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INFINITELY MANY SOLUTIONS FOR A PERTURBED NONLINEAR FRACTIONAL BOUNDARY-VALUE PROBLEM

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Abstract. Using variational methods, we prove the existence of infinitely many solutions for a class of nonlinear fractional boundary-value problems depending on two parameters.

1. Introduction

In recent years, some fixed point theorems and monotone iterative methods have been applied successfully to investigate the existence of solutions for nonlinear fractional boundary-value problems, see for example, [2, 3, 4, 5, 7, 8, 9, 23, 26, 27, 31, 32, 33] and the references therein. But till now, there are few results on the solutions to fractional boundary value problems which are established by the variational methods. It is often very difficult to establish a suitable space and variational functional for fractional boundary value problem for several reasons. First and foremost, the composition rule in general fails to be satisfied by fractional integral and fractional derivative operators. Furthermore, the fractional integral is a singular integral operator and fractional derivative operator is non-local. Besides, the adjoint of a fractional differential operator is not the negative of itself. Recently, by using critical point theory, Jiao and Zhou [24] studied the fractional BVP

$$\begin{gathered} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = 0, \quad u(T) = 0, \end{gathered}$$

where ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives of order $0 < \alpha \le 1$ respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function satisfying some assumptions and $\nabla F(t,x)$ is the gradient of F at x.

In [6], by using a local minimum theorem established by Bonanno in [16], we provided a new approach to investigated the existence of solutions for the following fractional boundary-value problem

$$\begin{gathered} \frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = 0, \quad u(T) = 0, \end{gathered}$$

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where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order $0 < \alpha \leq 1$ respectively, λ is a positive real parameter, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

The purpose of this article is to establish the existence of infinitely many solutions for the following perturbed nonlinear fractional boundary value problem

$$\begin{gathered} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \lambda a(t)f(u(t)) + \mu g(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) = 0, \quad u(T) = 0, \end{gathered} \tag{1.1}$$

where ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives of order $0 < \alpha \leq 1$ respectively, λ and μ are non-negative parameters, $a : [0, T] \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are three given continuous functions.

Precisely, under appropriate hypotheses on the nonlinear term f, g, the existence of two intervals Λ and J such that, for each $\lambda \in \Lambda$ and $\mu \in J$, BVP (1.1) admits a sequence of pairwise distinct solutions is proved. Our analysis is mainly based on a recent critical point theorem of Bonanno and Molica Bisci [10], which is a more precise version of Ricceri's Variational Principle [29]. This theorem and its variations have been used in several works in order to obtain existence of infinitely many solutions for different kinds of problems (see, for instance, [1, 10, 11, 12, 13, 14, 15] and references therein). The technique used in this paper in order to approach perturbed nonlinearity depending on two parameters has been adopted first in the paper [17]. Moreover, among authors that follow this technique, we recall the papers [18, 19, 20, 21, 22].

This article is organized as follows. In section 2, we present some necessary preliminary facts that will be needed in the paper. In section 3, we establish our main two existence results and give an example to show the effectiveness of the our results.

2. Preliminaries

In this section, we first introduce some necessary definitions and properties of the fractional calculus which are used in this paper.

Definition 2.1. Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order α for function f denoted by ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t \in [a, b],\ \alpha > 0, \\ {}_tD_b^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1}f(s)ds, \quad t \in [a, b],\ \alpha > 0, \end{aligned}$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2. Let f be a function defined on $[a, b]$. For $n-1 \leq \gamma < n (n \in \mathbb{N})$, the left and right Riemann-Liouville fractional derivatives of order γ for function f denoted by ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$, respectively, are defined by

$${}_aD_t^{\gamma}f(t) = \frac{d^n}{dt^n}{}_aD_t^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\gamma-1}f(s)ds, \quad t \in [a, b]$$

and

$$ {}_tD_b^\gamma f(t) = (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\gamma-n} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\gamma-1} f(s)ds, \quad t \in [a,b]. $$

