathbb{R}^N), \quad u \neq 0, \end{gathered} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < p < \frac{2N}{N-2\alpha}$, $N \geq 2$, $\lambda > 0$ and V is a positive continuous function. The fractional Laplacian can be characterized as $\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha} \mathcal{F}(u)(\xi)$, where $\mathcal{F}$ is the Fourier transform.

Equation (1.1) arises in the study of the fractional Schrödinger equation

$$i\psi_t + (-\Delta)^\alpha \psi + (V(x) + \omega)\psi = \lambda |\psi|^p \psi, \tag{1.2}$$

when looking for standing waves solutions that have the form $\psi(x,t) = e^{i\omega t} u(x)$, where $\omega \in \mathbb{R}$ and $u \in H^\alpha$, $u \neq 0$. This equation is a fundamental equation of fractional quantum mechanics; see, e.g., [11, 12, 13]. The fractional quantum mechanics has been discovered as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In recent years, more and more attention has been focusing on the study of the fractional Schrödinger equation (1.2) from a pure mathematical point of view, see [6, 9, 10, 16, 17, 19] and the references therein.

As is known, for $\alpha = 1$ the Lévy motion becomes Brownian motion and the fractional Schrödinger equation becomes the standard Schrödinger equation

$$i\psi_t + \Delta\psi + (V(x) + \omega)\psi = \lambda |\psi|^p \psi. \tag{1.3}$$

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The corresponding elliptic equation reads

$$-\Delta u+V(x)u=\lambda|u|^{p}u. \tag{1.4}$$

There exist a considerable amount of results in the physics and mathematics literature for those classical Schrödinger equations. In particular, the standing wave solutions have been investigated by many authors. We only refer to [1, 14, 15, 20, 3].

Recently, fractional nonlinear Schrödinger equation (1.1) has begun to receive increasing attention; for example, see [4, 5, 7, 8]. In [7], by using mountain pass lemma, the authors derived the existence of positive solutions to nonlinear fractional Schrödinger equation

$$(-\Delta)^{\alpha}u+u=f(x,u). \tag{1.5}$$

In particular, they used a comparison argument to overcome the difficulty that the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}^N$. They also analyzed regularity, decay, and symmetry properties of these solutions. Those results heavily rely on the representation formula

$$u=\mathcal{K}*f=\int_{\mathbb{R}^N}\mathcal{K}(x-\xi)f(\xi)d\xi,$$

for solutions of the equation

$$(-\Delta)^{\alpha}u+u=f\quad\text{in }\mathbb{R}^N,$$

where $\mathcal{K}$ is the Bessel kernel

$$\mathcal{K}=\mathcal{F}^{-1}\Big(\frac{1}{1+|\xi|^{2\alpha}}\Big).$$

However, there is not a similar representation formula for equation (1.1) with general potential V. In [5], following the argument of [1], the existence and symmetry results for bound state solutions to (1.1) with $V\equiv 1$ have been obtained by applying symmetric decreasing rearrangement. However, this method fails to work for equation (1.1) except that potential V is a spherically symmetric function. The existence of bound state solutions to (1.1) with unbounded potential have been derived by Lagrange multiplier method and Nehari's manifold approach in [4]. It is worth noticing that under the assumption that potential $V(x)\to\infty$ as $|x|\to\infty$, the embedding $H\hookrightarrow L^q(\mathbb{R}^N)$ is compact, where $H=\{u\in L^2;\int_{\mathbb{R}^N}|\xi|^{2\alpha}|\hat{u}(\xi)|^2d\xi+\int_{\mathbb{R}^N}V(x)|u(x)|^2dx<\infty\}$ and $2\le q<\frac{2N}{N-2\alpha}$. However, when potential V is a bounded function, it is well known that the embedding $H\hookrightarrow L^q(\mathbb{R}^N)$ is not compact. Therefore, how to overcome lack of compactness in variational problem, which is of particular interest, is one of main technique challenges in this paper.

