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RATE OF DECAY FOR SOLUTIONS OF VISCOELASTIC EVOLUTION EQUATIONS

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Abstract. In this article we consider a Cauchy problem of a nonlinear viscoelastic equation of order four. Under suitable conditions on the initial data and the relaxation function, we prove polynomial and logarithmic decay of solutions.

1. Introduction

In this work, we are concerned with the Cauchy problem

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t - \Delta u_{tt} = \operatorname{div}\varphi(\nabla u), \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where u_0, u_1 are initial data and g is the relaxation function subjected to some conditions to be specified later. The nonlinear function φ is a conservative vector field on $\mathbb{R}^n$. It is the gradient of some scalar function (potential) G. This type of a nonlinear evolution equations of fourth order arises in the study of strain solitary waves.

In a nonlinear elastic rods the longitudinal wave equation takes the form

$$u_{tt} - [b_0 + b_1 n(u_x)^{n-1}]u_{xx} - b_2 u_{xxtt} = 0, \tag{1.2}$$

where $b_0, b_2 > 0$ are constants, b_1 is arbitrary real, n is a natural number (see [24, 25]). In one-dimension ($n = 1$) and for a nonlinear function φ, equation (1.2) takes the form

$$u_{tt} - \alpha u_{xx} - \beta u_{xxtt} = \varphi(u_x)_x. \tag{1.3}$$

Considering an additional damping of the form $a_2 u_{xxt}$, equation (1.3) takes the form

$$u_{tt} - a_1 u_{xx} - a_2 u_{xxt} - a_3 u_{xxtt} = \varphi(u_x)_x. \tag{1.4}$$

In 1872, Boussinesq described shallow-water waves and derived the equation

$$u_{tt} = u_{xx} + u_{xxxx} + (u^2)_{xx}. \tag{1.5}$$

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This equation was improved and took many different forms. While the propagation of longitudinal deformation waves in an elastic rod is modelled by the nonlinear partial differential equation

$$u_{tt} - u_{xxtt} - u_{xx} - \frac{1}{p}(u^p)_{xx} = 0,$$

with $p = 3$ or 5, this equation is called the nonlinear Pochhammer-Chree equation (see [6, 16]). The general class (Cauchy problem) for this type of problems takes the form

$$u_{tt} - \nabla^2 u_{tt} - \nabla^2 u = \nabla^2 f(u) \quad \text{in } W^{s,p}(\mathbb{R}^n). \tag{1.6}$$

Chen [4] studied equation (1.4) with the initial and boundary conditions

$$\begin{gathered} u(0,t) = u(1,t) = 0, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [0,t]. \end{gathered} \tag{1.7}$$

He proved the existence of a unique classical solution and gave sufficient conditions for the blow-up of solutions using the concavity method. Moreover, the same author proved that problem in (1.3) with conditions given in (1.7) admits a unique global classical solution. Later on, Chen al [5] considered problem (1.4) with the same conditions and studied the asymptotic behavior of the solution. Sufficient conditions for a blow-up result are given. They used the integral inequality given in [10, Theorem 8.1].

The Cauchy problem for this type of equations have been extensively studied by many authors. De Godefrroy [8] considered the Cauchy problem of equation (1.5) and proved the existence of a unique local solution and gave sufficient conditions for a blow-up result using the concavity method. Constantin et al [7] studied the local well-posedness to the Cauchy problem

$$u_{tt} = u_{xxtt} + [F(u)]_{xx}$$

and obtained the global existence of the solution by the extension theorem. Liu [16] proved by the contraction mapping principle that the Cauchy problem

$$u_{tt} = u_{xxtt} + f(u)_{xx},$$

admits a unique global solution in addition to a blow up result. Wang et al [22] proved that the multidimensional generalized equation of (1.4) has a unique global small amplitude solution. In [23], the same authors have proved that the Cauchy problem of (1.4) has a unique global generalized solution and a unique global classical solution. Later on, they discussed the blow-up of the solution using the concavity method. However, the asymptotic behavior of the solution has not been studied.

On the viscoelastic problems, we will mention here some results from the literature related to this type of problems. Cavalcanti et al [3] considered the problem

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma \Delta u_t = 0,$$

for $\rho > 0$, and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This result has been improved by Messaoudi and Tatar [17], for $\gamma = 0$, they established exponential and polynomial decay results in the absence, as well as in the presence, of a source term.

Kafini and Messaoudi [11] studied the Cauchy problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = 0, \quad x \in \mathbb{R}^n,\ t > 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n,$$

where u_0, u_1 are two compactly supported functions and g is a positive nonincreasing function defined on $\mathbb{R}^+$. They proved that the decay of the solution is polynomial (respectively logarithmic) if the rate of decay of the relaxation function is exponential (respectively polynomial). They used the multiplier method and a lemma by Martinez [17].

For more results related to stability and asymptotic behavior of viscoelastic equations, we refer the reader to the work of Renardy et al [21], Munoz and Oquendo [19], Fabrizio and Morro [9], Baretto et al [1], Kafini [12, 13, 14, 15].

In the present article, we study the asymptotic behavior of solutions to (1.1). To achieve this goal some conditions have to be imposed on the relaxation function g. Since Poincaré and some embedding inequalities are no longer valid, We will use the nature of the wave propagation to overcome this difficulties. The proof is based on the multiplier method and makes use of a lemma by Martinez [17]. This paper is organized as follows. In Section 2, we present conditions and materials needed for our work. In section 3, we state and proof our main result.

2. Preliminaries

In this section we present some material needed for the proof of our results. We will use the following assumptions:

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$1 - \int_0^\infty g(s)ds = l > 0.$$

(G2) There exists $a > 0$ such that

$$g'(t) \le -ag^p(t), \quad 1 \le p < 3/2, \quad t \ge 0.$$

(G3) $\varphi \in C^2(\mathbb{R}^n)$, and for $\lambda > 0$, $\beta \ge 1$, $(n-2)\beta \le n$, it satisfies

$$|\varphi(s)| \le \lambda |s|^\beta, \quad \forall s \in \mathbb{R}^n.$$

(G4) There exists a function $G : \mathbb{R}^n \to \mathbb{R}$ such that $G(w) \ge 0, \nabla G(w) = \varphi(w)$, and $2G(w) \le w.\varphi(w)$ for all $w \in R^n$.

