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EXISTENCE OF SOLUTIONS FOR QUASILINEAR PARABOLIC EQUATIONS AT RESONANCE

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Abstract. In this article, we show the existence of nontrivial solutions for a class of quasilinear parabolic differential equations. To obtain the solution in a weighted Sobolev space, we use the Galerkin method, Brouwer's theorem, and a compact Sobolev-type embedding theorem proved by Shapiro.

1. Introduction

Many results on the existence of solutions of the quasilinear parabolic resonance problems have been presented in [3, 5, 7, 8, 9] and their references cited therein. Shapiro [9] considered a weak solution of the following problem, in the Hilbert space $\widetilde{H}(\widetilde{\Omega},\Gamma)$,

$$\begin{gathered}\rho D_t u + \mathcal{Q}u = [\lambda_{j_0} u + f(x,u) + g(x,t,u)]\rho, \quad (x,t)\in\widetilde{\Omega},\\ u\in\widetilde{H}(\widetilde{\Omega},\Gamma),\end{gathered}\tag{1.1}$$

where

$$\mathcal{Q}u = -\sum_{i=1}^{N} D_i\big[p_i^{1/2}(x)A_i(x,u,Du)\big] + qB_0(x,u,Du)u.$$

Kuo [5] also discussed the existence of a nontrivial solution for a quasilinear parabolic equation in the Hilbert space $\widetilde{H}_0^m$:

$$\begin{gathered}D_t u + \widetilde{\mathcal{Q}}u - \lambda_1 u + f(x,t,u) = h(x,t), \quad (x,t)\in\widetilde{\Omega},\\ u = 0, \quad (x,t)\in\partial\widetilde{\Omega},\end{gathered}\tag{1.2}$$

where

$$\widetilde{\mathcal{Q}}(u)(v) = \sum_{|\alpha|\le m}\int_{\widetilde{\Omega}} A_\alpha(x,\xi(u))D^\alpha v,$$

and λ_1 is the first eigenvalue of $\widetilde{\mathcal{Q}}$.

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Motivated by Shapiro [9], in this paper, we show the existence of solutions for the quasilinear parabolic equation in the weighted Sobolev space $\widetilde{H}(\widetilde{\Omega},\Gamma)$:

$$\begin{gathered}\rho D_t u+\widetilde{\mathcal{M}}u=[\lambda_{j_0}u+f(x,u)+g(x,t,u)]\rho-G,\quad (x,t)\in\widetilde{\Omega},\\ u\in\widetilde{H}(\widetilde{\Omega},\Gamma),\end{gathered}\tag{1.3}$$

where

$$\widetilde{\mathcal{M}}u=-\sum_{i,j=1}^{N}D_i[p_i^{1/2}(x)p_j^{1/2}(x)\sigma_i^{1/2}(u)\sigma_j^{1/2}(u)b_{ij}(x)D_ju]+b_0(x)\sigma_0(u)qu,\tag{1.4}$$

and λ_{j_0} is an eigenvalue of $\mathcal{L}$.

In fact, (1.3) is one of the most useful sets of Navier-Stokes equations which describe the motion of viscous fluid substances. They are widely used in the design of aircrafts and cars, the study of blood flow and the design of power stations, etc. Furthermore, coupled with Maxwell's equations, the Navier-Stokes equations can also be used to model and study magnetohydrodynamics.

The method of this paper is based on the Galerkin method [6], the generalized Brouwer's theorem [4] and a weighted compact Sobolev-type embedding theorem [9] established by Shapiro. The nonlinearity in (1.3) satisfies the generalized Landesman-Lazer type conditions [6]. Compared with the problem (1.1) in [9], the operator $\widetilde{\mathcal{M}}$ in (1.3) has an extensive presentation format and wider applications.

Now, we give the assumptions and definitions which are needed for the proof of Theorem 1.5.

Let $\Omega\subset R^N, N\geq 1$, be an open (possibly unbounded) set and let $\rho(x),p_i(x)\in C^0(\Omega)$ be positive functions with the property that

$$\int_\Omega\rho(x)dx<\infty,\quad \int_\Omega p_i(x)dx<\infty,\quad i=1,2,\dots,N.\tag{1.5}$$

Let $q(x)\in C^0(\Omega)$ be a nonnegative function and $\Gamma\subset\partial\Omega$ be a fixed closed set. Note that Γ may be an empty set and $q(x)$ may be zero. On the other hand, $q(x)$ will satisfy: There exists $K>0$, such that

$$0\leq q(x)\leq K\rho(x),\quad \text{for all } x\in\Omega.\tag{1.6}$$

Here $\mathcal{A}$ is a set of real-valued functions defined as

$$\mathcal{A}=\{u\in C^0(\bar{\Omega}\times R): u(x,t+2\pi)=u(x,t),\ \text{for all } (x,t)\in\bar{\Omega}\times R\}.$$

Setting $\widetilde{\Omega}=\Omega\times T$, $T=(-\pi,\pi)$, $p=(p_1,\dots,p_N)$ and $D_i=\frac{\partial u}{\partial x_i}, i=1,2,\dots,N$, we consider the following pre-Hilbert spaces (see [9]):

$$\widetilde{C}^0_\rho(\widetilde{\Omega})=\big\{u\in C^0(\widetilde{\Omega}):\int_{\widetilde{\Omega}}|u(x,t)|^2\rho(x)\,dx\,dt<\infty\big\},$$

with the inner product

$$\langle u,v\rangle^{\sim}_\rho=\int_{\widetilde{\Omega}}u(x,t)v(x,t)\rho(x)\,dx\,dt,$$

and the space

$$\begin{gathered}\widetilde{C}^1_{p,\rho}(\widetilde{\Omega},\Gamma)=\Big\{u\in\mathcal{A}\cap C^1(\Omega\times R): u(x,t)=0,\ \text{for all } (x,t)\in\Gamma\times R;\\ \int_{\widetilde{\Omega}}[\sum_{i=1}^{N}|D_iu|^2p_i+(u^2+|D_tu|^2)\rho]<\infty\Big\}\end{gathered}$$

with inner product

$$\langle u, v\rangle_{\widetilde{H}} = \int_{\widetilde{\Omega}} \big[\sum_{i=1}^{N} p_i D_i u D_i v + (uv + D_t u D_t v)\rho\big]\,dx\,dt. \tag{1.7}$$

Let $\widetilde{L}^2_\rho = L^2_\rho(\widetilde{\Omega})$ denote the Hilbert space obtained from the completion of $\widetilde{C}^0_\rho$ with the norm $\|u\|_\rho = (\langle u,u\rangle^{\sim}_\rho)^{1/2}$ by using Cauchy sequences, and $\widetilde{H} = \widetilde{H}(\widetilde{\Omega},\Gamma)$ denote the completion of the space $\widetilde{C}^1_{p,\rho}$ with the norm $\|u\|_{\widetilde{H}} = \langle u,u\rangle^{1/2}_{\widetilde{H}}$. Similarly, we have $\widetilde{L}^2_{p_i}, (i = 1,2,\cdots,N)$ and $\widetilde{L}^2_q$.

It is assumed throughout this paper that $\sigma_i(u)(i = 0,1,\dots,N)$ meets:

- (S1) $\sigma_i(u) : \widetilde{H} \to \mathbb{R}$ is weakly sequentially continuous;
- (S2) there are $\eta_0, \eta_1 > 0$, such that $\eta_0 \le \sigma_i(u) \le \eta_1$, and $\sigma_i(u)$ is measurable, for $u \in \widetilde{H}$.