Motivated by the above discussion, the goal of the current paper is to consider the existence of ground states to (1.1) with a bounded potential V. Equation (1.1) involves the fractional Laplacian $(-\Delta)^{\alpha}$, $0<\alpha<1$, which is a nonlocal operator. A general approach to deal with this problem is to transform (1.1) into a local problem via the Dirichlet-Neumann map, see [2, 18]. That is, for $u\in H^{\alpha}$, one considers the problem

$$\begin{gathered}-\operatorname{div}(y^{1-2\alpha}\nabla v)=0\quad\text{in }\mathbb{R}^{N+1}_+,\\ v(x,0)=u\quad\text{on }\mathbb{R}^N,\end{gathered} \tag{1.6}$$

from where the fractional Laplacian is obtained by

$$(-\Delta)^{\alpha}u(x) = -b_{\alpha} \lim_{y\to 0^+} y^{1-2\alpha} v_y,$$

where b_{α} is an appropriate constant.

In this article, we prefer to investigate equation (1.1) directly in $H^{\alpha}(\mathbb{R}^N)$. This enables us to prove the existence of solutions to (1.1) by an analogue argument as the case $\alpha = 1$. It is well known that concentration compactness principle due to Lions is a powerful tool which is designed to pass to limit in variational problem with lack of compactness. Recently, a version of concentration compactness of Lions was used in [7] to treat fractional nonlinear Schrödinger equation (1.5). Thus, there is no doubt that the method can be adapted and applied to deal with our problem.

Firstly, we consider the case where V is a constant. Without loss of generality, we may assume $\lambda = 1$ and consider the following minimization problem:

$$-c = \inf\{E(u);\ u \in M\}, \tag{1.7}$$

where the constricted set is

$$M := \{u \in H^{\alpha};\int_{\mathbb{R}^N} |u(x)|^2 dx = \mu\},$$

and energy is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}(\xi)|^2 d\xi - \frac{1}{p+2}\int_{\mathbb{R}^N} |u(x)|^{p+2} dx.$$

Any minimizer $u \in H^{\alpha}$ of (1.7) has to satisfy the corresponding Euler-Lagrange equation

$$(-\Delta)^{\alpha}u + \omega u = |u|^p u, \tag{1.8}$$

with some Lagrange multiplier $\omega > 0$. Notice that the existence of bound state solutions of (1.8) have been obtained in [5], by using concentration compactness principle in fractional Sobolev spaces, we obtain the existence of ground state solutions to (1.8). More precisely, our result is as follows:

Theorem 1.1. *Let $0 < p < 4\alpha/N$, $N \geq 2$, and $\mu > 0$. Then, the minimizing problem (1.7) has a positive ground state solution $u \in H^{\alpha}$, and it satisfies (1.8) for some $\omega > 0$. Moreover, every minimizing sequence $(u_n)_{n>0}$ of (1.7) is relatively compact in H^{α} up to translations; i.e., there exist a subsequence $(u_{n_k})_{k>0}$ and $(y_k)_{k>0} \subset \mathbb{R}^N$ such that $u_{n_k}(\cdot - y_k) \to u$ in H^{α} as $k \to \infty$. In particular, u is a solution of (1.7).*

Next, let us consider general situation. Let $V : \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous function $V > 0$ and assume that

$$\lim_{|x|\to\infty} V(x) = V_{\infty} > 0. \tag{1.9}$$

Note that solutions of (1.1) correspond to critical points of the functional

$$E_1(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}(\xi)|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u(x)|^2 dx \tag{1.10}$$

on H^{α}, restricted to the unit sphere

$$M_1 = \{u \in H^{\alpha};\int_{\mathbb{R}^N} |u(x)|^{p+2} dx = 1\}$$

in L^{p+2}. In addition, if $V(x) = V_\infty$, then E_1 is invariant under translations
$$u \mapsto u_{x_0} = u(x - x_0).$$
In general, for any $u \in H^\alpha$, after a substitution of variables
$$\begin{aligned} E_1(u_{x_0}) &= \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^N} V(x + x_0)|u|^2 dx \\ &\to \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty|u|^2 dx \end{aligned}$$
as $|x_0| \to \infty$. Hence, we call
$$E_1^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty|u|^2 dx.$$
the functional at infinity associated with E_1.