Proposition 2.1 ([4, 5]). *Assume that* (G1)–(G2) *hold and* $u_0, u_1 \in H^1(\mathbb{R}^n)$*, with compact support, then problem* (1.1) *has a unique global solution such that*

$$u \in C^1((0,\infty); H^1(\mathbb{R}^n)), \quad u_{tt} \in C((0,\infty); L^2(\mathbb{R}^n)).$$

Proposition 2.2 ([17]). *Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be non-increasing function, and* $\phi : \mathbb{R}_+ \to \mathbb{R}$ *an increasing function in* C^2 *such that* $\phi(0) = 0$ *and* $\phi(t) \to +\infty$ *as* $t \to +\infty$*. Assume that there exist* $q \ge 0$ *and* $A > 0$ *such that*

$$\int_S^{+\infty} E^{q+1}(t)\phi'(t)dt \le AE(S), \quad 0 \le S < +\infty.$$

Then

$$E(t) \le CE(0)(1+\phi(t))^{-1/q} \quad \forall t \ge 0, \quad \textit{if } q > 0,$$

and

$$E(t) \le CE(0)e^{-\omega\phi(t)} \qquad \forall t \ge 0, \quad \text{if } q = 0,$$

where C and ω are positive constants independent of the initial energy $E(0)$.

Lemma 2.3 (Sobolev, Gagliardo, Nirenberg [2, Thm. 9.9])**.** *Suppose that $1 \le p < n$. If $u \in W^{1,p}(\mathbb{R}^n)$, then $u \in L^{p*}(\mathbb{R}^n)$, with*

$$\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}.$$

Moreover there exists a constant $C = C(n,p)$ such that

$$\|u\|_{p^*} \le C\|\nabla u\|_p, \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Lemma 2.4. *If u is the solution of* (1.1)*, then*

$$\|u(t)\|_2 \le C(L+t)\|\nabla u(t)\|_2.$$

where $L > 0$ is such that

$$\operatorname{supp}\{u_0(x), u_1(x)\} \subset B(L) = \{x \in \mathbb{R}^n \mid |x| < L\}.$$

Proof. Using Lemma 2.3, for $p = 2$, we have

$$\|u\|_{p^*} \le C\|\nabla u\|_2, \quad p^* = \frac{2n}{n-2} \quad , \text{if } n \ge 3.$$

By using the finite-speed propagation property and Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^2 dx &= \int_{B(L+t)} |u|^2 dx \\
&\le \Big(\int_{B(L+t)} 1 dx\Big)^{1-\frac{2}{p*}} \Big(\int_{B(L+t)} (|u|^2)^{\frac{p^*}{2}} dx\Big)^{2/p*} \\
&\le C(L+t)^2 \|u(t)\|_{p^*}^2.
\end{aligned}$$

Hence,

$$\|u(t)\|_2 \le C(L+t)\|u(t)\|_{p^*} \le C(L+t)\|\nabla u(t)\|_2.$$

□

Now, we introduce the "modified" energy functional

$$\begin{aligned}
E(t) = \frac{1}{2}\Big[&\int_{\mathbb{R}^n} |u_t|^2 dx + \Big(1 - \int_0^t g(s)ds\Big)\int_{\mathbb{R}^n} |\nabla u|^2 dx \\
&+ \int_{\mathbb{R}^n} |\nabla u_t|^2 dx + (g \circ \nabla u)\Big] + \int_{\mathbb{R}^n} G(\nabla u) dx,
\end{aligned}$$

where

$$(g \circ u)(t) = \int_0^t g(t-s)\int_{\mathbb{R}^n} |u(s) - u(t)|^2 \, dx \, ds.$$

Lemma 2.5. *If u is a solution of* (1.1)*, then the modified energy satisfies*

$$E'(t) = \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\|\nabla u_t\|_2^2 \le \frac{1}{2}(g' \circ \nabla u) \le 0. \tag{2.1}$$

The proof of the above lemma follows by multiplying equation (1.1) by u_t and integrating over $\mathbb{R}^n$, using integration by parts, and repeating the same computations as in [20].

Corollary 2.6. *Under the assumptions* (G1)–(G2)*, we have*

$$(g^p \circ \nabla u)(t) \le (-g' \circ \nabla u)(t) \le -2E'(t). \tag{2.2}$$

The proof of the above corollary follows by using (G2) and (2.1).

Lemma 2.7. *Let $1 < p < 2$ and u be the solution of* (1.1)*, then for any $0 < \theta < 2 - p$, there exists $C(\theta, p)$ such that*

$$(g \circ \nabla u)^{\frac{p-1+\theta}{\theta}}(t) \le C(\theta)(g^p \circ \nabla u)(t). \tag{2.3}$$

Proof. Using Hölder's inequality, it is easy to see that

$$\begin{aligned}(g \circ \nabla u)(t) &= \int_0^t g^{\frac{(p-1)(1-\theta)}{p-1+\theta}}(t-s)\|\nabla u(t) - \nabla u(s)\|_2^{\frac{2(p-1)}{p-1+\theta}} \\ &\quad \times g^{\frac{p\theta}{p-1+\theta}}(t-s)\|\nabla u(t) - \nabla u(s)\|_2^{\frac{2\theta}{p-1+\theta}} ds \\ &\le \Big\{ \int_0^t g^{1-\theta}(s)ds \|\nabla u(t) - \nabla u(s)\|_2^2 \Big\}^{\frac{p-1}{p-1+\theta}} \big((g^p \circ \nabla u)(t)\big)^{\frac{\theta}{p-1+\theta}}.\end{aligned} \tag{2.4}$$