According to [25, 28], if $f \in AC^n([a,b],\mathbb{R}^N)$, then by performing repeatedly integration by parts and differentiation, for $n-1 \le \gamma < n$, we have

$$ {}_aD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds + \sum_{j=0}^{n-1} \frac{f^j(a)}{\Gamma(j-\gamma+1)} (t-a)^{j-\gamma}, \tag{2.1} $$

and

$$ {}_tD_b^\gamma f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b \frac{f^{(n)}(s)}{(s-t)^{\gamma+1-n}} ds + \sum_{j=0}^{n-1} \frac{(-1)^j f^j(b)}{\Gamma(j-\gamma+1)} (b-t)^{j-\gamma}, \tag{2.2} $$

where $t \in [a,b]$.

From [25], [30], we have the following property of fractional integration.

Proposition 2.3. *If $f \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ and $p \ge 1$, $q \ge 1$, $1/p + 1/q \le 1 + \gamma$ or $p \ne 1$, $q \ne 1$, $1/p + 1/q = 1 + \gamma$, then*

$$ \int_a^b [{}_aD_t^{-\gamma} f(t)] g(t) dt = \int_a^b [{}_tD_b^{-\gamma} g(t)] f(t) dt, \quad \gamma > 0. \tag{2.3} $$

By properties (2.1)–(2.3), one has (see [24])

Proposition 2.4. *If $f(a) = f(b) = 0$, $f' \in L^\infty([a,b],\mathbb{R}^N)$ and $g \in L^1([a,b],\mathbb{R}^N)$, or $g(a) = g(b) = 0$, $g' \in L^\infty([a,b],\mathbb{R}^N)$ and $f \in L^1([a,b],\mathbb{R}^N)$, then*

$$ \int_a^b [{}_aD_t^\alpha f(t)]\, g(t) dt = \int_a^b [{}_tD_b^\alpha g(t)]\, f(t) dt, \quad 0 < \alpha \le 1. \tag{2.4} $$

To establish a variational structure for BVP (1.1), it is necessary to construct appropriate function spaces.

Definition 2.5. Let $0 < \alpha \le 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$ \|u\|_\alpha = \Big(\int_0^T |{}_0D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt \Big)^{1/2}, \quad \forall u \in E_0^\alpha. $$

It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2[0,T]$ having an α-order fractional derivative ${}_0D_t^\alpha u \in L^2[0,T]$ and $u(0) = u(T) = 0$.

Proposition 2.6 ([24]). *Let $0 < \alpha \le 1$. The fractional derivative space E_0^α is reflexive and separable Banach space.*

Proposition 2.7 ([24]). *Let $1/2 < \alpha \le 1$. For all $u \in E_0^\alpha$, we have*

(i)

$$ \|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^2}. \tag{2.5} $$

(ii)

$$ \|u\|_\infty = \max_{t \in [0,T]} |u(t)| \le \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}} \|{}_0D_t^\alpha u\|_{L^2}. \tag{2.6} $$

By (2.5), we can consider E_0^α with respect to the norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0D_t^\alpha u(t)|^2 dt\Big)^{1/2} = \|{}_0D_t^\alpha u\|_{L^2}, \quad \forall u \in E_0^\alpha \tag{2.7}$$

in the following analysis.

We are now in a position to give the definition for the solution of BVP (1.1).

Definition 2.8. A function $u : [0,T] \to \mathbb{R}$ is called a solution of BVP (1.1) if

(i) ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t))$ and ${}_0D_t^{\alpha-1}u(t)$ exist for almost every $t \in [0,T]$, and
(ii) u satisfies (1.1).

By using (2.4) and Definition 2.8, we can give the definition of weak solution for BVP (1.1) as follows.