Theorem 1.2. *Let $0 < p < 4\alpha/(N - 2\alpha)$, $N \geq 2$ and V satisfy* (1.9). *Assume*
$$c = \inf_{u \in M_1} E_1(u) < \inf_{u \in M_1} E_1^\infty(u) := c^\infty. \tag{1.11}$$
Then, there exists a positive solution $u \in H^\alpha$ of (1.1) *for some $\lambda > 0$. Moreover, condition* (1.11) *is necessary and sufficient for the relative compactness of all minimizing sequences for E_1 in M_1.*

Notation. Throughout this paper, $C > 0$ will stand for a constant that may be different from line to line when it does not cause any confusion. Since we deal with $\mathbb{R}^N$, we often use the abbreviations $L^r = L^r(\mathbb{R}^N)$, $H^\alpha = H^\alpha(\mathbb{R}^N)$.

2. Preliminaries

In this section, we state and prove some preliminary results that will be used later. First, we recall some definitions about the fractional Laplacian operator. For any $\alpha \in (0, 1)$, the fractional Sobolev space H^α is defined by
$$H^\alpha = \big\{u \in L^2; \int_{\mathbb{R}^N} (1 + |\xi|^{2\alpha})|\hat{u}(\xi)|^2 d\xi < \infty\big\},$$
endowed with the norm
$$\|u\|_{H^\alpha} = \big\{\int_{\mathbb{R}^N} |u|^2 dx + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}}\, dx\, dy\big\}^{1/2},$$
where the symbol ˆ stands for Fourier transform and the term
$$[u]_{H^\alpha} = \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2} := \Big\{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}}\, dx\, dy\Big\}^{1/2}$$
is the so-called Gagliardo semi-norm of u. Therefore, we often use the explicit formula
$$\langle(-\Delta)^\alpha u, u\rangle := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}}\, dx\, dy.$$
Next, we recall the definition of weak solutions $u \in H^\alpha$ to (1.1).

Definition 2.1. We say that $u \in H^\alpha$ is a weak solution of (1.1) if
$$\int_{\mathbb{R}^N} |\xi|^{2\alpha}\hat{u}\hat{\varphi} d\xi + \int_{\mathbb{R}^N} V(x)u\varphi dx = \int_{\mathbb{R}^N} |u|^p u\varphi dx,$$
for any $\varphi \in C_c^\infty(\mathbb{R}^N)$.

Following the argument of [14, Theorem 7.13], we can obtain the following lemma which can be used to derive the existence of positive solution to (1.1).

Lemma 2.2. *Let f, g are two real-valued functions in H^α with $f \neq 0$. Then*

$$\Big\langle (-\Delta)^\alpha \sqrt{f^2+g^2}, \sqrt{f^2+g^2} \Big\rangle \leq \langle (-\Delta)^\alpha f, f\rangle + \langle (-\Delta)^\alpha g, g\rangle.$$

Equality holds if and only if f has a definite sign and $g(x) = Cf(x)$ for a.e $x \in \mathbb{R}^N$ for some constant C.

By the similar proof as [14, Theorem 7.16], we have the following lemma:

Lemma 2.3. *Let ψ be a bounded function in $C^\infty(\mathbb{R}^N)$ with bounded derivatives and $f \in H^\alpha$. Then the pointwise product of ψ and f,*

$$(\psi \cdot f)(x) = \psi(x) f(x),$$

is also a function in H^α and

$$\|\psi f\|_{H^\alpha} \leq C(\|\psi\|_{L^\infty} + \|\nabla \psi\|_{L^\infty}) \|f\|_{H^\alpha}.$$

Finally, we extend the concentration compactness principle in H^1 to fractional Sobolev spaces H^α. The proof follows the argument of [3, Proposition 1.7.6], but we need to operate some modifications due to the non-locality of the fractional operators $(-\Delta)^\alpha$.