The functions $a_{ij}(i,j = 1,2,\dots,N)$ and $a_0(x)$ satisfy (also $b_{ij}(x)$ and $b_0(x)$):

$$\begin{gathered} a_0(x), a_{ij}(x) \in C^0(\Omega) \cap L^\infty(\Omega), \quad i,j = 1,2,\dots,N;\\ a_{ij}(x) = a_{ji}(x), \quad \forall x \in \Omega,\ i,j = 1,2,\dots,N;\\ a_0(x) \ge \beta_0 > 0 \quad (b_0(x) \ge \beta_1 > 0), \quad \text{for } x \in \Omega;\\ \text{there is a } c_0 > 0\ (c_1 > 0) \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^N, \text{ such that}\\ \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2 \quad (\sum_{i,j=1}^{N} b_{ij}(x)\xi_i\xi_j \ge c_1|\xi|^2). \end{gathered} \tag{1.8}$$

The function $g(x,t,s)$ meets the following conditions:

- (G1) $g(x,t,s)$ satisfies the Caratheodory assumptions;
- (G2) for any $\varepsilon > 0$, there is a $g_\varepsilon(x,t) \in \widetilde{L}^2_\rho$, such that $|g(x,t,s)| \le \varepsilon|s| + g_\varepsilon(x,t)$, for a.e. $(x,t) \in \widetilde{\Omega}$, and all $s \in \mathbb{R}$.

Definition 1.1. For the quasilinear differential operator $\widetilde{\mathcal{M}}$, the two-form is

$$\widetilde{\mathcal{M}}(u,v) = \sum_{i,j=1}^{N} \int_{\widetilde{\Omega}} \big[p_i^{\frac12} p_j^{\frac12} \sigma_i^{\frac12}(u)\sigma_j^{\frac12}(u) b_{ij} D_j u D_i v\big] + \int_{\widetilde{\Omega}} q\sigma_0(u) b_0 uv, \tag{1.9}$$

for $u, v \in \widetilde{H}(\widetilde{\Omega},\Gamma)$.

Defining

$$\mathcal{L}u = -\sum_{i,j=1}^{N} D_i\big[p_i^{\frac12} p_j^{\frac12} a_{ij} D_j u\big] + a_0 q u, \tag{1.10}$$

for $u \in H_{p,q,\rho} = H_{p,q,\rho}(\Omega,\Gamma)$ (as described in [9]), and

$$\widetilde{\mathcal{L}}u = -\sum_{i,j=1}^{N} D_i\big[p_i^{\frac12} p_j^{\frac12} a_{ij} D_j u\big] + a_0 q u, \quad u \in \widetilde{H}(\widetilde{\Omega},\Gamma),$$

then the two-form of $\mathcal{L}$ is

$$\mathcal{L}(u,v) = \sum_{i,j=1}^{N} \int_{\Omega} p_i^{1/2} p_j^{1/2} a_{ij}(x) D_j u D_i v + \int_{\Omega} a_0 uvq, \quad u,v \in H_{p,q,\rho}(\Omega,\Gamma), \tag{1.11}$$

and the two-form of $\widetilde{\mathcal{L}}$ is

$$\widetilde{\mathcal{L}}(u,v)=\sum_{i,j=1}^{N}\int_{\widetilde{\Omega}}p_i^{1/2}p_j^{1/2}a_{ij}(x)D_juD_iv+\int_{\widetilde{\Omega}}a_0uvq,\quad u,v\in\widetilde{H}(\widetilde{\Omega},\Gamma). \tag{1.12}$$

Definition 1.2. We say that $\widetilde{\mathcal{M}}$ is $\#\widetilde{H}$-related to $\widetilde{\mathcal{L}}$ if the following condition holds:

$$\lim_{\|u\|_{\widetilde{H}}\to\infty}\frac{[\widetilde{\mathcal{M}}(u,v)-\widetilde{\mathcal{L}}(u,v)]}{\|u\|_{\widetilde{H}}}=0,\quad \text{uniformly for } \|v\|_{\widetilde{H}}\le 1.$$

Definition 1.3. The pair (Ω,Γ) is a $V_L(\Omega,\Gamma)$ if

- (VL1) there is a complete orthonormal system $\{\varphi_n\}_{n=1}^\infty$ in L^2_ρ. Also $\varphi_n\in H_{p,q,\rho}\cap C^2$ for all n;
- (VL2) there is a sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ with $0<\lambda_1<\lambda_2\le\lambda_3\le\dots\le\lambda_n\to\infty$ such that $\mathcal{L}(\varphi_n,v)=\lambda_n\langle\varphi_n,v\rangle_\rho$ for all $v\in H_{p,q,\rho}(\Omega,\Gamma)$. Also $\varphi_1>0$ in Ω.

We set

$$\gamma=(\lambda_{j_0+j_1}-\lambda_{j_0})/2, \tag{1.13}$$

where λ_{j_0} is an eigenvalue of $\mathcal{L}$ of multiplicity j_1. So $\lambda_{j_0+j_1}$ is the next eigenvalue strictly greater than λ_{j_0}. Also, we set

$$\mathcal{F}^{\pm}(x)=\limsup_{s\to\pm\infty}f(x,s)/s,\quad \mathcal{F}_{\pm}(x)=\liminf_{s\to\pm\infty}f(x,s)/s\,. \tag{1.14}$$

For $f(x,s)$, we have:

- (F1) $f(x,s)$ satisfies the Caratheodory conditions;
- (F2) $|f(x,s)-\gamma s|\le\gamma|s|+f_0(x)$ for all $s\in\mathbb{R}$, a.e. $x\in\Omega$, where $f_0\in L^2_\rho$;
- (F3)

$$\int_{\Omega\cap(v>0)}(\lambda_{j_0+j_1}-\lambda_{j_0}-\mathcal{F}^+)v^2\rho+\int_{\Omega\cap(v<0)}(\lambda_{j_0+j_1}-\lambda_{j_0}-\mathcal{F}^-)v^2\rho>0, \tag{1.15}$$

for every nontrivial $\lambda_{j_0+j_1}$-eigenfunction v of $\mathcal{L}$, and

$$\int_{\Omega\cap(w>0)}\mathcal{F}_+w^2\rho+\int_{\Omega\cap(w<0)}\mathcal{F}_-w^2\rho>0, \tag{1.16}$$

for every nontrivial λ_{j_0}-eigenfunction w of $\mathcal{L}$.

Remark 1.4. If $\widetilde{\mathcal{M}}$, as defined by (1.4), satisfies (S1)–(S2), then

$$\widetilde{M}(v,D_tv)=0,\quad \forall v\in\widetilde{C}^{1b}_{p,\rho}=\{v\in\widetilde{C}^1_{p,\rho}:D_tv\in\widetilde{C}^1_{p,\rho}\}. \tag{1.17}$$

Now, we state the main result of this article.

Theorem 1.5. *Let $\Omega\subset R^N(N\ge1)$, $T=(-\pi,\pi)$, $\widetilde{\Omega}=\Omega\times T$, $p=(p_1,\dots,p_N)$, ρ and $p_i(i=1,\dots,N)$ be positive functions in $C^0(\Omega)$ satisfying* (1.5)*, $q\in C^0(\Omega)$ be a nonnegative function satisfying* (1.6)*, and $\Gamma\subset\partial\Omega$ be a closed set. Let $\mathcal{L}$ and $\widetilde{\mathcal{M}}$ be given by* (1.9) *and* (1.4) *satisfying* (1.8), (S1), (S2) *respectively and $\mathcal{L}$ satisfies the conditions of $V_L(\Omega,\Gamma)$. If λ_{j_0} is an eigenvalue of $\mathcal{L}$ of multiplicity j_1, $\widetilde{\mathcal{M}}$ is $\#\widetilde{H}$- related to $\widetilde{\mathcal{L}}$, and* (F1)–(F3), (G1)–(G2) *hold, then problem* (1.3) *has at least one weak solution; i.e., there exits $u^*\in\widetilde{H}$ such that*

$$\langle D_tu^*,v\rangle^{\sim}_\rho+\widetilde{\mathcal{M}}(u^*,v)=\lambda_{j_0}\langle u^*,v\rangle^{\sim}_\rho+\langle f(x,u^*)+g(x,t,u^*),v\rangle^{\sim}_\rho-G(v)\,. \tag{1.18}$$

The rest of this article is arranged as follows. In section 2, we will give some preliminary lemmas; In section 3, we will prove the main results on the quasilinear parabolic differential equations.