Definition 2.9. By the weak solution of BVP (1.1), we mean that the function $u \in E_0^\alpha$ such that $a(\cdot)f(u(\cdot)) \in L^1[0,T]$, $g(\cdot,u(\cdot)) \in L^1[0,T]$ and satisfies

$$\int_0^T [{}_0D_t^\alpha u(t) \cdot {}_0D_t^\alpha v(t) - \lambda a(t)f(u(t)) \cdot v(t) - \mu g(t,u(t)) \cdot v(t)]dt = 0$$

for all $v \in C_0^\infty([0,T],\mathbb{R})$.

By [24, Theorem 5.1], we have

Theorem 2.10. *Let $0 < \alpha \le 1$ and $u \in E_0^\alpha$. If u is a non-trivial weak solution of* (1.1)*, then u is also a non-trivial solution of* (1.1).

Our main tools is an infinitely many critical points theorem [10] which is recalled below.

Theorem 2.11. *Let X be a reflexive real Banach space; $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) = \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma = \liminf_{r\to+\infty} \varphi(r), \quad \delta = \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

(1) If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either the functional $\Phi - \lambda\Psi$ has a global minimum, or there exists a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(2) If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either there exists a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or there exists a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.

3. Main results

We define the functional $\Phi, \Psi : E_0^\alpha \to R$ by setting, for every $u \in E_0^\alpha$,

$$\Phi(u) := \frac{1}{2}\|u\|_\alpha^2, \tag{3.1}$$

$$\Psi(u) := \int_0^T \Big[a(t)F(u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big]\,dt, \tag{3.2}$$

where $F(u) = \int_0^u f(s)ds$ and $G(t,u) = \int_0^u g(t,x)dx$. Clearly, Φ and Ψ are Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in E_0^\alpha$ are given by

$$\Phi'(u)v = \int_0^T {}_0D_t^\alpha u(t) \cdot {}_0D_t^\alpha v(t)dt,$$
$$\Psi'(u)v = \int_0^T \Big(a(t)f(u(t)) + \frac{\mu}{\lambda}g(t,u(t))\Big)\,v(t)dt$$

for every $v \in E_0^\alpha$. Hence, a critical point of $I_\lambda = \Phi - \lambda\Psi$, gives us a weak solution of (1.1), which is also a solution of (1.1) by Theorem 2.10.

If $\alpha > 1/2$, by Proposition 2.8 and (2.7), one has

$$\|u\|_\infty \le M\Big(\int_0^T |{}_0D_t^\alpha u(t)|^2dt\Big)^{1/2} = M\|u\|_\alpha, \quad u \in E_0^\alpha, \tag{3.3}$$

where

$$M = \frac{T^{\alpha-\frac12}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}. \tag{3.4}$$

Given a constant $0 < h < 1/2$, put

$$\begin{aligned} A(\alpha,h) := \frac{1}{2h^2T^2}\Big[&\frac{1+h^{3-2\alpha}+(1-h)^{3-2\alpha}}{3-2\alpha}T^{3-2\alpha} \\ &- 2\int_{(1-h)T}^T t^{1-\alpha}(t-(1-h)T)^{1-\alpha}dt - 2\int_{hT}^T t^{1-\alpha}(t-hT)^{1-\alpha}dt \\ &+ 2\int_{(1-h)T}^T (t-hT)^{1-\alpha}(t-(1-h)T)^{1-\alpha}dt\Big], \end{aligned}$$

$$K := \frac{\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T} a(t)dt}{2M^2A(\alpha,h)\int_0^T a(t)dt}, \tag{3.5}$$

$$\lambda_1 = \begin{cases} \frac{A(\alpha,h)}{\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T} a(t)dt\cdot\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}}, & \text{if } \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} < +\infty, \\ 0, & \text{if } \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} = +\infty \end{cases} \tag{3.6}$$

$$\lambda_2 = \frac{1}{2M^2\int_0^T a(t)dt\cdot\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}}, \tag{3.7}$$

where M as in (3.4).