Lemma 2.4. *Let $N \geq 2$. Suppose $(u_n)_{n>0} \subset H^\alpha$ and satisfy*

$$\int_{\mathbb{R}^N} |u_n(x)|^2 dx = \mu > 0, \tag{2.1}$$

$$\sup_{n>0} \|u_n\|_{H^\alpha} < \infty. \tag{2.2}$$

Then there exists a subsequence $(u_{n_k})_{k>0}$, for which one of the following properties holds.

(i) *Compactness: There exists a sequence $(y_k)_{k>0}$ in $\mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists $0 < r < \infty$ with*

$$\int_{|x-y_k| \leq r} |u_{n_k}(x)|^2 dx \geq \mu - \varepsilon. \tag{2.3}$$

(ii) *Vanishing: For all $r < \infty$, it follows that*

$$\lim_{k\to\infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y|\leq r} |u_{n_k}(x)|^2 dx = 0.$$

(iii) *Dichotomy: There exist a constant $\beta \in (0, \mu)$ and two bounded sequences $(v_k)_{k>0}$, $(w_k)_{k>0} \subset H^\alpha$ such that*

$$\operatorname{supp} v_k \cap \operatorname{supp} w_k = \emptyset, \tag{2.4}$$

$$|v_k| + |w_k| \leq |u_{n_k}|, \tag{2.5}$$

$$\|v_k\|_{L^2}^2 \to \beta, \quad \|w_k\|_{L^2}^2 \to (\mu - \beta) \quad \text{as } k \to \infty, \tag{2.6}$$

$$\|u_{n_k} - v_k - w_k\|_{L^p} \to 0 \quad \text{for } 2 \leq p < \frac{2N}{N-2\alpha}, \tag{2.7}$$

$$\liminf_{k\to\infty} \big\{ \langle (-\Delta)^\alpha u_{n_k}, u_{n_k}\rangle - \langle (-\Delta)^\alpha v_k, v_k\rangle - \langle (-\Delta)^\alpha w_k, w_k\rangle \big\} \geq 0. \tag{2.8}$$

Proof. We proceed along the lines of [3, Proposition 1.7.6]. Let $(u_n)_{n>0} \subset H^\alpha$ satisfy (2.1) and (2.2). We define the sequence, $(Q_n)_{n>0}$, of Lévy concentration functions by

$$Q_n(r) := \sup_{y\in\mathbb{R}^N} \int_{|x-y|<r} |u_n(x)|^2 dx, \quad \text{for all } r \geq 0. \tag{2.9}$$

By a similar argument as that of [3, Proposition 1.7.6], there exists a subsequence, $(Q_{n_k})_{k>0}$, such that

$$Q_{n_k}(r) \to Q(r) \quad \text{as } k \to \infty \text{ for all } r \geq 0,$$

where $Q(r)$ is a nonnegative, nondecreasing function. Clearly, we see that

$$\beta := \lim_{r\to\infty} Q(r) \in [0, \mu]. \tag{2.10}$$

If $\beta = 0$, then the vanishing property is a direct consequence of the definition of $Q_n(r)$. If $\beta = \mu$, then compactness holds, see [3] for details.

Assume that $\beta \in (0, \mu)$ holds. We fix $\phi, \varphi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \phi, \varphi \leq 1$ and

$$\begin{aligned} \phi(x) &\equiv 1 \quad \text{for } 0 \leq |x| \leq 1, \quad \phi(x) \equiv 0 \quad \text{for } |x| \geq 2, \\ \varphi(x) &\equiv 0 \quad \text{for } 0 \leq |x| \leq 2, \quad \varphi(x) \equiv 1 \quad \text{for } |x| \geq 3. \end{aligned}$$

We set $v_k = \phi_k u_{n_k}$ and $w_k = \varphi_k u_{n_k}$, where

$$\phi_k(x) = \phi(\frac{x - y_k}{r_k}), \quad \varphi_k(x) = \varphi(\frac{x - y_k}{r_k}).$$