2. Preliminary lemmas

In this section, we introduce some lemmas, and concepts which will be used later. If both (1.8) and the conditions of $V_L(\Omega,\Gamma)$ hold, we have

$$\{\widetilde{\varphi}^c_{jk}\}_{j=1,k=0}^{\infty,\infty} \cup \{\widetilde{\varphi}^s_{jk}\}_{j=1,k=1}^{\infty,\infty} \text{is a CONS for } \widetilde{L}^2_\rho, \tag{2.1}$$

where

$$\begin{gathered}\widetilde{\varphi}^c_{jk}(x,t) = \begin{cases} \varphi_j(x)/\sqrt{2\pi} & k=0,\ j=1,2,\dots,\\ \varphi_j(x)\cos(kt)/\sqrt{\pi} & j,k=1,2,\dots, \end{cases}\\ \widetilde{\varphi}^s_{jk}(x,t) = \varphi_j(x)\sin(kt)/\sqrt{\pi} \quad j,k=1,2,\dots.\end{gathered} \tag{2.2}$$

Obviously, both $\widetilde{\varphi}^c_{jk}$ and $\widetilde{\varphi}^s_{jk}$ are in $\widetilde{H}(\widetilde{\Omega},\Gamma)$. Define

$$\mathcal{L}_1(u,v) = \widetilde{\mathcal{L}}(u,v) + \langle u,v\rangle^\sim_\rho, \quad \forall u,v \in \widetilde{H}. \tag{2.3}$$

It is clear that $\mathcal{L}_1(u,v)$ is an inner product on $\widetilde{H}$ and from (1.6)-(1.8), (1.12) and (2.3), there are $K_1,K_2>0$ such that

$$K_1\|v\|^2_{\widetilde{H}} \le \mathcal{L}_1(v,v) + \|D_t v\|^2_\rho \le K_2\|v\|^2_{\widetilde{H}}, \quad \forall v\in\widetilde{H}. \tag{2.4}$$

For $v\in\widetilde{L}^2_\rho$, setting

$$\widehat{v}^c(j,k) = \langle v,\widetilde{\varphi}^c_{jk}\rangle^\sim_\rho, \quad \widehat{v}^s(j,k) = \langle v,\widetilde{\varphi}^s_{jk}\rangle^\sim_\rho, \tag{2.5}$$

and from (VL2), (1.12) and (2.3), we see that for $v\in\widetilde{H}$,

$$\mathcal{L}_1(v,\widetilde{\varphi}^s_{jk}) = (\lambda_j+1)\widehat{v}^s(j,k), \quad \mathcal{L}_1(v,\widetilde{\varphi}^c_{jk}) = (\lambda_j+1)\widehat{v}^c(j,k). \tag{2.6}$$

Lemma 2.1. *If $\{\widetilde{\varphi}^c_{jk}\}_{j=1,k=0}^{\infty,\infty} \cup \{\widetilde{\varphi}^s_{jk}\}_{j=1,k=1}^{\infty,\infty}$ is a CONS for $L^2_\rho(\widetilde{\Omega})$ defined by (2.2), setting*

$$\tau_n(v) = \sum_{j=1}^n \widehat{v}^c(j,0)\widetilde{\varphi}^c_{j0} + \sum_{j=1}^n\sum_{k=1}^n [\widehat{v}^c(j,k)\widetilde{\varphi}^c_{jk} + \widehat{v}^s(j,k)\widetilde{\varphi}^s_{jk}], \tag{2.7}$$

we have

$$\lim_{n\to\infty} \|\tau_n(v) - v\|_{\widetilde{H}} = 0, \quad \text{for all } v\in\widetilde{H}. \tag{2.8}$$

Lemma 2.2. (i) *If $v\in\widetilde{H}$, then*

$$\begin{aligned}&\mathcal{L}_1(v,v) + \|D_t v\|^2_\rho\\ &= \sum_{j=1}^\infty |\widehat{v}^c(j,0)|^2(\lambda_j+1) + \sum_{j=1}^\infty\sum_{k=1}^\infty [|\widehat{v}^c(j,k)|^2 + |\widehat{v}^s(j,k)|^2](\lambda_j+1+k^2).\end{aligned} \tag{2.9}$$

(ii) *If $v\in L^2_\rho(\widetilde{\Omega})$ and $\mathcal{L}_1(v,v)+\|D_t v\|^2_\rho<\infty$, then $v\in\widetilde{H}$.*

Lemma 2.3. *Let $\widetilde{\Omega},\rho,p,q,$ and $\mathcal{L}$ be as in the hypothesis of Theorem 1.5 and assume that (Ω,Γ) is a $V_L(\Omega,\Gamma)$. Then $\widetilde{H}$ is compactly imbedded in $L^2_\rho(\widetilde{\Omega})$.*

The proofs of Lemmas 2.1–2.3, can be found in [9]. We define

$$S_n = \Big\{v\in\widetilde{H} : v = \sum_{j=1}^n \eta^c_{j0}\widetilde{\varphi}^c_{j0} + \sum_{j=1}^n\sum_{k=1}^n \eta^c_{jk}\widetilde{\varphi}^c_{jk} + \eta^s_{jk}\widetilde{\varphi}^s_{jk},\ \eta^c_{jk},\eta^s_{jk}\in\mathbb{R}\Big\}. \tag{2.10}$$

Remark 2.4. *If $u_n \in S_n$, then $\widetilde{\mathcal{M}}(u_n, D_t u_n) = 0$.*

3. Proof of main results

In this section, we will give the proof of Theorem 1.5. To do this, we divide the proof into three parts. In part 1, we construct approximation solutions in a finite dimension space S_n. That is,

Lemma 3.1. *Assume that all the conditions in the hypothesis of Theorem 1.5 hold except for* (F3). *Let S_n be the subspace of $\widetilde{H}$ defined by* (2.10). *Taking $n_0 = j_0 + j_1$, then for $n \geq n_0$, there is a $u_n \in S_n$ with the property that*

$$\begin{aligned}&\langle D_t u_n, v\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, v)\\&= (\lambda_{j_0} + \gamma n^{-1})\langle u_n, v\rangle_\rho^\sim + (1 - n^{-1})\langle f(x, u_n) + g(x, t, u_n), v\rangle_\rho^\sim - G(v), \quad \forall v \in S_n.\end{aligned} \tag{3.1}$$

Proof. First observe that from (2.10),

$$\begin{gathered}(1) \quad v \in S_n \Rightarrow D_t v \in S_n,\\(2) \quad \langle D_t(\alpha\widetilde{\varphi}_{jk}^c + \beta\widetilde{\varphi}_{jk}^s), \alpha\widetilde{\varphi}_{jk}^c + \beta\widetilde{\varphi}_{jk}^s\rangle_\rho^\sim = 0, \quad \text{for } j, k \geq 1, \alpha, \beta \in \mathbb{R}.\end{gathered} \tag{3.2}$$

Let $\{\psi_i\}_{i=1}^{2n^2+n}$ be an enumeration of $\{\widetilde{\varphi}_{jk}^c\}_{j=1,k=0}^{n,n} \cup \{\widetilde{\varphi}_{jk}^s\}_{j=1,k=1}^{n,n}$, and set

$$n^* = (j_0 + j_1 - 1)(2n + 1). \tag{3.3}$$

So $\{\psi_i\}_{i=1}^{n^*}$ is an enumeration of $\{\widetilde{\varphi}_{jk}^c\}_{j=1,k=0}^{j_0+j_1-1,n} \cup \{\widetilde{\varphi}_{jk}^s\}_{j=1,k=1}^{j_0+j_1-1,n}$, where $n \geq n_0$.