Using (G1) and (G2), we easily arrive at

$$\begin{aligned}\int_0^t g^{1-\theta}(s)ds \le \int_0^\infty g^{1-\theta}(s)ds &\le -\int_0^\infty g^{1-\theta-p}(s)g'(s)ds \\ &= \frac{-1}{(2-p-\theta)}[g^{2-\theta-p}(s)]_0^\infty = \frac{g^{2-\theta-p}(0)}{(2-p-\theta)} \\ &= C_0 < \infty,\end{aligned}$$

where $2 - p - \theta > 0$. Therefore, (2.4) becomes

$$\begin{aligned}(g \circ \nabla u)(t) &\le \Big\{2C_0 \sup_{0 \le t < \infty} \|\nabla u\|_2^2\Big\}^{\frac{p-1}{p-1+\theta}} \big((g^p \circ \nabla u)(t)\big)^{\frac{\theta}{p-1+\theta}} \\ &\le \Big\{2C_0 \sup_{0 \le t < \infty} E(t)\Big\}^{\frac{p-1}{p-1+\theta}} \big((g^p \circ \nabla u)(t)\big)^{\frac{\theta}{p-1+\theta}} \\ &\le \{2C_0 E(0)\}^{\frac{p-1}{p-1+\theta}} \big((g^p \circ \nabla u)(t)\big)^{\frac{\theta}{p-1+\theta}}.\end{aligned} \tag{2.5}$$

Thus, (2.3) is established, with $C = \{2C_0 E(0)\}^{\frac{p-1}{\theta}}$. □

Lemma 2.8. *Let $1 < p < 3/2$, then*

$$\int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|ds \Big) dx \le \Big(\int_0^t g^{2-p}(s)ds \Big)^{1/2} \big((g^p \circ \nabla u)(t)\big)^{1/2}.$$

Proof. Using Hölder's inequality, direct calculations lead to

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|ds \Big) dx \\ &= \int_{\mathbb{R}^n} \Big(\int_0^t g^{1-p/2}(t-s) g^{p/2}(t-s)|\nabla u(t) - \nabla u(s)|ds \Big) dx \\ &\le \Big(\int_0^t g^{2-p}(s)ds \Big)^{1/2} \big((g^p \circ \nabla u)(t)\big)^{1/2}.\end{aligned}$$

□

3. Decay of solutions

In this section, we establish two lemmas then we state and prove our main result. We take $\phi(t) = \ln(1+t)$, in order to apply Proposition 2.2.

Lemma 3.1. *Under assumptions* (G1)–(G4), *for* $\delta, \delta_1, \delta_2 > 0$, *the solution of* (1.1) *satisfies*

$$\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\Big[\Big(1-(\delta+C\delta_1+\delta_2)-\int_0^t g(s)ds\Big)|\nabla u|^2 \\
&\quad -\big(1+\frac{C}{4\delta_1}\big)|u_t|^2-\big(1+\frac{1}{4\delta_2}\big)|\nabla u_t|^2\Big]+(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}2G(\nabla u)dx \\
&\quad +(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}2G(\nabla u)dx \\
&\le -\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}uu_tdx\Big]-\frac{1}{2}\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}|\nabla u|^2dx\Big] \\
&\quad -\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx\Big] \\
&\quad +C\Big[\Big(\frac{1}{4\delta}+1\Big)(1+t)^{-1}+1\Big]E^{q-1}(t)\big(-E'(t)\big),
\end{aligned} \tag{3.1}$$

where C *is a generic positive constant independent of* δ, δ_1 *and* δ_2.

Proof. Multiplying equation (1.1) by $(1+t)^{-1}E^q(t)u(t)$ and integrating by parts over $\mathbb{R}^n$, we obtain

$$\begin{aligned}
&(1+t)^{-1}E^q(t)\Big[\int_{\mathbb{R}^n}uu_{tt}dx-\int_{\mathbb{R}^n}u\Delta u\,dx+\int_{\mathbb{R}^n}u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx \\
&\quad -\int_{\mathbb{R}^n}u\Delta u_tdx-\int_{\mathbb{R}^n}u\Delta u_{tt}dx\Big] \\
&=(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}u\,\mathrm{div}\,\big(\varphi(\nabla u)\big)dx.
\end{aligned} \tag{3.2}$$

A direct integration by parts yields

$$\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\big(|\nabla u|^2-|u_t|^2-|\nabla u_t|^2\big)\,dx \\
&\quad +(1+t)^{-1}E^q(t)\Big(\int_0^t g(t-s)\int_{\mathbb{R}^n}u(t)\Delta u(s)\,dx\,ds\Big) \\
&\quad -q(1+t)^{-1}E^{q-1}(t)E'(t)\Big(\int_{\mathbb{R}^n}uu_tdx+\int_{\mathbb{R}^n}|\nabla u|^2dx+\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx\Big) \\
&\quad +(1+t)^{-2}E^q(t)\Big(\int_{\mathbb{R}^n}uu_tdx+\int_{\mathbb{R}^n}|\nabla u|^2dx+\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx\Big) \\
&=-\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}uu_tdx\Big]-\frac{1}{2}\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}|\nabla u|^2dx\Big] \\
&\quad -\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx\Big] \\
&\quad -(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u.\varphi(\nabla u)dx,
\end{aligned} \tag{3.3}$$

the second term in the left side of (3.3) can be estimated as

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} u(t)\Delta u(s)\,dx\,ds \\
&= -(1+t)^{-1}E^q(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} \nabla u(t).\nabla u(s)\,dx\,ds \\
&= -(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \nabla u(t).\int_0^t g(t-s)\left(\nabla u(s)-\nabla u(t)\right)\,dx\,ds \\
&\quad - (1+t)^{-1}E^q(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} |\nabla u(t)|^2\,dx\,ds \\
&\geq -(1+t)^{-1}E^q(t)\left(\delta\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)(g^p\circ\nabla u)(t)\right) \\
&\quad - (1+t)^{-1}E^q(t)\int_0^t g(s)ds\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx.
\end{aligned}
$$