Therefore, properties (2.4) and (2.5) are immediate. By using Lemma 2.3, we find that $(v_k)_{k>0}$ and $(w_k)_{k>0}$ are two bounded sequences H^α. Following the argument of [3, Proposition 1.7.6], we can obtain (2.6). (2.7) follows by interpolation inequality. Finally, we show (2.8). One easily verifies that

$$\begin{aligned} |v_k(x) - v_k(y)|^2 &= |\phi_k(x)u_{n_k}(x) - \phi_k(y)u_{n_k}(y)|^2 \\ &\leq \frac{1}{2}|\phi_k(x) - \phi_k(y)|^2(|u_{n_k}(x)|^2 + |u_{n_k}(y)|^2) \\ &\quad + \frac{1}{2}(|\phi_k(x)|^2 + |\phi_k(y)|^2)|u_{n_k}(x) - u_{n_k}(y)|^2. \end{aligned} \tag{2.11}$$

Similarly,

$$\begin{aligned} |w_k(x) - w_k(y)|^2 &\leq \frac{1}{2}|\varphi_k(x) - \varphi_k(y)|^2(|u_{n_k}(x)|^2 + |u_{n_k}(y)|^2) \\ &\quad + \frac{1}{2}(|\varphi_k(x)|^2 + |\varphi_k(y)|^2)|u_{n_k}(x) - u_{n_k}(y)|^2. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), we derive

$$\begin{aligned} &\langle(-\Delta)^\alpha u_{n_k}, u_{n_k}\rangle - \langle(-\Delta)^\alpha v_k, v_k\rangle - \langle(-\Delta)^\alpha w_k, w_k\rangle \\ &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_{n_k}(x) - u_{n_k}(y)|^2 - |v_k(x) - v_k(y)|^2 - |w_k(x) - w_k(y)|^2}{|x - y|^{N+2\alpha}}\,dx\,dy \\ &\geq -\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varphi_k(x) - \varphi_k(y)|^2|u_{n_k}(x)|^2 + |\phi_k(x) - \phi_k(y)|^2|u_{n_k}(x)|^2}{|x - y|^{N+2\alpha}}\,dx\,dy. \end{aligned} \tag{2.13}$$

Therefore, to prove (2.8), it suffices to show that the last term in (2.13) converges to zero as k approaches infinity. Indeed, note the fact that $\operatorname{supp}\phi_k \cap \operatorname{supp}\varphi_k = \emptyset$, by mean value theorem, we can estimate as follows:

$$
\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\phi_k(x)-\phi_k(y)|^2|u_{n_k}(x)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\le \int_{|x-y|\le r_k} \frac{|\phi_k(x)-\phi_k(y)|^2|u_{n_k}(x)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\quad + \int_{|x-y|> r_k} \frac{|\phi_k(x)-\phi_k(y)|^2|u_{n_k}(x)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\le \frac{1}{r_k^2}\int_{|x-y|\le r_k} \frac{|u_{n_k}(x)|^2}{|x-y|^{N+2\alpha-2}}\,dx\,dy \\
&\quad + \frac{1}{r_k^\alpha}\int_{|x-y|> r_k} \frac{|\phi_k(x)-\phi_k(y)|^2|u_{n_k}(x)|^2}{|x-y|^{N+\alpha}}\,dx\,dy \\
&\le \frac{1}{r_k^2}\int_{\mathbb{R}^N} |u_{n_k}(x)|^2dx \int_{|z|\le r_k} |z|^{-(N+2\alpha-2)}dz \\
&\quad + \frac{C}{r_k^\alpha}\int_{\mathbb{R}^N} |u_{n_k}(x)|^2dx \int_{|x-y|> r_k} \frac{1}{|x-y|^{N+\alpha}}dy \\
&\le C(r_k^{-2\alpha}+r_k^{-\alpha})\int_{\mathbb{R}^N} |u_{n_k}(x)|^2dx \to 0 \quad \text{as } k\to\infty.
\end{aligned}
$$

Similarly,

$$
\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varphi_k(x)-\varphi_k(y)|^2|u_{n_k}(x)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \le C(r_k^{-2\alpha}+r_k^{-\alpha})\int_{\mathbb{R}^N} |u_{n_k}(x)|^2dx
$$

which converges to zero as $k\to\infty$. This completes the proof. □

3. Proof of main results

Proof of Theorem 1.1. We proceed in three steps.