Theorem 3.1. *Let $1/2 < \alpha \le 1$, $0 < h < 1/2$, $a : [0,T] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be two nonnegative continuous functions, and assume that*

$$0 < \liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} < K\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}, \tag{3.8}$$

where K is given in (3.5). For every $\lambda \in \Lambda :=]\lambda_1, \lambda_2[$ (λ_1 and λ_2 are given in (3.6) and (3.7) respectively) and for every $g \in C([0,T] \times \mathbb{R})$ such that

$$\begin{gathered} G(t,u) \geq 0, \quad \forall (t,u) \in [0,T] \times [0,+\infty[, \\ G_\infty := \limsup_{\xi \to +\infty} \frac{\int_0^T \max_{|x| \leq \xi} G(t,x)dt}{\xi^2} < +\infty, \end{gathered} \tag{3.9}$$

if we put

$$\mu_* := \frac{1}{2M^2 G_\infty}\Big(1 - 2M^2\lambda \int_0^T a(t)dt \cdot \lim\inf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}\Big),$$

with $\mu_* = +\infty$ when $G_\infty = 0$, then (1.1) possesses an unbounded sequence of solutions in E_0^α for every $\mu \in J := [0, \mu_*[$.

Proof. Our aim is to apply part (1) of Theorem 2.11. Let Φ, Ψ be the functionals defined in (3.1) and (3.2) respectively. By the Lemma 5.1 in [24], Φ is continuous and convex, so it is weakly sequentially lower semicontinuous, moreover, Φ is continuously Gâteaux differentiable and coercive. The functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, hence it is sequentially weakly upper semicontinuous. It is well known that the critical point of the functional $\Phi - \lambda\Psi$ in E_0^α is exactly the solution of (1.1).

Let ρ_n be a sequence of positive numbers such that $\lim_{n\to\infty} \rho_n = +\infty$ and

$$\lim_{n\to\infty} \frac{F(\rho_n)}{\rho_n^2} = \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$

Let $r_n = \frac{\rho_n^2}{2M^2}$ for all $n \in \mathbb{N}$. By (3.3), for all $v \in E_0^\alpha$ such that $\Phi(v) \leq r_n$, one has $\|v\|_\infty \leq \rho_n$. Thus,

$$\begin{aligned} \varphi(r_n) &= \inf_{u \in \Phi^{-1}(]-\infty, r_n[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r_n[)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}(]-\infty, r_n[)} \Psi(v)}{r_n} \\ &\leq 2M^2 \int_0^T a(t)dt \cdot \frac{F(\rho_n)}{\rho_n^2} + \frac{2M^2\mu}{\lambda} \frac{\int_0^T \max_{|\xi| \leq \rho_n} G(t,\xi)dt}{\rho_n^2}. \end{aligned}$$

So,

$$\gamma \leq \lim\inf_{n \to +\infty} \varphi(r_n) \leq 2M^2 \int_0^T a(t)dt \cdot \lim\inf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} + \frac{2M^2\mu}{\lambda} G_\infty < +\infty.$$

Thus, it is easy to verify that when $G_\infty > 0$, for every $\mu \in J$,

$$\gamma < 2M^2 \int_0^T a(t)dt \cdot \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} + \frac{2M^2\mu_*}{\lambda} G_\infty = \frac{1}{\lambda},$$

while, when $G_\infty = 0$, we have by $\lambda \in]\lambda_1, \lambda_2[$ that,

$$\gamma < 2M^2 \int_0^T a(t)dt \cdot \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{\lambda}.$$

Thus, we conclude that

$$\Lambda \subset]0, \frac{1}{\gamma}[,$$

by the definition of Λ. Now, we claim that the functional $\Phi - \lambda\Psi$ is unbounded from below. Let $\{\eta_n\}$ be a positive real sequence such that $\lim_{n\to\infty} \eta_n = +\infty$. We consider a function v_n defined by setting

$$v_n(t) = \begin{cases} \frac{\Gamma(2-\alpha)\eta_n}{hT} t, & t \in [0, hT[, \\ \Gamma(2-\alpha)\eta_n, & t \in [hT, (1-h)T], \\ \frac{\Gamma(2-\alpha)\eta_n}{hT}(T-t), & t \in](1-h)T, T], \end{cases} \tag{3.10}$$