Thus (3.3) becomes

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\Big[\Big(1-\delta-\int_0^t g(s)ds\Big)|\nabla u|^2 - |u_t|^2 - |\nabla u_t|^2\Big]dx \\
&- \frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(1+t)^{-1}E^q(t)(g^p\circ\nabla u) \\
&- q(1+t)^{-1}E^{q-1}E'(t)\Big(\int_{\mathbb{R}^n} uu_t dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u_t.\nabla u\,dx\Big)dt \\
&+ (1+t)^{-2}E^q(t)\Big(\int_{\mathbb{R}^n} uu_t dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u_t.\nabla u\,dx\Big)dt \\
&\leq -\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} uu_t dx\Big] - \frac{1}{2}\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big] \\
&- \frac{d}{dt}\Big[\int_{\mathbb{R}^n}(1+t)^{-1}E^q(t)\nabla u_t.\nabla u\,dx\Big] \\
&- (1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u.\varphi(\nabla u)dx.
\end{aligned}
\tag{3.4}
$$

Adding $(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} 2G(\nabla u)dx$ to both sides of (3.4), we obtain

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\Big[\Big(1-\delta-\int_0^t g(s)ds\Big)|\nabla u|^2 - |u_t|^2 - |\nabla u_t|^2\Big]dx \\
&\quad + (1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} 2G(\nabla u)dx \\
&\leq \frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(1+t)^{-1}E^q(t)(g^p\circ\nabla u)(t) \\
&\quad + q(1+t)^{-1}E^{q-1}E'(t)\left(\int_{\mathbb{R}^n} uu_t dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u_t.\nabla u\,dx\right) \\
&\quad - (1+t)^{-2}E^q(t)\Big(\int_{\mathbb{R}^n} uu_t dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u_t.\nabla u\,dx\Big) \\
&\quad - \frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} uu_t dx\Big] - \frac{1}{2}\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{d}{dt}\Big[\int_{\mathbb{R}^n} (1+t)^{-1} E^q(t) \nabla u_t . \nabla u \, dx \Big] \\
&+ (1+t)^{-1} E^q(t) \int_{\mathbb{R}^n} (2G(\nabla u) - \nabla u . \varphi(\nabla u)) dx.
\end{aligned}
$$

By assumption (G4), the above inequality yields

$$
\begin{aligned}
&(1+t)^{-1} E^q(t) \int_{\mathbb{R}^n} \Big[\Big(1 - \delta - \int_0^t g(s) ds \Big) |\nabla u|^2 - |u_t|^2 - |\nabla u_t|^2 \Big] dx \\
&\quad + (1+t)^{-1} E^q(t) \int_{\mathbb{R}^n} 2G(\nabla u) dx \\
&\le -(1+t)^{-2} E^q(t) \Big(\int_{\mathbb{R}^n} u u_t dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u_t . \nabla u \, dx \Big) \\
&\quad + q(1+t)^{-1} E^{q-1} E'(t) \Big(\int_{\mathbb{R}^n} u u_t dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u_t . \nabla u \, dx \Big) \\
&\quad + \frac{1}{4\delta} \Big(\int_0^t g^{2-p}(s) ds \Big) (1+t)^{-1} E^q(t) (g^p \circ \nabla u)(t) \\
&\quad - \frac{d}{dt} \Big[(1+t)^{-1} E^q(t) \int_{\mathbb{R}^n} u u_t dx \Big] - \frac{1}{2} \frac{d}{dt} \Big[\int_{\mathbb{R}^n} (1+t)^{-1} E^q(t) |\nabla u|^2 dx \Big] \\
&\quad - \frac{d}{dt} \Big[\int_{\mathbb{R}^n} (1+t)^{-1} E^q(t) \nabla u_t . \nabla u \, dx \Big].
\end{aligned} \tag{3.5}
$$

Terms in the right side of (3.5) can be estimated using the non-increasing property of $E(t)$, Cauchy's inequality, assumption (G2), Lemma 2.4 and Lemma 2.8, the first term is handled as follows

$$
\begin{aligned}
-(1+t)^{-2} E^q(t) \int_{\mathbb{R}^n} u u_t dx &\le (1+t)^{-2} E^q(t) \Big(\int_{\mathbb{R}^n} |u|^2 dx \Big)^{1/2} \Big(\int_{\mathbb{R}^n} |u_t|^2 dx \Big)^{1/2} \\
&\le (1+t)^{-2} E^q(t) C (1+t) \|\nabla u\|_2 \|u_t\|_2 \\
&\le C (1+t)^{-1} E^q(t) \Big[\delta_1 \|\nabla u\|_2^2 + \frac{1}{4\delta_1} \|u_t\|_2^2 \Big].
\end{aligned}
$$

The second term satisfies

$$
-(1+t)^{-2} E^q(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx \le 0, \tag{3.6}
$$

The third term satisfies

$$
\begin{aligned}
-(1+t)^{-2} E^q(t) \int_{\mathbb{R}^n} \nabla u_t . \nabla u \, dx &\le (1+t)^{-2} E^q(t) \big[\delta_2 \|\nabla u\|_2^2 + \frac{1}{4\delta_2} \|\nabla u_t\|_2^2 \big] \\
&\le (1+t)^{-1} E^q(t) \big[\delta_2 \|\nabla u\|_2^2 + \frac{1}{4\delta_2} \|\nabla u_t\|_2^2 \big].
\end{aligned} \tag{3.7}
$$

The fourth term satisfies

$$
\begin{aligned}
&\frac{1}{4\delta} \Big(\int_0^t g^{2-p}(s) ds \Big) (1+t)^{-1} E^q(t) (g^p \circ \nabla u) \\
&\le \frac{C}{4\delta} (1+t)^{-1} E^q(t) (g^p \circ \nabla u) \le \frac{-C}{4\delta} (1+t)^{-1} E^q(t) (g' \circ \nabla u) \\
&\le \frac{-C}{4\delta} (1+t)^{-1} E^q(t) E'(t),
\end{aligned} \tag{3.8}
$$

where, by the assumptions on g,

$$\int_0^t g^{2-p}(s)ds < \int_0^\infty g^{2-p}(s)ds < \infty.$$

The fifth term satisfies

$$\begin{aligned} &q(1+t)^{-1}E^{q-1}E'(t)\int_{\mathbb{R}^n} uu_t dx \\ &\leq -C(1+t)^{-1}E^{q-1}E'(t)\int_{\mathbb{R}^n}\big(|u|^2+|u_t|^2\big)dx \\ &\leq -CE^{q-1}(t)E'(t)\int_{\mathbb{R}^n}\big(|\nabla u|^2+|\nabla u_t|^2\big)dx \\ &\leq -CE^{q-1}(t)E'(t)E(t) \leq -CE^q(t)E'(t). \end{aligned}$$