Step 1. $0 < c < \infty$. It is clear that $M \neq \emptyset$. Let $u \in M$ and $\gamma > 0$, set $u_\gamma = \gamma^{N/2}u(\gamma x)$. It is straightforward to check that $u_\gamma \in M$ and

$$
E(u_\gamma) = \frac{\gamma^{2\alpha}}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2d\xi - \frac{\gamma^{\frac{Np}{2}}}{p+2}\int_{\mathbb{R}^N} |u|^{p+2}dx.
$$

Since $Np < 4\alpha$, we derive $E(u_\gamma) < 0$ for γ small, and so $c > 0$.

On the other hand, we deduce from an interpolation inequality, the embedding theorem $\|u\|_{L^{\frac{2N}{N-2\alpha}}} \le C\|(-\Delta)^{\alpha/2}u\|_{L^2}$ and the Young inequality with ε that

$$
\begin{aligned}
E(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2d\xi - C\Big(\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2d\xi\Big)^{\frac{Np}{4\alpha}}\|u\|_{L^2}^{p+2-\frac{Np}{2\alpha}} \\
&\ge \frac{1-\varepsilon}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2d\xi - K
\end{aligned} \tag{3.1}
$$

for some $0 < K < \infty$. This implies $c < \infty$.

Step 2. Estimates on the minimizing sequence $(u_n)_{n>0}$ of (1.7). Due to $u_n \in M$, $(u_n)_{n>0}$ is bounded in L^2. It follows from (3.1) that $(u_n)_{n>0}$ is bounded in H^α.

Furthermore, thanks to $c > 0$, we obtain $E(u_n) \le -c/2$ for n sufficiently large. We consequently derive that

$$\int_{\mathbb{R}^N} |u_n|^{p+2} dx \ge \frac{p+2}{2} c. \tag{3.2}$$

Step 3. Conclusion. Let $(u_n)_{n>0}$ be a minimizing sequence of (1.7). Note that by scaling we may assume $\mu = 1$. In view of Lemma 2.2, $(|u_n|)_{n>0}$ is also a minimizing sequence of (1.7). So, without loss of generality, we may suppose that u_n is nonnegative. Let us now apply Lemma 2.4 to the minimizing sequence $(u_n)_{n>0}$.

Firstly, we claim vanishing cannot occur. Indeed, if not, applying [7, Lemma 2.2], we have $u_k \to 0$ in L^{p+2}, which is a contradiction with (3.2).

Next, we show dichotomy cannot occur. If not, there exist a constant $\beta \in (0, 1)$, two sequences $(v_k)_{k>0}$ and $(w_k)_{k>0}$ introduced in Lemma 2.4. It follows from (2.7) and (2.8) that

$$\liminf_{k\to\infty} (E(u_k) - E(v_k) - E(w_k)) \ge 0.$$

Hence,

$$\limsup_{k\to\infty} (E(v_k) + E(w_k)) \le -c. \tag{3.3}$$

On the other hand, given $u \in H^\alpha$ and $a > 0$, we have

$$E(u) = \frac{1}{a^2} E(au) + \frac{a^p - 1}{p+2} \int_{\mathbb{R}^N} |u|^{p+2} dx.$$

Applying the above inequality with v_k and $a_k = 1/\|v_k\|_{L^2}$, due to $a_k v_k \in M$, we obtain

$$E(v_k) \ge \frac{-c}{a_k^2} + \frac{a_k^p - 1}{p+2} \int_{\mathbb{R}^N} |v_k|^{p+2} dx. \tag{3.4}$$