For $\alpha = (\alpha_1, \dots, \alpha_{2n^2+n})$, setting

$$u = \sum_{i=1}^{2n^2+n} \alpha_i \psi_i, \quad \widetilde{u} = \sum_{i=1}^{2n^2+n} \delta_i \alpha_i \psi_i, \tag{3.4}$$

where

$$\delta_i = \begin{cases}-1, & 1 \leq i \leq n^*,\\ 1, & n^* + 1 \leq i \leq 2n^2 + n,\end{cases} \tag{3.5}$$

we define

$$\begin{aligned}F_i(\alpha) = &\langle D_t u, \delta_i\psi_i\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u, \delta_i\psi_i) - (\lambda_{j_0} + \gamma n^{-1})\langle u, \delta_i\psi_i\rangle_\rho^\sim\\&- (1 - n^{-1})\langle f(x, u) + g(x, t, u), \delta_i\psi_i\rangle_\rho^\sim + G(\delta_i\psi_i).\end{aligned} \tag{3.6}$$

It is clear from the orthogonality that $\langle D_t u, \widetilde{u}\rangle_\rho^\sim = 0$. From (3.4) and (3.6), we obtain

$$\begin{aligned}\sum_{i=1}^{2n^2+n} F_i(\alpha)\alpha_i = &\widetilde{\mathcal{M}}(u, \widetilde{u}) - (\lambda_{j_0} + \gamma n^{-1})\langle u, \widetilde{u}\rangle_\rho^\sim\\&- (1 - n^{-1})\langle f(x, u) + g(x, t, u), \widetilde{u}\rangle_\rho^\sim + G(\widetilde{u}).\end{aligned} \tag{3.7}$$

From (3.3)–(3.5), we have

$$\begin{gathered}
\widetilde{u}=\sum_{j=1}^{n}\delta_j\widehat{u}^c(j,0)\widetilde{\varphi}^c_{j0}+\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_j[\widehat{u}^c(j,k)\widetilde{\varphi}^c_{jk}+\widehat{u}^s(j,k)\widetilde{\varphi}^s_{jk}],\\
u=\sum_{j=1}^{n}\widehat{u}^c(j,0)\widetilde{\varphi}^c_{j0}+\sum_{j=1}^{n}\sum_{k=1}^{n}[\widehat{u}^c(j,k)\widetilde{\varphi}^c_{jk}+\widehat{u}^s(j,k)\widetilde{\varphi}^s_{jk}], \qquad (3.8)\\
\delta_j=\begin{cases}-1, & 1\le j\le j_0+j_1-1,\\ 1, & j_0+j_1\le j\le n.\end{cases}
\end{gathered}$$

Consequently, we obtain

$$\widetilde{L}(u,\widetilde{u})=\sum_{j=1}^{n}\delta_j\lambda_j|\widehat{u}^c(j,0)|^2+\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\delta_j[|\widehat{u}^c(j,k)|^2+|\widehat{u}^s(j,k)|^2].$$

Adding and subtracting $-\gamma\langle u,\widetilde{u}\rangle^{\sim}_{\rho}+\widetilde{\mathcal{L}}(u,\widetilde{u})$ to the right-hand side of (3.7), we see that

$$\begin{aligned}
\sum_{i=1}^{2n^2+n}F_i(\alpha)\alpha_i&=\sum_{j=1}^{n}\delta_j(\lambda_j-\lambda_{j_0}-\gamma)|\widehat{u}^c(j,0)|^2\\
&\quad+\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_j(\lambda_j-\lambda_{j_0}-\gamma)[|\widehat{u}^c(j,k)|^2+|\widehat{u}^s(j,k)|^2] \qquad (3.9)\\
&\quad-(1-n^{-1})\langle f(x,u)-\gamma u,\widetilde{u}\rangle^{\sim}_{\rho}-(1-n^{-1})\langle g(x,t,u),\widetilde{u}\rangle^{\sim}_{\rho}\\
&\quad+G(\widetilde{u})+\widetilde{\mathcal{M}}(u,\widetilde{u})-\widetilde{\mathcal{L}}(u,\widetilde{u}).
\end{aligned}$$

By (3.8), it is obvious that $\delta_j(\lambda_j-\lambda_{j_0}-\gamma)\ge\gamma$, for $j=1,\dots,n$. From (F2), (2.1) and (3.9) there exists is a $K>0$, such that

$$\begin{aligned}
\sum_{i=1}^{2n^2+n}F_i(\alpha)\alpha_i&\ge\gamma n^{-1}\|u\|^2_{\rho}-(1-n^{-1})\langle g(x,t,u),\widetilde{u}\rangle^{\sim}_{\rho}\\
&\quad-K\|u\|_{\rho}+G(\widetilde{u})+\widetilde{\mathcal{M}}(u,\widetilde{u})-\widetilde{\mathcal{L}}(u,\widetilde{u}).
\end{aligned} \qquad (3.10)$$

Now from (G2), it follows that

$$\lim_{\|u\|_{\rho}\to\infty}|\langle g(x,t,u),\widetilde{u}\rangle^{\sim}_{\rho}|/\|u\|^2_{\rho}=0. \qquad (3.11)$$

For a fixed n , it follows from (2.4), (2.9), and (3.8) that there is a $K>0$, such that

$$\|u\|_{\widetilde{H}}\le K\|u\|_{\rho},\quad u\in S_n, \qquad (3.12)$$

and since $\widetilde{\mathcal{M}}$ is $\#\widetilde{H}$- related to $\widetilde{\mathcal{L}}$,

$$\lim_{\|u\|_{\rho}\to\infty}|\widetilde{\mathcal{M}}(u,\widetilde{u})-\widetilde{\mathcal{L}}(u,\widetilde{u})|/\|u\|^2_{\rho}=0,\quad u\in S_n. \qquad (3.13)$$

From $\|u\|^2_{\rho}=|\alpha|^2$ and $G\in(\widetilde{H})'$, we conclude from (3.10)–(3.13) that there is an $s_0>0$ such that

$$\sum_{i=1}^{2n^2+n}F_i(\alpha)\alpha_i\ge\frac{\gamma|\alpha|^2}{2n},\quad\text{for }|\alpha|>s_0. \qquad (3.14)$$

From the generalized Brouwer's theorem [4], there exists $\alpha^* = (\alpha_1^*, \dots, \alpha_{2n^2+n}^*)$ satisfying $F_i(\alpha^*) = 0$. Thus, setting $u_n = \sum_{i=1}^{2n^2+n} \alpha_i^* \psi_i$, and from (3.6), we have

$$\begin{aligned} &\langle D_t u_n, \psi_i\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, \psi_i) \\ &= (\lambda_{j_0} + \gamma n^{-1})\langle u_n, \psi_i\rangle_\rho^\sim + (1 - n^{-1})\langle f(x, u_n) + g(x, t, u_n), \psi_i\rangle_\rho^\sim - G(\psi_i), \end{aligned}$$

for $i = 1, \dots, 2n^2 + n$. The proof of Lemma 3.1 is completed by the definition of S_n. □

Lemma 3.2. *Assume that the conditions in Lemma 3.1 hold. If (F3) holds, then the sequence $\{u_n\}$ obtained in Lemma 3.1 is uniformly bounded in $\widetilde{H}$ with respect to the norm $\|u_n\|_{\widetilde{H}} = \langle u_n, u_n\rangle_{\widetilde{H}}^{1/2}$.*

Proof. We assume that $\lambda_{j_0+j_1}$ is an eigenvalue of $\mathcal{L}$ of multiplicity j_2. By Lemma 3.1, for $u_n \in S_n$, we have

$$\begin{aligned} &\langle D_t u_n, v\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, v) \\ &= (\lambda_{j_0} + \gamma n^{-1})\langle u_n, v\rangle_\rho^\sim + (1 - n^{-1})\langle f(x, u_n) + g(x, t, u_n), v\rangle_\rho^\sim - G(v), \end{aligned} \tag{3.15}$$

for all $v \in S_n$, $n \geq n_1 = j_0 + j_1 + j_2$. We claim that there is a constant K such that

$$\|u_n\|_{\widetilde{H}} \leq K, \quad \text{for all } n \geq n_1. \tag{3.16}$$

Suppose that (3.16) fails. For ease of notation and without loss of generality, we assume

$$\lim_{n\to\infty} \|u_n\|_{\widetilde{H}} = \infty. \tag{3.17}$$

Taking $v = u_n$ in (3.15), from (3.11), (3.2)(1), (F2), (G2), $G \in (\widetilde{H})'$ and Schwarz's inequality, there exists a $K > 0$ such that

$$\widetilde{\mathcal{M}}(u_n, u_n) \leq K\|u_n\|_\rho^2 + K\|u_n\|_\rho, \quad \text{for } n \geq n_1. \tag{3.18}$$