where $0 < h < 1/2$. Clearly $v_n(0) = v_n(T) = 0$ and $v_n \in L^2[0,T]$. A direct calculation shows that

$${}_0D_t^\alpha v_n(t) = \begin{cases} \frac{\eta_n}{hT} t^{1-\alpha}, & t \in [0, hT[, \\ \frac{\eta_n}{hT}\left(t^{1-\alpha} - (t-hT)^{1-\alpha}\right), & t \in [hT, (1-h)T], \\ \frac{\eta_n}{hT}\left(t^{1-\alpha} - (t-hT)^{1-\alpha} - (t-(1-h)T)^{1-\alpha}\right), & t \in](1-h)T, T] \end{cases}$$

and

$$\begin{aligned}
&\int_0^T ({}_0D_t^\alpha v_n(t))^2 dt \\
&= \int_0^{hT} + \int_{hT}^{(1-h)T} + \int_{(1-h)T}^T ({}_0D_t^\alpha v_n(t)^2 dt \\
&= \frac{\eta_n^2}{h^2T^2}\Big[\int_0^T t^{2(1-\alpha)} dt + \int_{hT}^T (t-hT)^{2(1-\alpha)} dt \\
&\quad + \int_{(1-h)T}^T (t-(1-h)T)^{2(1-\alpha)} dt \\
&\quad - 2\int_{hT}^T t^{1-\alpha}(t-hT)^{1-\alpha} dt - 2\int_{(1-h)T}^T t^{1-\alpha}(t-(1-h)T)^{1-\alpha} dt \\
&\quad + 2\int_{(1-h)T}^T (t-hT)^{1-\alpha}(t-(1-h)T)^{1-\alpha} dt\Big] \\
&= \frac{\eta_n^2}{h^2T^2}\Big[\frac{1+h^{3-2\alpha}+(1-h)^{3-2\alpha}}{3-2\alpha} T^{3-2\alpha} \\
&\quad - 2\int_{(1-h)T}^T t^{1-\alpha}(t-(1-h)T)^{1-\alpha} dt - 2\int_{hT}^T t^{1-\alpha}(t-hT)^{1-\alpha} dt \\
&\quad + 2\int_{(1-h)T}^T (t-hT)^{1-\alpha}(t-(1-h)T)^{1-\alpha} dt\Big] \\
&= 2A(\alpha,h)\eta_n^2,
\end{aligned}$$

for each $n \in \mathbb{N}$. Thus, $v_n \in E_0^\alpha$, and

$$\Phi(v_n) = \frac{1}{2}\|v_n\|_\alpha^2 = A(\alpha,h)\eta_n^2. \tag{3.11}$$

Putting together (3.9) and the nonnegative of f, one has

$$\begin{aligned} \Psi(v_n) &= \int_0^T \big[a(t)F(v_n(t)) + \frac{\mu}{\lambda}G(t, v_n(t))\big]dt \\ &\geq \int_0^T a(t)F(v_n(t))dt \\ &= \int_0^{hT} a(t)F\Big(\frac{\Gamma(2-\alpha)\eta_n}{hT}t\Big)dt + \int_{hT}^{(1-h)T} a(t)F\Big(\Gamma(2-\alpha)\eta_n\Big)dt \\ &\quad + \int_{(1-h)T}^{T} a(t)F\Big(\frac{\Gamma(2-\alpha)\eta_n}{hT}(T-t)\Big)dt \\ &\geq F(\Gamma(2-\alpha)\eta_n)\int_{hT}^{(1-h)T} a(t)dt. \end{aligned} \tag{3.12}$$

Therefore, from (3.11) and (3.12) we achieve

$$\Phi(v_n) - \lambda\Psi(v_n) \leq A(\alpha, h)\eta_n^2 - \lambda F\left(\Gamma(2-\alpha)\eta_n\right)\int_{hT}^{(1-h)T} a(t)dt.$$