The sixth term satisfies

$$q(1+t)^{-1}E^{q-1}E'(t)\int_{\mathbb{R}^n}|\nabla u|^2dx \leq -C(1+t)^{-1}E^qE'(t). \tag{3.9}$$

The seventh term satisfies

$$\begin{aligned} &q(1+t)^{-1}E^{q-1}E'(t)\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx \\ &\leq -C(1+t)^{-1}E^{q-1}E'(t)\int_{\mathbb{R}^n}\Big(|\nabla u_t|^2+|\nabla u|^2\Big)dx \\ &\leq -C(1+t)^{-1}E^{q-1}E'(t)E(t)dt \leq -C(1+t)^{-1}E^q(t)E'(t). \end{aligned} \tag{3.10}$$

Combining (3.5)–(3.10) and the Lemma is proved. ☐

Lemma 3.2. *Under assumptions* (G1), (G4), *for* $\delta > 0$, *the solution of* (1.1) *satisfies*

$$\begin{aligned} &(1+t)^{-1}E^q(t)\Big(\int_0^t g(s)ds-\delta(1+C)\Big)\|u_t\|_2^2-(1+t)^{-1}E^q(t)\delta(2-l+k)\|\nabla u\|_2^2 \\ &-C(1+t)^{-1}E^q(t)\Big(\delta-\int_0^t g(s)ds\Big)\|\nabla u_t\|_2^2 \\ &\leq \frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}u_t\int_0^t g(t-s)\,(u(t)-u(s))\;ds\,dx\Big] \\ &\quad+\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\,(\nabla u(t)-\nabla u(s))\;ds\,dx\Big] \\ &\quad+C\Big[\frac{1}{\delta}(1+t)^{-1}+1+\frac{1}{4\delta}\Big]E^q(t)\,(-E'(t))\,, \end{aligned} \tag{3.11}$$

where C *is a generic positive constant independent of* δ.

Proof. Multiplying equation (1.1) by

$$(1+t)^{-1}E^q(t)\int_0^t g(t-s)\,(u(t)-u(s))\,ds,$$

and integrating over $\mathbb{R}^n$ we obtain

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\Big(\int_0^t g(s)ds\Big)\|u_t\|_2^2\\
&=\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx\Big]\\
&\quad+(1+t)^{-2}E^q(t)\int_{\mathbb{R}^n} u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx\\
&\quad-q(1+t)^{-1}E^{q-1}(t)E'(t)\int_{\mathbb{R}^n} u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx\\
&\quad-(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} u_t\int_0^t g'(t-s)\left(u(t)-u(s)\right)\,ds\,dx\\
&\quad+(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \nabla u\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\quad-(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)\nabla u(s)ds\Big)\\
&\quad\times\Big(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\Big)dx\\
&\quad+(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \nabla u_t\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\quad+(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \nabla u_{tt}\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\quad+(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \operatorname{div}\varphi(\nabla u)\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx.
\end{aligned}
\tag{3.12}
$$

Terms in the right hand side of the above expression can be treated in a similar way as in (3.5).

The second term satisfies

$$
\begin{aligned}
&(1+t)^{-2}E^q(t)\int_{\mathbb{R}^n} u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx\\
&\le(1+t)^{-2}E^q(t)\Big(\int_{\mathbb{R}^n}|u_t|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\Big)^2dx\Big)^{1/2}\\
&\le(1+t)^{-2}E^q(t)\|u_t\|_2C(1+t)\Big(\int_{\mathbb{R}^n}|\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)ds|^2dx\Big)^{1/2}\\
&\le C(1+t)^{-1}E^q(t)\|u_t\|_2\Big(\Big[\int_0^t g^{2-p}(s)ds\Big](g^p\circ\nabla u)\Big)^{1/2}\\
&\le C(1+t)^{-1}E^q(t)\Big[\delta\|u_t\|_2^2+\frac{C}{4\delta}(g^p\circ\nabla u)\Big]\\
&\le C(1+t)^{-1}E^q(t)\Big[\delta\|u_t\|_2^2+\frac{C}{4\delta}(-E'(t))\Big].
\end{aligned}
\tag{3.13}
$$

The third term satisfies

$$
-q(1+t)^{-1}E^{q-1}(t)E'(t)\int_{\mathbb{R}^n} u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx
$$

$$
\begin{aligned}
&\le -q(1+t)^{-1}E^{q-1}(t)E'(t)\Big(\int_{\mathbb{R}^n} u_t^2 dx\Big)^{1/2}\\
&\quad\times \Big(\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^2 dx\Big)^{1/2}\\
&\le -C(1+t)^{-1}E^{q-1}(t)E'(t)\|u_t\|_2(1+t)\\
&\quad\times \Big(\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2 dx\Big)^{1/2}\\
&\le -CE^{q-1}(t)E'(t)\|u_t\|_2(1-l)\Big(\int_{\mathbb{R}^n}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big)^{1/2}\\
&\le -CE^{q-1}(t)E'(t)E^{1/2}(t)E^{1/2}(t)\\
&\le -CE^q(t)E'(t).
\end{aligned}
$$

The fourth term satisfies

$$
\begin{aligned}
&-(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&\le (1+t)^{-1}E^q(t)\big[\delta\|u_t\|_2^2+\frac{Cg(0)}{4\delta}(1+t)(-g'\circ\nabla u)\big]\\
&\le \delta(1+t)^{-1}E^q(t)\|u_t\|_2^2-\frac{C}{4\delta}E^q(t)E'(t).
\end{aligned}\tag{3.14}
$$

Using (2.2), the fifth term satisfies

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\le (1+t)^{-1}E^q(t)\Big[\delta\|\nabla u\|_2^2+\frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(g^p\circ\nabla u)\Big]\\
&\le (1+t)^{-1}E^q(t)\big[\delta\|\nabla u\|_2^2+\frac{C}{4\delta}(-E'(t))\big].
\end{aligned}\tag{3.15}
$$