We observe from (3.2)(1) and Remark 2.4 that

$$\widetilde{\mathcal{M}}(u_n, D_t u_n) = 0, \quad \text{for } n \geq n_1. \tag{3.19}$$

Thus, replacing v by $D_t u_n$ in (3.15), from (F2), (G2) and $G \in (\widetilde{H})'$, there is a $K > 0$ such that

$$\|D_t u_n\|_\rho \leq K\|u_n\|_\rho + K. \tag{3.20}$$

From (1.8) and (S2), there is a $K > 0$ such that

$$K\Big(\sum_{i=1}^N \int_{\widetilde{\Omega}} p_i |D_i u_n|^2 + \int_{\widetilde{\Omega}} q u_n^2\Big) \leq \widetilde{\mathcal{M}}(u_n, u_n). \tag{3.21}$$

Therefore, from (1.7), (3.18), (3.20) and (3.21), it is easy to obtain from (3.17) that

$$\lim_{n\to\infty} \|u_n\|_\rho = \infty, \tag{3.22}$$

and there exist $n_2 > 0$ and $K > 0$ such that

$$\|u_n\|_{\widetilde{H}} \leq K\|u_n\|_\rho, \quad \text{for } n \geq n_2. \tag{3.23}$$

Set

$$u_n = u_{n1} + u_{n2} + u_{n3} + u_{n4}, \tag{3.24}$$

where

$$
\begin{aligned}
u_{n1} &= \sum_{j=1}^{j_0-1} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=1}^{j_0-1}\sum_{k=1}^{n}[\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s],\\
u_{n2} &= \sum_{j=j_0}^{j_0+j_1-1} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=j_0}^{j_0+j_1-1}\sum_{k=1}^{n}[\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s],\\
u_{n3} &= \sum_{j=j_0+j_1}^{j_0+j_1+j_2-1} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=j_0+j_1}^{j_0+j_1+j_2-1}\sum_{k=1}^{n}[\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s],\\
u_{n4} &= \sum_{j=j_0+j_1+j_2}^{n} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=j_0+j_1+j_2}^{n}\sum_{k=1}^{n}[\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s].
\end{aligned}
$$

Step 1: We claim that

$$
\begin{gathered}
(1) \quad \lim_{n\to\infty}(\|u_{n1}\|_\rho^2 + \|u_{n4}\|_\rho^2)/\|u_n\|_\rho^2 = 0,\\
(2) \quad \lim_{n\to\infty}(\|u_{n4}\|_{\widetilde{H}})/\|u_n\|_\rho = 0.
\end{gathered} \tag{3.25}
$$

Defining

$$
\widetilde{u}_n = -u_{n1} - u_{n2} + u_{n3} + u_{n4}, \tag{3.26}
$$

and from (3.2)(2), we have

$$
\langle D_t u_n, \widetilde{u}_n\rangle_\rho^\sim = 0. \tag{3.27}
$$

As a result, from (3.15) with $v = \widetilde{u}_n$, we obtain

$$
\begin{aligned}
&\widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - (\lambda_{j_0}+\gamma)\langle u_n,\widetilde{u}_n\rangle_\rho^\sim\\
&= (1-n^{-1})\langle f(x,u_n) - \gamma u_n, \widetilde{u}_n\rangle_\rho^\sim + \langle g(x,t,u_n),\widetilde{u}_n\rangle_\rho^\sim - G(\widetilde{u}_n)\\
&\quad + \widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - \widetilde{\mathcal{M}}(u_n,\widetilde{u}_n).
\end{aligned} \tag{3.28}
$$

Set

$$
\begin{gathered}
I = \widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - (\lambda_{j_0}+\gamma)\langle u_n,\widetilde{u}_n\rangle_\rho^\sim,\\
II = (1-n^{-1})\langle f(x,u_n)-\gamma u_n,\widetilde{u}_n\rangle_\rho^\sim + \langle g(x,t,u_n),\widetilde{u}_n\rangle_\rho^\sim\\
- G(\widetilde{u}_n) + \widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - \widetilde{\mathcal{M}}(u_n,\widetilde{u}_n).
\end{gathered}
$$

Now from (3.8) and $\lambda_{j_0}+\gamma = \lambda_{j_0+j_1}-\gamma$, we see that

$$
I \ge \gamma\|u_n\|_\rho^2 + I^*, \tag{3.29}
$$

where

$$
\begin{aligned}
I^* &= \sum_{j=1}^{j_0-1}(\lambda_{j_0}-\lambda_j)|\widehat{u}_n^c(j,0)|^2 + \sum_{j=1}^{j_0-1}(\lambda_{j_0}-\lambda_j)\sum_{k=1}^{n}[|\widehat{u}_n^c(j,k)|^2 + |\widehat{u}_n^s(j,k)|^2]\\
&\quad + \sum_{j=j_0+j_1+j_2}^{n}(\lambda_j - \lambda_{j_0+j_1})|\widehat{u}_n^c(j,0)|^2 + \sum_{j=j_0+j_1+j_2}^{n}(\lambda_j-\lambda_{j_0+j_1})\sum_{k=1}^{n}[|\widehat{u}_n^c(j,k)|^2\\
&\quad + |\widehat{u}_n^s(j,k)|^2].
\end{aligned}
$$

Hence, from (3.24), we see that

$$
I \ge \gamma\|u_n\|_\rho^2 + (\lambda_{j_0}-\lambda_{j_0-1})\|u_{n1}\|_\rho^2 + (\lambda_{j_0+j_1+j_2}-\lambda_{j_0+j_1})\|u_{n4}\|_\rho^2. \tag{3.30}
$$

On the other hand, for any $\varepsilon > 0$, from (F2), (G2) and $G \in (\widetilde{H})'$ we see that

$$\begin{aligned} II \le [(\gamma+\varepsilon)\|u_n\|_\rho + \|f_0\|_\rho + \|g_\varepsilon\|_\rho]\|\widetilde{u}_n\|_\rho \\ + K_0\|\widetilde{u}_n\|_{\widetilde{H}} + |\widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - \widetilde{\mathcal{M}}(u_n,\widetilde{u}_n)|, \end{aligned} \tag{3.31}$$

and $\widetilde{\mathcal{M}}$ being $\#\widetilde{H}$-related to $\widetilde{\mathcal{L}}$, we have

$$\lim_{n\to\infty} |\widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - \widetilde{\mathcal{M}}(u_n,\widetilde{u}_n)|/\|u_n\|_\rho^2 = 0. \tag{3.32}$$

Therefore, from (3.28) and (3.32), we see on dividing both (3.30) and (3.31) by $\|u_n\|_\rho^2$ and passing to the limit as $n \to \infty$, that

$$\lim_{n\to\infty} [(\lambda_{j_0} - \lambda_{j_0-1})\|u_{n1}\|_\rho^2 + (\lambda_{j_0+j_1+j_2} - \lambda_{j_0+j_1})\|u_{n4}\|_\rho^2]/\|u_n\|_\rho^2 = 0.$$

Since $\lambda_{j_0} - \lambda_{j_0-1} > 0$ and $\lambda_{j_0+j_1+j_2} - \lambda_{j_0+j_1} > 0$, we see that claim (1) in (3.25) is true.