From (3.8), we know that $\lambda_1 < \lambda_2$. Let

$$B = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}. \tag{3.13}$$

If $B < +\infty$, we set $\epsilon \in \big]0, B - \frac{A(\alpha,h)}{\lambda\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T} a(t)dt}\big[$, then from (3.13) there exists N_1 such that

$$F\left(\Gamma(2-\alpha)\eta_n\right) > (B-\epsilon)\Gamma^2(2-\alpha)\eta_n^2, \quad \forall n > N_1.$$

Hence,

$$\begin{aligned} \Phi(v_n) - \lambda\Psi(v_n) &< A(\alpha, h)\eta_n^2 - \lambda(B-\epsilon)\Gamma^2(2-\alpha)\eta_n^2\int_{hT}^{(1-h)T} a(t)dt \\ &= \eta_n^2\Big(A(\alpha, h) - \lambda(B-\epsilon)\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T} a(t)dt\Big), \end{aligned} \tag{3.14}$$

for $n > N_1$. From the choice of ϵ, we have

$$\lim_{n\to+\infty}(\Phi(v_n) - \lambda\Psi(v_n)) = -\infty.$$

On the other hand, if $B = +\infty$, we fix $\Theta > \frac{A(\alpha,h)}{\lambda\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T} a(t)dt}$, then from (3.13) there exists N_Θ such that

$$F(\Gamma(2-\alpha)\eta_n) > \Theta\Gamma^2(2-\alpha)\eta_n^2, \quad \forall n > N_\Theta.$$

Therefore,

$$\begin{aligned} \Phi(v_n) - \lambda\Psi(v_n) &\leq A(\alpha, h)\eta_n^2 - \lambda F\left(\Gamma(2-\alpha)\eta_n\right)\int_{hT}^{(1-h)T} a(t)dt \\ &< A(\alpha, h)\eta_n^2 - \lambda\Theta\Gamma^2(2-\alpha)\eta_n^2\int_{hT}^{(1-h)T} a(t)dt \\ &= \eta_n^2\Big(A(\alpha, h) - \lambda\Theta\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T} a(t)dt\Big), \quad n > N_\Theta. \end{aligned}$$

Taking into account the choice of Θ, one has

$$\lim_{n\to+\infty}(\Phi(v_n)-\lambda\Psi(v_n))=-\infty.$$

By Theorem 2.11, the functional $\Phi-\lambda\Psi$ admits a sequence u_n of critical points such that $\lim_{n\to+\infty}\Phi(u_n)=+\infty$. It follows from (3.1) that

$$\|u_n\|_\alpha=\sqrt{2\Phi(u_n)},$$

which implies $\lim_{n\to+\infty}\|u_n\|_\alpha=+\infty$. This completes the proof in view of the relation between the critical points of $\Phi-\lambda\Psi$ and the solutions of problem (1.1) pointed out in Theorem 2.10. □

In the following, arguing in a similar way, but applying case (2) of Theorem 2.11, we can establish the existence of infinitely many solutions of (1.1) converging at zero. For convenience, let

$$\lambda_3=\begin{cases}\frac{A(\alpha,h)}{\Gamma^2(2-\alpha)\int_{hT}^{(1-h)T}a(t)dt\cdot\limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^2}}, & \text{if } \limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^2}<+\infty,\\ 0, & \text{if } \limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^2}=+\infty,\end{cases}\tag{3.15}$$

$$\lambda_4=\frac{1}{2M^2\int_0^T a(t)dt\cdot\liminf_{\xi\to0^+}\frac{F(\xi)}{\xi^2}}.\tag{3.16}$$