Using Young's inequality and lemma 2.8, he sixth term satisfies

$$
\begin{aligned}
&-(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)\nabla u(s)ds.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)dx\\
&\le (1+t)^{-1}E^q(t)\Big[\delta(1-l)^2\|\nabla u\|_2^2+\frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(g^p\circ\nabla u)\Big]\\
&\le (1+t)^{-1}E^q(t)\big[\delta(1-l)^2\|\nabla u\|_2^2+\frac{C}{4\delta}(g^p\circ\nabla u)\big]\\
&\le (1+t)^{-1}E^q(t)\big[\delta(1-l)\|\nabla u\|_2^2+\frac{C}{4\delta}(-E'(t))\big].
\end{aligned}\tag{3.16}
$$

The seventh term satisfies

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\le (1+t)^{-1}E^q(t)\Big[\delta\|\nabla u_t\|_2^2+\frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(g^p\circ\nabla u)\Big]\\
&\le (1+t)^{-1}E^q(t)\big[\delta\|\nabla u_t\|_2^2+\frac{C}{4\delta}(-E'(t))\big].
\end{aligned}\tag{3.17}
$$

The eighth term satisfies

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_{tt}.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&=\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\Big]\\
&\quad+(1+t)^{-2}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\quad-q(1+t)^{-1}E^{q-1}(t)E'(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\quad-(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g'(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\\
&\quad-(1+t)^{-1}E^q(t)\Big(\int_0^t g(s)ds\Big)\|\nabla u_t\|_2^2\\
&\le\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\Big]\\
&\quad+C(1+t)^{-2}E^q(t)\big[\delta\|\nabla u_t\|_2^2+\frac{C}{4\delta}(-E'(t))\big]\\
&\quad-C(1+t)^{-1}E^q(t)E'(t)-C(1+t)^{-1}E^q(t)E'(t)\\
&\quad-C(1+t)^{-1}E^q(t)\Big(\int_0^t g(s)ds\Big)\|\nabla u_t\|_2^2\\
&\le\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\Big]\\
&\quad+C(1+t)^{-1}E^q(t)\big[\delta\|\nabla u_t\|_2^2+\frac{C}{4\delta}(-E'(t))\big]-C(1+t)^{-1}E^q(t)E'(t)\\
&\quad-C(1+t)^{-1}E^q(t)\Big(\int_0^t g(s)ds\Big)\|\nabla u_t\|_2^2.
\end{aligned}
$$

The ninth term satisfies

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\operatorname{div}\varphi(\nabla u)\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx\\
&=-(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\varphi(\nabla u).\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,dx\,ds\\
&\le(1+t)^{-1}E^q(t)\Big\{\delta\int_{\mathbb{R}^n}|\varphi(\nabla u(t))|^2dx+\frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(g^p\circ\nabla u)\Big\}\\
&\le(1+t)^{-1}E^q(t)\Big\{\delta\lambda^2\int_{\mathbb{R}^n}|\nabla u(t)|^{2\beta}dx+\frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(g^p\circ\nabla u)\Big\}\\
&\le(1+t)^{-1}E^q(t)\Big\{\delta\lambda^2\left(\frac{2E(0)}{l}\right)^{\beta-1}\|\nabla u(t)\|_2^2+\frac{1}{4\delta}\Big(\int_0^t g^{2-p}(s)ds\Big)(g^p\circ\nabla u)\Big\}\\
&\le(1+t)^{-1}E^q(t)\Big\{\delta C\|\nabla u(t)\|_2^2+\frac{C}{4\delta}\left(-E'(t)\right)\Big\}.
\end{aligned}
\tag{3.18}
$$

Combining (3.12)–(3.18), the assertion of the lemma is proved. □

Now, we multiply (3.11) by N and add it to (3.1) to obtain

$$
\begin{aligned}
&(1+t)^{-1}E^q(t)\Big[N\Big(\int_0^t g(s)ds-\delta(1+C)\Big)-(1+\frac{C}{4\delta_1})\Big]\|u_t(t)\|_2^2\\
&+(1+t)^{-1}E^q(t)\Big[1-\int_0^t g(s)ds-(C\delta_1+\delta_2)-\delta\left(1+N(2-l+k)\right)\Big]\|\nabla u(t)\|_2^2\\
&+(1+t)^{-1}E^q(t)\Big(N\Big(\int_0^t g(s)ds-\delta\Big)-\Big(1+\frac{1}{4\delta_2}\Big)\Big)\|\nabla u_t(t)\|_2^2\\
&+(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}2G(\nabla u)dx\\
&\le -\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}uu_tdx\Big]-\frac{1}{2}\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}|\nabla u|^2dx\Big]\\
&\quad-\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx\Big]\\
&\quad+N\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\,ds\,dx\Big]\\
&\quad+N\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx\Big]\\
&\quad+\Big[NC\big(\frac{3}{4\delta}(1+t)^{-1}+1+\frac{1}{4\delta}\big)+1+\big(\frac{1}{4\delta}+1\big)(1+t)^{-1}\Big]E^q(t)\left(-E'(t)\right).
\end{aligned}
\tag{3.19}
$$

Since g is positive, continuous and $g(0)>0$, then for any $t_0>0$, we have

$$\int_0^t g(s)ds\ge\int_0^{t_0}g(s)ds=g_0>0,\quad \forall t\ge t_0.$$

At this point we choose δ_1, δ_2 small such that

$$1-\int_0^t g(s)ds-(C\delta_1+\delta_2)>l-(C\delta_1+\delta_2)>0,$$

and we choose N large enough so that

$$Ng_0-(1+\frac{C}{4\delta_1})>0,\quad Ng_0-(1+\frac{1}{4\delta_2})>0.$$

Whence δ_1, δ_2 and N are fixed, we pick δ small enough such that

$$
\begin{gathered}
Ng_0-(1+\frac{1}{4\delta_1})-\delta(1+C)>0,\\
1-\int_0^t g(s)ds-(C\delta_1+\delta_2)-\delta(1+N(2-l+k))>0,\\
N\Big(\int_0^t g(s)ds-\delta\Big)-\Big(1+\frac{1}{4\delta_2}\Big)>0.
\end{gathered}
$$