Set $\delta = 1 - \lambda_{j_0+j_1}/\lambda_{j_0+j_1+j_2}$. It implies

$$\lambda_j - \lambda_{j_0+j_1} \ge \delta\lambda_j, \quad \text{for } j \ge j_0 + j_1 + j_2. \tag{3.33}$$

Then, (3.24) and (3.33) give

$$I^* \ge \delta\widetilde{\mathcal{L}}(u_{n4}, u_{n4}). \tag{3.34}$$

Hence, from (3.28)-(3.30), (3.34), we obtain

$$\begin{aligned} \gamma\|u_n\|_\rho^2 + \delta\widetilde{\mathcal{L}}(u_{n4},u_{n4}) \le |\widetilde{\mathcal{L}}(u_n,\widetilde{u}_n) - \widetilde{\mathcal{M}}(u_n,\widetilde{u}_n)| + [(\gamma+\varepsilon)\|u_n\|_\rho + \|f_0\|_\rho \\ + \|g_\varepsilon\|_\rho]\|\widetilde{u}_n\|_\rho + K_0\|\widetilde{u}_n\|_{\widetilde{H}}, \end{aligned}$$

and on dividing by $\|u_n\|_\rho^2$ on both sides and letting $n \to \infty$, we see that

$$\lim_{n\to\infty} \widetilde{\mathcal{L}}(u_{n4}, u_{n4})/\|u_n\|_\rho^2 = 0. \tag{3.35}$$

Since

$$\langle f(x,v), D_t v\rangle_\rho^\sim = \int_{\widetilde{\Omega}} f(x,v(x,t))D_t v(x,t)\rho\,dt\,dx = 0, \tag{3.36}$$

for $v \in \widetilde{C}^{1b}_{p,\rho}$, we conclude from (3.36), (F2) and the definition of S_n that

$$\langle f(x,u_n), D_t u_n\rangle_\rho^\sim = 0, \quad \text{for } u_n \in S_n,\ n \ge n_2.$$

Hence, replacing v by $D_t u_n$ in (3.15) and from Remark 1.4, Schwarz's inequality and $G \in (\widetilde{H})'$, we obtain $\|D_t u_n\|_\rho \le \|g(x,t,u_n)\|_\rho + K$ and

$$\lim_{n\to\infty} \|D_t u_n\|_\rho^2/\|u_n\|_\rho^2 = 0. \tag{3.37}$$

From claim (3.25)(1), (2.4), (3.35), (3.37) and $\|D_t u_{n4}\|_\rho^2 \le \|D_t u_n\|_\rho^2$, claim (3.25)(2) is true.

Step 2: We show that $W(x) = W_{(2)}(x) + W_{(3)}(x)$. Setting

$$W_n(x) = u_n/\|u_n\|_\rho, \quad W_{ni}(x) = u_{ni}/\|u_n\|_\rho, \quad \text{for } i = 1,\dots,4, \tag{3.38}$$

from (3.23) there is a K such that

$$\|W_n\|_{\widetilde{H}} \le K \quad \text{and} \quad \|W_{ni}\|_{\widetilde{H}} \le K, \tag{3.39}$$

for $i = 1, \dots, 4$, and $n \ge n_2$. From (3.39) and Lemma 3.1, we obtain that there is a $W \in \widetilde{H}$ such that

$$\begin{gathered} (1) \quad \lim_{n\to\infty} \|W_n - W\|_\rho = 0, \\ (2) \quad \lim_{n\to\infty} W_n(x,t) = W(x,t), \quad \text{a.e. in } \overline{\Omega}, \\ (3) \quad \lim_{n\to\infty} \langle W_n, v\rangle_{\widetilde{H}} = \langle W, v\rangle_{\widetilde{H}}, \quad \text{for } v \in \widetilde{H}. \end{gathered} \tag{3.40}$$

Since $\widetilde{\mathcal{M}}$ is $\#\widetilde{H}$-related to $\widetilde{\mathcal{L}}$, we obtain from (3.39) that

$$\lim_{n\to\infty} |\widetilde{\mathcal{L}}(u_n, W_{ni}(x)) - \widetilde{\mathcal{M}}(u_n, W_{ni}(x))|/\|u_n\|_\rho = 0, \text{for } i = 1, \dots, 4. \tag{3.41}$$

We observe from (3.25) that $\lim_{n\to\infty} \|W_{n4}(x)\|_\rho = 0$. Hence, if $n \to \infty$, then

$$\langle W_n, \widetilde{\varphi}^c_{jk}\rangle^\sim_\rho = \langle W_{n4}, \varphi^c_{jk}\rangle^\sim_\rho \to 0, \quad \text{for } j \ge j_0 + j_1 + j_2,$$

and from (3.40)(3), we obtain $\widehat{W}^c(j,k) = 0$, for $j \ge j_0 + j_1 + j_2$ and all k. In similar way, we have $\widehat{W}^s(j,k) = 0$, for $j \ge j_0 + j_1 + j_2$ and all k. Also, we observe from (3.25) that $\lim_{n\to\infty} \|W_{n1}(x)\|_\rho = 0$. So we obtain $\widehat{W}^c(j,k) = 0$ and $\widehat{W}^s(j,k) = 0$ for $1 \le j \le j_0 - 1$ and all k. Therefore, we have

$$\widehat{W}^c(j,k) = 0 \quad \text{and} \quad \widehat{W}^s(j,k) = 0, \quad \text{for } j \ge j_0 + j_1 + j_2 \text{ and all } k, \tag{3.42}$$

$$\widehat{W}^c(j,k) = 0 \quad \text{and} \quad \widehat{W}^s(j,k) = 0, \quad \text{for } 1 \le j \le j_0 - 1 \text{ and all } k. \tag{3.43}$$

Next, for $k \ge 1$ and $j_0 \le j \le j_0 + j_1 + j_2 - 1$, from (2.2) and (3.37) , we have

$$k\widehat{W}^c(j,k) = -\lim_{n\to\infty} \int_{\widetilde{\Omega}} D_t W_n(x,t) \widetilde{\varphi}^s_{jk}(x,t)\rho(x)\, dx\, dt = 0.$$

A similar situation prevails for $k\widehat{W}^s(j,k)$. So we have

$$\widehat{W}^c(j,k) = 0 \quad \text{and} \quad \widehat{W}^s(j,k) = 0,$$

for $k \ge 1$ and $j_0 \le j \le j_0 + j_1 + j_2 - 1$. Hence, from (3.42), (3.43) and the above formula, we see that $W(x,t)$ is a function unrelated to t; i.e.,

$$\begin{gathered} W(x,t) \equiv W(x), \\ W(x) = W_{(2)}(x) + W_{(3)}(x), \\ W_{(2)}(x) = \sum_{j=j_0}^{j_0+j_1-1} \widehat{W}^c(j,0)\widetilde{\varphi}^c_{j0}, \\ W_{(3)}(x) = \sum_{j=j_0+j_1}^{j_0+j_1+j_2-1} \widehat{W}^c(j,0)\widetilde{\varphi}^c_{j0}. \end{gathered} \tag{3.44}$$

Step 3: We show that $\langle f^*, W_{(2)}\rangle^\sim_\rho = 0$ and $\langle f^*, W_{(3)}\rangle^\sim_\rho = (\lambda_{j_0+j_1} - \lambda_{j_0})\|W_{(3)}\|^2_\rho$. From (3.38) and orthogonality we observe that

$$\|W_n - W\|^2_\rho = \|W_{n1}\|^2_\rho + \|W_{n2} - W_{(2)}\|^2_\rho + \|W_{n3} - W_{(3)}\|^2_\rho + \|W_{n4}\|^2_\rho.$$

From (3.40)(1), we conclude that

$$\lim_{n\to\infty} \|W_{n2} - W_{(2)}\|^2_\rho = 0 \quad \text{and} \quad \lim_{n\to\infty} \|W_{n3} - W_{(3)}\|^2_\rho = 0. \tag{3.45}$$

Putting W_{ni} in place of v in (3.15), we obtain

$$\begin{aligned} &\widetilde{\mathcal{L}}(u_n, W_{ni}) \\ &= (\lambda_{j_0} + \gamma n^{-1})\langle u_n, W_{ni}\rangle_\rho^\sim + (1 - n^{-1})\langle f(x, u_n) + g(x, t, u_n), W_{ni}\rangle_\rho^\sim \\ &\quad - G(W_{ni}) + \widetilde{\mathcal{L}}(u_n, W_{ni}) - \widetilde{\mathcal{M}}(u_n, W_{ni}), \quad i = 1, 2. \end{aligned} \tag{3.46}$$

Dividing by $\|u_n\|_\rho^2$ on both sides of (3.46) and letting $n \to \infty$, from (3.24), (3.38), (3.40), (3.41), (3.45), Schwarz's inequality, $G \in (\widetilde{H})'$ and (G2) we obtain

$$\lim_{n\to\infty} \langle f(x, u_n), W_{n3}\rangle_\rho^\sim / \|u_n\|_\rho = (\lambda_{j_0+j_1} - \lambda_{j_0})\|W_{(3)}\|_\rho^2. \tag{3.47}$$