Similarly,

$$E(w_k) \ge \frac{-c}{b_k^2} + \frac{b_k^p - 1}{p+2} \int_{\mathbb{R}^N} |w_k|^{p+2} dx, \tag{3.5}$$

where $b_k = 1/\|w_k\|_{L^2}$. Therefore, collecting (3.4) and (3.5), we see that

$$E(v_k) + E(w_k) \ge -c(a_k^{-2} + b_k^{-2}) + \frac{a_k^p - 1}{p+2} \int_{\mathbb{R}^N} |v_k|^{p+2} dx + \frac{b_k^p - 1}{p+2} \int_{\mathbb{R}^N} |w_k|^{p+2} dx.$$

Note that $a_k^{-2} \to \beta$ and $b_k^{-2} \to 1 - \beta$ by (2.6). It follows from $0 < \beta < 1$ that

$$\theta := \min\{\beta^{-p/2}, (1-\beta)^{-p/2}\} > 1.$$

Therefore, we deduce from (2.7) and (3.2) that

$$\liminf_{k\to\infty} (E(v_k) + E(w_k)) \ge -c + \frac{\theta - 1}{p+2} \liminf_{k\to\infty} \int_{\mathbb{R}^N} |u_k|^{p+2} dx \ge -c + \frac{\theta - 1}{2} > -c,$$

which contradicts (3.3).

Finally, since we have ruled out both vanishing and dichotomy, then we conclude that indeed compactness occur. Applying Lemma 2.4, we deduce that for some sequence $(y_k) \subset \mathbb{R}^N$ and some $u \in H^\alpha$, such that $u_{n_k}(\cdot - y_k) \to u$ in L^2 and in L^{p+2}. Together with the weak lower semicontinuity of the H^α norm, this implies

$$E(u) \le \lim_{k\to\infty} E(u_{n_k}) = -c.$$

In view of definition of c, we have $E(u) = -c$. In particular, $E(u_{n_k}) \to E(u)$ and it holds that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}_{n_k}|^2 d\xi \to \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2 d\xi \quad \text{as } k \to \infty,$$

which implies $u_{n_k}(\cdot - y_k) \to u$ in H^α. This completes the proof. □

Proof of Theorem 1.2. Firstly, we show that (1.11) is necessary for the relative compactness of all minimizing sequences for E_1 in M_1. Indeed, if $c \geq c^\infty$, let $(u_n)_{n>0}$ be a minimizing sequence for E_1^∞. Then $(\tilde{u}_n)_{n>0}$, defined by $\tilde{u}_n = u_n(\cdot + x_n)$, is also a minimizing sequence for E_1^∞. One easily verifies that

$$|E_1(\tilde{u}_n) - E_1^\infty(\tilde{u}_n)| = \frac{1}{2}\int_{\mathbb{R}^N} (V(x) - V_\infty)|\tilde{u}_n|^2 dx \to 0 \quad \text{as } |x_n| \to \infty.$$

This implies $(\tilde{u}_n)$ is a minimizing sequence for E_1.

On the other hand, it is straightforward to check that

$$\tilde{u}_n \to 0 \quad \text{in } L^p_{\rm loc} \text{ as } |x_n| \to \infty.$$

Thus, $(\tilde{u}_n)_{n>0}$ cannot be relatively compact, which is a contradiction with the fact $\|\tilde{u}_n\|_{L^p} = 1$.

We now show that condition (1.11) is sufficient. Let $(u_n)_{n>0}$ be a minimizing sequence for E_1 in M_1 such that

$$E_1(u_n) \to c \quad \text{as } n \to \infty.$$

By Lemma 2.2, we may assume u_n is nonnegative and $u_n \rightharpoonup u$ weakly in L^{p+2}. In view of assumption of V, V is positive on $\mathbb{R}^N$. Hence we have

$$\|u_n\|^2_{H^\alpha} \leq C_1 E_1(u_n) \leq C < \infty, \tag{3.6}$$

and we may assume that $u_n \rightharpoonup u$ weakly in H^α and pointwise almost everywhere. Denote $u_n = v_n + u$. Applying Bresiz-Lieb lemma, we have

$$\Big|\int_{\mathbb{R}^N} |u_n|^{p+2}dx - \int_{\mathbb{R}^N} |v_n|^{p+2}dx - \int_{\mathbb{R}^N} |u|^{p+2}dx\Big| \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\int_{\mathbb{R}^N} |v_n|^{p+2}dx + \int_{\mathbb{R}^N} |u|^{p+2}dx \to 1 \quad \text{as } n \to \infty.$$