Consequently, we obtain from (3.19), for some constants α, $C>0$,

$$
\begin{aligned}
&(1+t)^{-1}E^{q+1}(t)\\
&\le -\alpha\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}uu_tdx\Big]-\frac{\alpha}{2}\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}|\nabla u|^2dx\Big]\\
&\quad-\alpha\frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n}\nabla u_t.\nabla u\,dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha N \frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} u_t \int_0^t g(t-s)\,(u(t)-u(s))\;ds\,dx\Big]\\
&+ \alpha N \frac{d}{dt}\Big[(1+t)^{-1}E^q(t)\int_{\mathbb{R}^n} \nabla u_t.\int_0^t g(t-s)\,(\nabla u(t)-\nabla u(s))\;ds\,dx\Big]\\
&+ \alpha\Big[NC\Big(\frac{3}{4\delta}(1+t)^{-1}+1+\frac{1}{4\delta}\Big)+1+\Big(\frac{1}{4\delta}+1\Big)(1+t)^{-1}\Big]E^q(t)\,(-E'(t))\\
&+ C(1+t)^{-1}E^q(t)(g\circ\nabla u).
\end{aligned}
$$

Integrating over (S,T), where $T > S \geq t_0$, gives

$$
\begin{aligned}
&\int_S^T (1+t)^{-1}E^{q+1}(t)dt\\
&\leq -\alpha(1+T)^{-1}E^q(T)\int_{\mathbb{R}^n} uu_t(T)dx + \alpha(1+S)^{-1}E^q(S)\int_{\mathbb{R}^n} uu_t(S)dx\\
&\quad - \alpha(1+T)^{-1}E^q(T)\int_{\mathbb{R}^n} |\nabla u(T)|^2dx + \alpha(1+S)^{-1}E^q(S)\int_{\mathbb{R}^n} |\nabla u(S)|^2dx\\
&\quad - \alpha(1+T)^{-1}E^q(T)\int_{\mathbb{R}^n} \nabla u_t.\nabla u(T)dx + \alpha(1+S)^{-1}E^q(S)\int_{\mathbb{R}^n} \nabla u_t.\nabla u(S)dx\\
&\quad + \alpha N(1+T)^{-1}E^q(T)\int_{\mathbb{R}^n} u_t(x,T)\int_0^t g(T-s)\,(u(T)-u(s))\;ds\,dx\\
&\quad - \alpha N(1+S)^{-1}E^q(S)\int_{\mathbb{R}^n} u_t(x,S)\int_0^t g(S-s)\,(u(S)-u(s))\;ds\,dx\\
&\quad + \alpha N(1+T)^{-1}E^q(T)\int_{\mathbb{R}^n} \nabla u_t(x,T)\int_0^t g(T-s)\,(\nabla u(T)-\nabla u(s))\;ds\,dx\\
&\quad - \alpha N(1+S)^{-1}E^q(S)\int_{\mathbb{R}^n} \nabla u_t(x,S)\int_0^t g(S-s)\,(\nabla u(S)-\nabla u(s))\;ds\,dx\\
&\quad - \alpha C\left[E^{q+1}(T)-E^{q+1}(S)\right]\\
&\quad + C\int_S^T \big((1+t)^{-1}E^q(t)(g\circ\nabla u)\big)\,dt.
\end{aligned}
$$

Using the estimates

$$
(1+t)^{-1}\int_{\mathbb{R}^n} uu_tdx \leq C\|\nabla u\|_2\|u_t\|_2 \leq CE(t)^{1/2}E(t)^{1/2} \leq CE(t),
$$

$$
\int_{\mathbb{R}^n} |\nabla u|^2dx \leq CE(t),
$$

$$
\int_{\mathbb{R}^n} \nabla u_t.\nabla u\,dx \leq C\big[\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\big] \leq CE(t),
$$

$$
\begin{aligned}
&(1+t)^{-1}\int_{\mathbb{R}^n} u_t(x,t)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&\leq C(1+t)^{-1}\|u_t(t)\|_2(1+t)(g\circ\nabla u)^{1/2}\\
&\leq CE(t)^{1/2}E(t)^{1/2} \leq CE(t),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u_t.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\leq C\big[\|\nabla u_t(t)\|_2^2 + (g\circ\nabla u)\big] \leq CE(t),
\end{aligned}
$$

we obtain, for all $S \geq t_0$,

$$\int_S^T (1+t)^{-1} E^{q+1}(t)dt \leq CE^{q+1}(S) + C\int_S^T (1+t)^{-1} E^q(t)(g \circ \nabla u)dt. \quad (3.20)$$

At this point we have two possible cases:

Theorem 3.3. *(case $p = 1$) Let $u_{0,}u_1 \in H_0^1(\mathbb{R}^n)$ and assume that* (G1), (G2) *hold. Then, there exist positive constants C and ω such that, for any $t_0 > 0$,*

$$E(t) \leq \frac{C}{(1+t)^\omega}, \quad \forall t \geq t_0.$$

Proof. Estimates (3.20) and (2.2) yield

$$\int_S^T (1+t)^{-1} E^{q+1}(t)dt \leq AE^{q+1}(S), \quad \forall S \geq t_0,$$

taking $q = 0$, we obtain

$$\int_S^T (1+t)^{-1} E(t)dt \leq AE(S), \quad \forall t \geq t_0.$$

Then let $T \to +\infty$ to obtain

$$\int_S^{+\infty} (1+t)^{-1} E(t)dt \leq AE(S), \quad \forall t \geq t_0\,.$$

Thus Proposition 2.2. yields

$$E(t) \leq CE(0)e^{-\omega\phi(t)} \leq CE(0)e^{-\omega\ln(1+t)} \leq \frac{C}{(1+t)^\omega}, \quad \forall t \geq t_0.$$