In a similar way, from (3.45), we have

$$\lim_{n\to\infty} \langle f(x, u_n), W_{n2}\rangle_\rho^\sim / \|u_n\|_\rho = 0. \tag{3.48}$$

Next, from (F2) and (3.22) that there are K and n_3 such that

$$\|f(x, u_n)\|_\rho / \|u_n\|_\rho \le K, \quad \text{for } n \ge n_3, \tag{3.49}$$

where $n_3 \ge n_2$. Using the Banach-Saks theorem and other facts about Hilbert spaces (see [2, p. 181]), we obtain that there exits $f^*(x, t) \in \widetilde{L}_\rho^2$ such that

$$\begin{gathered} (1) \quad \lim_{n\to\infty} \langle \frac{f(x, u_n)}{\|u_n\|_\rho}, v\rangle_\rho^\sim = \langle f^*, v\rangle_\rho^\sim, \quad \forall v \in \widetilde{L}_\rho^2; \\ (2) \quad \lim_{n\to\infty} \|\frac{1}{n}\sum_{k=n_2}^{n_2+n} \frac{f(x, u_k)}{\|u_k\|_\rho} - f^*\|_\rho = 0; \\ (3) \quad \text{there is } \{n_j\} \subset \{n\}, \text{ such that} \\ \lim_{j\to\infty} \frac{1}{n_j}\sum_{k=n_2}^{n_2+n_j} \frac{f(x, u_k)}{\|u_k\|_\rho} = f^*(x, t), \quad \text{a.e. in } \widetilde{\Omega}. \end{gathered} \tag{3.50}$$

From (3.45), (3.47), (3.49) and (3.50)(2) we obtain

$$\langle f^*, W_{(3)}\rangle_\rho^\sim = (\lambda_{j_0+j_1} - \lambda_{j_0})\|W_{(3)}\|_\rho^2. \tag{3.51}$$

In a similar manner, from (3.48) we obtain

$$\langle f^*, W_{(2)}\rangle_\rho^\sim = 0. \tag{3.52}$$

Step 4: We show that $\langle f^*, W_{(2)}\rangle_\rho^\sim > 0$ and $\langle f^*, W_{(3)}\rangle_\rho^\sim < (\lambda_{j_0+j_1} - \lambda_{j_0})\|W_{(3)}\|_\rho^2$ under assumption (3.17).

From (3.38), (F2), (3.40)(2) and (3.50)(3) it follows that

$$f^*(x, t) = 0, \quad \text{a.e. in } \widetilde{\Omega}_0 = \Omega_0 \times T, \tag{3.53}$$

where $\Omega_0 = \{x \in \Omega : W(x) = 0\}$. From (F2), for $s \neq 0$, a.e. $x \in \Omega$, we have

$$-\frac{f_0(x)}{|s|} \le \frac{f(x, s)}{s} \le 2\gamma + \frac{f_0(x)}{|s|},$$

and for a.e. $x \in \Omega$, from (1.14) we see that

$$0 \le \mathcal{F}_+(x) \le \mathcal{F}^+(x) \le 2\gamma \quad \text{and} \quad 0 \le \mathcal{F}_-(x) \le \mathcal{F}^-(x) \le 2\gamma. \tag{3.54}$$

Setting

$$\widetilde{\Omega}_+ = \Omega_+ \times T \quad \text{and} \quad \widetilde{\Omega}_- = \Omega_- \times T, \tag{3.55}$$

where $\Omega_+ = \{x \in \Omega : W(x) > 0\}$ and $\Omega_- = \{x \in \Omega : W(x) < 0\}$, let $(x_0, t_0) \in \widetilde{\Omega}_+$ be such that $f^*(x_0, t_0)$ is finite, (3.40)(2) and (3.50)(3) hold, and x_0 be a value such that (3.54) holds. Then given $\varepsilon > 0$, we see that there is an $s^* > 0$ such that $f(x_0, s) \le \mathcal{F}^+(x_0)s + \varepsilon s$ for $s \ge s^*$. Since $u_n(x_0, t_0) = \|u_n\|_\rho W_n(x_0, t_0)$, from (3.22), (3.40)(2) and (3.50)(3), we obtain $f^*(x_0, t_0) \le \mathcal{F}^+(x_0)W(x_0)$. Similarly, we have $\mathcal{F}_+(x_0)W(x_0) \le f^*(x_0, t_0)$. Also, we can prevail for $\widetilde{\Omega}_-$. Hence, we conclude that

$$\begin{aligned} &(1)\quad \mathcal{F}_+(x)W(x) \le f^*(x,t) \le \mathcal{F}^+(x)W(x), \quad \text{a.e. in } \widetilde{\Omega}_+;\\ &(2)\quad \mathcal{F}^-(x)W(x) \le f^*(x,t) \le \mathcal{F}_-(x)W(x), \quad \text{a.e. in } \widetilde{\Omega}_-. \end{aligned} \tag{3.56}$$

Since $\widetilde{\Omega} = \widetilde{\Omega}_0 \cup \widetilde{\Omega}_+ \cup \widetilde{\Omega}_-$, we define

$$f^{**}(x,t) = \begin{cases} 0, & (x,t) \in \widetilde{\Omega}_0,\\ f^*(x,t)/W(x), & (x,t) \in \widetilde{\Omega}_+ \cap \widetilde{\Omega}_-. \end{cases}$$

From (3.53)–(3.56), we have

$$f^*(x,t) = f^{**}(x,t)W(x), \quad \text{a.e. in } \widetilde{\Omega}, \tag{3.57}$$

$$0 \le f^{**}(x,t) \le 2\gamma, \quad \text{a.e. in } \widetilde{\Omega}. \tag{3.58}$$

Furthermore, from (3.44), (3.51), (3.52), (3.57) and (3.58) we see that

$$\langle (\lambda_{j_0+j_1} - \lambda_{j_0} - f^{**})W_{(3)}, W_{(3)}\rangle_\rho^\sim + \langle f^{**}W_{(2)}, W_{(2)}\rangle_\rho^\sim = 0,$$

and

$$\begin{gathered} \langle (\lambda_{j_0+j_1} - \lambda_{j_0} - f^{**})W_{(3)}, W_{(3)}\rangle_\rho^\sim = 0,\\ \langle f^{**}W_{(2)}, W_{(2)}\rangle_\rho^\sim = 0. \end{gathered} \tag{3.59}$$

Setting

$$\widetilde{\Omega}_2 = \Omega_2 \times T \quad \text{and} \quad \widetilde{\Omega}_3 = \Omega_3 \times T, \tag{3.60}$$

where $\Omega_i = \{x \in \Omega : W_{(i)}(x) \ne 0\}$, $i = 2, 3$, from (3.58) and (3.59), we see that $f^{**}(x,t) = \lambda_{j_0+j_1} - \lambda_{j_0}$, a.e. in $\widetilde{\Omega}_3$. Then, $\widetilde{\Omega}_2 \cap \widetilde{\Omega}_3$ is a set of Lebesgue measure zero. Also, both $W_{(2)}$ and $W_{(3)}$ are continuous functions in Ω by (3.44). Therefore, both Ω_2 and Ω_3 are open sets, and we see that Ω_2 and Ω_3 are disjoint sets. Since $W = W_{(2)} + W_{(3)}$, we find

$$W = W_{(2)} \text{ on } \Omega_2, \quad W = W_{(3)} \text{ on } \Omega_3. \tag{3.61}$$

Defining

$$\widetilde{\Omega}_{i+} = \Omega_{i+} \times T \quad \text{and} \quad \widetilde{\Omega}_{i-} = \Omega_{i-} \times T, \tag{3.62}$$

where $\Omega_{i+} = \{x \in \Omega : W_{(i)}(x) > 0\}$, $\Omega_{i-} = \{x \in \Omega : W_{(i)}(x) < 0\}$, $i = 2, 3$, from (3.44) we see that $W_{(2)}(x)$ is a λ_{j_0}-eigenfuction for $\mathcal{L}$. If $W_{(2)}(x)$ is nontrivial, then from (1.16) we have

$$0 < \int_{\widetilde{\Omega}_{2+}} \mathcal{F}_+(x)W(x)W_{(2)}(x)\rho(x)\,dx\,dt + \int_{\widetilde{\Omega}_{2-}} \mathcal{F}_-(x)W(x)W_{(2)}(x)\rho(x)\,dx\,dt. \tag{3.63}$$