It follows that

$$\begin{aligned} E_1(u_n) &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_n|^2dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{v}_n|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\hat{u}|^2 d\xi \\ &\quad + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(v_n(x) - v_n(y))(u(x) - u(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(x)(|u|^2 + 2uv_n + |v_n|^2)dx \\ &= E_1(v_n) + E_1(u) + \langle(-\Delta)^\alpha v_n, u\rangle + \int_{\mathbb{R}^N} V(x)uv_n dx. \end{aligned} \tag{3.7}$$

We deduce from $v_n \rightharpoonup 0$ in H^α that

$$\langle(-\Delta)^\alpha v_n, u\rangle + \int_{\mathbb{R}^N} V(x)uv_n\,dx \to 0. \tag{3.8}$$

Moreover, for any $\varepsilon > 0$, let

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N; |V(x) - V_\infty| \geq \varepsilon\}.$$

In view of the definition of V, Ω_ε is a bounded compact subset in $\mathbb{R}^N$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (V(x) - V_\infty)|v_n|^2 dx &\leq V_\infty \int_{\Omega_\varepsilon} |v_n|^2 dx + \varepsilon \int_{\mathbb{R}^N} |v_n|^2 dx \\ &\leq C\varepsilon + o(1), \end{aligned} \tag{3.9}$$

where $o(1)$ denotes error terms such that $o(1) \to 0$ as $n \to \infty$. Collecting (3.7)-(3.9), we have

$$E_1(u_n) = E_1(u) + E_1(v_n) + o(1) = E_1(u) + E_1^\infty(v_n) + o(1). \tag{3.10}$$

By homogeneity, and setting $\beta = \int_{\mathbb{R}^N} |u|^{p+2} dx$, if $\beta > 0$

$$E_1(u) = \beta^{2/(p+2)} E_1(\beta^{-1/(p+2)} u) \geq \beta^{2/(p+2)} c \tag{3.11}$$

if $0 \leq \beta < 1$

$$E_1^\infty(v_n) = (1-\beta)^{2/(p+2)} E_1^\infty((1-\beta)^{-1/(p+2)} u) \geq (1-\beta)^{2/(p+2)} c^\infty + o(1). \tag{3.12}$$

Therefore, we deduce (3.10)-(3.12) that for all $\beta \in [0,1]$

$$\begin{aligned} c = E_1(u_n) + \circ(1) &= E_1(u) + E_1^\infty(v_n) + o(1) \\ &\geq \beta^{2/(p+2)} c + (1-\beta)^{2/(p+2)} c^\infty + o(1), \end{aligned}$$

which, together with (1.11), implies $\beta \in \{0, 1\}$. If $\beta = 0$, then

$$c \geq c^\infty + o(1).$$

Thus $\beta = 1$; that is, $u_n \to u$ in L^{p+2} and $u \in M_1$. By convexity of E_1,

$$E_1(u) \leq \liminf_{n\to\infty} E_1(u_n) = c,$$

and u minimizes E_1 in M_1. Hence, $E_1(u_n) \to E_1(u)$. Combining this, (3.6) and (3.10), we have

$$\|u_n - u\|_{H^\alpha}^2 \leq C_1 E_1(u_n - u) = C_1(E_1(u_n) - E_1(u)) + o(1).$$

This implies $u_n \to u$ in H^α as $n \to \infty$. The existence of solution to (1.1) follows by Lagrange multiplier methods for some $\lambda > 0$. ☐

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