□

Theorem 3.4. *(case $p > 1$) Let $u_{0,}u_1 \in H_0^1(\mathbb{R}^n)$ be given, and assume that* (G1), (G2) *hold. Then there exists positive constant C such that, for any $0 < \theta < 2 - p$ and $t_0 > 0$,*

$$E(t) \leq C(1+\ln(1+t))^{-\theta/(p-1)}.$$

Proof. By Lemma 2.7 and using Young's inequality, for $0 < \theta < 1$, we have

$$\begin{aligned}
&(1+t)^{-1}E^q(t)(g \circ \nabla u) \leq C(1+t)^{-1}E^q(t)(g^p \circ \nabla u)^{\frac{\theta}{p-1+\theta}} \\
&\leq C(1+t)^{-1}\Big[\varepsilon E^{\frac{q(p-1+\theta)}{p-1}}(t) + C(\varepsilon)(g^p \circ \nabla u)\Big],
\end{aligned}$$

we choose $q = (p-1)/\theta$ so that $\frac{q(p-1+\theta)}{p-1} = q + 1$. Consequently,

$$\int_S^T (1+t)^{-1} E^{q+1}(t)dt \leq AE(S), \quad \forall S \geq t_0.$$

Then letting $T \to +\infty$, we obtain

$$\int_S^{+\infty} (1+t)^{-1} E^{q+1}(t)dt \leq AE(S).$$

Thus Proposition 2.2. yields

$$E(t) \leq CE(0)(1+\phi(t))^{\frac{-1}{q}} \leq C(1+\ln(1+t))^{\frac{-\theta}{p-1}}, \quad \forall t \geq t_0.$$

This completes the proof. □

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References

[1] Barreto, R.; Munoz Rivera, J. E.; *Uniform rates of decay in nonlinear viscoelasticity for polynomial decaying kernels*, Appl. Anal. **60** (1996), 263–283.

[2] Brezis, H.; *Functional analysis*, Springer Science + Business Media, LLC 2011.

[3] Cavalcanti, M. M.; Domingos Cavalcanti, V. N.; Ferreira, J.; *Existence and uniform decay for nonlinear viscoelastic equation with strong damping*, Math. Methods Appl. Sci., **24** (2001), 1043-1053.

[4] Chen, X.; *Existence and nonexistence of global solutionsfor nonlinear evolution equation of fourth order*, Appl. Math. J. Chinese Univ. Ser. B, **16** (3) (2001), 251-258.

[5] Chen, X.; Chen, G.; *Asymptotic behavior and blow-up of solutions to a nonlinear evolution equation of fourth order*, Nonlinear Anal, **68** (2008), 892-904.

[6] Clarkson P.A., LeVeque R.J., Saxton R., Solitary-wave interaction in elastic rods, Stud. Appl. Math., **75** (1986), 95-122.

[7] Constantin, A.; Molinet, L.; *The initial value problem for a generalized Boussinesq equation*, Differential Integral Equations, **15** (9) (2002), 1061-1072.

[8] Dé Godefroy, A.; *Blow up of solutions of a generalized Boussinesq equation*, IMA J. Appl. Math., **60** (2) (1998), 123-138.

[9] Fabrizio, M.; Morro, A.; *Mathematical Problems in Linear Viscoelasticity*, SIAM Studies in Appl. Math., Vol. 12, Philadelphia 1992.

[10] Komornik, V.; *Exact Controllability and Stabilization, The Multiplier Method*, Masson John Wiley, Paris, 1994.

[11] Kafini, M.; Messaoudi, S.; *On the uniform decay in viscoelastic problems in* $\mathbb{R}^n$, Applied Mathematics and Computation, **215** (2009), 1161–1169.

[12] Kafini, M.; *On the uniform decay in Cauchy viscoelastic problems*, Afrika Matematika, DOI10.1007/s13370-011-0020-3.

[13] Kafini, M.; *Uniform decay of solutions to Cauchy viscoelastic problems with density*, Electron. J. Diff. Eqs., Vol. 2011 (2011), No. 93, pp. 1-9.

[14] Kafini, M.; Messaoudi, S.; Tatar N-E.; *Decay rates of solutions to a nonlinear viscoelastic evolution equation of fourth order in* $\mathbb{R}^n$. Indagationes Mathematicae - New Series., **22** (2011), 103–115.

[15] Kafini, M.; *Decay results for viscoelastic diffusion equations in absence of instantaneous elasticity*, Electron. J. Diff. Equ., Vol. 2012 (2012), No. 73, pp. 1-14.

[16] Liu, Y.; *Existence and blow up of solution of a nonlinear Pochhammer Chree equation*, Indiana Univ. Math. J., **45** (1996), 797-816.

[17] Martinez, P.; *A new method to decay rate estimates for dissipative systems*, ESAIM Control. Optim. Cal. Var., **4** (1999), 419-444.

[18] Messaoudi, S. A.; Tatar, N.-E.; *Global existence asymptotic behavior for a Nonlinear Viscoelastic Problem*, J. Math. Sci. Res., **7** (4) (2003), 136-149.

[19] Munoz Rivera, J. E.; Oquendo Portillo, H.; *Exponential stability to a contact problem of partially viscoelastic materials*, J. Elasticity, **63**, (2) (2001), 87-111.

[20] Messaoudi, S. A.; *Blow up and global existence in a nonlinear viscoelastic wave equation*, Math. Nachrich., **260** (2003), pp. 58-66.

[21] Renardy, M.; Hrusa, W.J.; and Nohel, J. A.; Mathematical Problems in Viscoelasticity, Pitman Monographs and Surveys in Pure and Appl. Math., **35**, Longman Scientific & Technical, Harlow, 1987.

[22] Wang, S.; Chen, G.; *Small amplitude solutions of the generalized IMBq equation*, J. Math. Anal. Appl., **274** (2002), 846-866.

[23] Wang, S.; Chen, G.; *The Cauchy problem for the generalized IMBq equation in* $W^{s,p}(\mathbb{R}^n)$, J. Math. Anal. Appl., **266** (2002), 38-54.

[24] Zhuang, W.; Guitong, Y.; *Propagation of solitary waves in the nonlinear rods*, Appl. Math. Mech., **7** (1986), 571-581.

[25] Zhang, S.; Zhuang, W.; *Strain solitary waves in the nonlinear elastic rods*, Acta Mech. Sinica, **20** (1988) 58-66.

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