If $(x,t) \in \widetilde{\Omega}_{2+}$, from (3.61) and (3.62), then $(x,t) \in \widetilde{\Omega}_+$. So, from (3.56) and (3.63) we obtain

$$0 < \int_{\widetilde{\Omega}_{2+}} f^*(x,t)W_{(2)}(x)\rho(x)\,dx\,dt + \int_{\widetilde{\Omega}_{2-}} f^*(x,t)W_{(2)}(x)\rho(x)\,dx\,dt;$$

i.e., $\langle f^*, W_{(2)}(x)\rangle_\rho^\sim > 0$. It is a contradiction of (3.52). As a result, we conclude that $W_{(2)}(x)$ is indeed trivial; i.e., $W_{(2)}(x) = 0$, for all $x \in \Omega$. Hence, from (3.44) we obtain

$$W(x) = W_{(3)}(x), \quad \text{for all } x \in \Omega. \tag{3.64}$$

Since $W_{(3)}(x)$ is a $\lambda_{j_0+j_1}$-eigenfuction for $\mathcal{L}$, so is W from (3.64). If W is a nontrivial function, then from (1.15) and (3.55) we obtain

$$2\gamma\|W\|_\rho^2 > \int_{\widetilde{\Omega}_+} \mathcal{F}^+(x)W^2(x)\rho(x)\,dx\,dt + \int_{\widetilde{\Omega}_-} \mathcal{F}^-(x)W^2(x)\rho(x)\,dx\,dt. \tag{3.65}$$

Therefore, we obtain from (3.56)(1)(2) and (3.65) that

$$2\gamma\|W\|_\rho^2 > \int_{\widetilde{\Omega}_+} f^*(x,t)W(x)\rho(x)\,dx\,dt + \int_{\widetilde{\Omega}_-} f^*(x,t)W(x)\rho(x)\,dx\,dt; \tag{3.66}$$

i.e., $\langle f^*, W\rangle_\rho^\sim < (\lambda_{j_0+j_1} - \lambda_{j_0})\|W\|_\rho^2$. It is a direct contradiction of (3.51). So we conclude that $W(x) = 0$, for all $x \in \Omega$. Next, from (3.40)(1), we obtain $\lim_{n\to\infty} \|W_n\|_\rho = 0$. However, from (3.38) we see that $\lim_{n\to\infty} \|W_n\|_\rho = 1$. Obviously, it is a contradiction, and (3.16) is indeed true. □

Proof of Theorem 1.5. Since $\widetilde{H}$ is a separable Hilbert space, from (3.16), (S1) and Lemma 3.2, there is a subsequence (still denoted by $\{u_n\}_{n=n_3}^\infty$ and a function $u^* \in \widetilde{H}$) such that

$$\lim_{n\to\infty} \|u_n - u^*\|_\rho = 0;$$

there exists $W^*(x,t) \in \widetilde{L}_\rho^2$, such that $|u_n(x,t)| \le W^*(x,t)$, a.e. $(x,t) \in \widetilde{\Omega}$, $n \ge n_3$;

$$\begin{gathered} (1) \quad \lim_{n\to\infty} u_n(x,t) = u^*(x,t), \quad \text{a.e. } (x,t) \in \widetilde{\Omega}; \\ (2) \quad \lim_{n\to\infty} \langle D_i u_n, v\rangle_{p_i}^\sim = \langle D_i u^*, v\rangle_{p_i}^\sim, \quad \forall v \in \widetilde{L}_{p_i}^2,\ i = 1,\dots,N; \\ (3) \quad \lim_{n\to\infty} \langle D_t u_n, v\rangle_\rho^\sim = \langle D_t u^*, v\rangle_\rho^\sim, \quad \forall v \in \widetilde{L}_\rho^2; \\ (4) \quad \lim_{n\to\infty} \sigma_i(u_n) = \sigma_i(u^*), \quad i = 0,1,\dots,N. \end{gathered} \tag{3.67}$$

Let $v \in \widetilde{H}$ and $\tau_J(v)$ be defined by (2.7). Then $\tau_J(v) \in S_J(J \ge n_3)$ and from (3.15), for $n \ge J$, we have that

$$\begin{aligned} &\langle D_t u_n, \tau_J(v)\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, \tau_J(v)) \\ &= (\lambda_{j_0} + \gamma n^{-1})\langle u_n, \tau_J(v)\rangle_\rho^\sim + (1 - n^{-1})\langle f(x,u_n) + g(x,t,u_n), \tau_J(v)\rangle_\rho^\sim \\ &\quad - G(\tau_J(v)). \end{aligned} \tag{3.68}$$

We conclude from (1.4) and (3.67) that

$$\lim_{n\to\infty} \widetilde{\mathcal{M}}(u_n, \tau_J(v)) = \widetilde{\mathcal{M}}(u^*, \tau_J(v)). \tag{3.69}$$

Next, from (F2), (G2), (3.67)(2) and the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty} \langle f(x,u_n) + g(x,t,u_n), \tau_J(v)\rangle_\rho^\sim = \langle f(x,u^*) + g(x,t,u^*), \tau_J(v)\rangle_\rho^\sim. \tag{3.70}$$

From (3.69), (3.70), (3.67)(1)(3), and (3.68), we obtain

$$\begin{aligned} &\langle D_t u^*, \tau_J(v)\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u^*, \tau_J(v)) \\ &= \lambda_{j_0}\langle u^*, \tau_J(v)\rangle_\rho^\sim + \langle f(x,u^*) + g(x,t,u^*), \tau_J(v)\rangle_\rho^\sim - G(\tau_J(v)). \end{aligned} \tag{3.71}$$

Passing to the limit as $J \to \infty$ on both sides of (3.71), we obtain

$$\langle D_t u^*, v\rangle_\rho^\sim + \widetilde{\mathcal{M}}(u^*, v) = \lambda_{j_0}\langle u^*, v\rangle_\rho^\sim + \langle f(x, u^*) + g(x, t, u^*), v\rangle_\rho^\sim - G(v),$$

for all $v \in \widetilde{H}$, and the proof of Theorem 1.5 is complete. □

4. An example

We give two functions to establish existence results for a function $f(x, s)$ that satisfies (F1), (F2) and a function $g(x, t, s)$ that satisfies (G1), (G2). Set $\Omega = \{x = (x_1, x_2) : x_1^2 + x_2^2 < 1\}$ and

$$f_*(x) = (x_1^2 + x_2^2)^{-\rho}, \quad 0 < \rho < 1/4, \quad x \in \Omega.$$

Also, $\gamma > 0$ is given, and set

$$f(x, s) = \begin{cases} -s^2 f_*(x) + \gamma s, & 0 \le s < 1, \\ -\sqrt{s} f_*(x) + \gamma s, & 1 \le s < +\infty, \end{cases}$$

for $x \in \Omega$ and $0 \le s \le +\infty$. For $-\infty < s < 0$, we set $f(x, s) = -f(x, -s)$. Clearly, $f(x, s)$ meets (F1), (F2). For $g(x, t, s)$, set $\Omega = \{x = (x_1, x_2) : x_1^2 + x_2^2 < 1\}$, $T = (-\pi, \pi)$, and

$$g_0(x, t) = |t|(x_1^2 + x_2^2)^{-\rho}, \quad 0 < \rho < 1/4, \quad (x, t) \in \widetilde{\Omega}.$$

Also, we set

$$g(x, t, s) = \begin{cases} -s^2 g_0(x, t), & 0 \le s < 1, \\ -\sqrt{s} g_0(x, t), & 1 \le s < +\infty, \end{cases}$$

for $(x, t) \in \widetilde{\Omega}$ and $0 \le s < +\infty$. For $-\infty < s < 0$, we set $g(x, t, s) = -g(x, t, -s)$. Clearly, $g(x, t, s)$ satisfies (G1)–(G2).

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