Theorem 3.2. *Let $1/2<\alpha\le1$, $0<h<1/2$, $a:[0,T]\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be two nonnegative continuous functions, and assume that*

$$0<\liminf_{\xi\to0^+}\frac{F(\xi)}{\xi^2}<K\limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^2},\tag{3.17}$$

where K is given in (3.5). For every $\lambda\in\Lambda_1:=]\lambda_3,\lambda_4[$ (λ_3 and λ_4 are given in (3.15) and (3.16) respectively) and for every $g\in C([0,T]\times\mathbb{R})$ such that

$$\begin{gathered}G(t,u)\ge0\quad\text{for all }(t,u)\in[0,T]\times[0,\tau],\text{for some }\tau>0,\\ G_0:=\limsup_{\xi\to0^+}\frac{\int_0^T\max_{|x|\le\xi}G(t,x)dt}{\xi^2}<+\infty,\end{gathered}\tag{3.18}$$

if we put

$$\mu_*:=\frac{1}{2M^2G_0}\Big(1-2M^2\lambda\int_0^T a(t)dt\cdot\lim\inf_{\xi\to0^+}\frac{F(\xi)}{\xi^2}\Big),$$

with $\mu_*=+\infty$ when $G_0=0$, then (1.1) admits a sequence $\{u_n\}$ of solutions such that $u_n\to0$ strongly in E_0^α for every $\mu\in J:=[0,\mu_*[$.

Proof. Fix $\lambda\in\Lambda_1$, and pick $\mu\in[0,\mu_*[$. We want to apply Theorem 2.11(2), with $X=E_0^\alpha$, and Φ,Ψ be the functionals defined in (3.1) and (3.2) respectively. Let k_n be a sequence of positive numbers such that $\lim_{n\to\infty}k_n=0$ and

$$\lim_{n\to\infty}\frac{F(k_n)}{k_n^2}=\lim\inf_{\xi\to0^+}\frac{F(\xi)}{\xi^2}.$$

Putting $r_n=\frac{k_n^2}{2M^2}$ for all $n\in\mathbb{N}$ and working as in the proof of Theorem 3.1, it follows that $\delta<+\infty$, where δ is as defined in Theorem 2.11, and also $\Lambda_1\subset\big]0,\frac{1}{\delta}\big[$. Now we claim that

$$\Phi-\lambda\Psi\quad\text{does not have a local minimum at zero.}\tag{3.19}$$

Let $\{\eta_n\}$ be a sequence of positive numbers in $]0,\eta[$ $(\eta>0)$ such that $\eta_n \to 0$, and $\{v_n\}$ be the sequence in E_0^α defined in (3.10). From (3.18) and the nonnegative of f one has that (3.12) holds.

By condition (3.17), we know that $\lambda_3 < \lambda_4$. Let

$$B_1 = \limsup_{\xi\to+0^+} \frac{F(\xi)}{\xi^2}. \tag{3.20}$$

For the case : $B_1 < +\infty$, one has (3.14) holds. From the choice of ε, we have

$$\lim_{n\to+\infty}(\Phi(v_n)-\lambda\Psi(v_n)) < 0 = \Phi(0)-\lambda\Psi(0).$$

for each $n\in\mathbb{N}$ large enough, which implies (3.19) holds in view of fact that $\|v_n\|\to 0$. Similarly, for the case $B_1=+\infty$, one has (3.19) holds.

Observing that $\min_X \Phi = \Phi(0)$, the conclusion follows from Theorem 2.11 case (2). $\square$

Finally, we give an example to show the effectiveness of the results obtained here.

Example 3.3. Let $\alpha = 0.8$ and $T=1$. Consider the following boundary-value problem

$$\begin{gathered} {}_tD_1^{0.8}({}_0D_t^{0.8}u(t)) = \lambda a(t)f(u(t)) + \mu g(t,u(t)), \quad \text{a.e. } t\in[0,1], \\ u(0)=u(1)=0, \end{gathered} \tag{3.21}$$

where $a:[0,1]\to\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$ and $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are the nonnegative and continuous functions defined as follows

$$a(t) = \begin{cases} \frac{t}{3}, & t\in[0,\frac13[\\ -12\big(t-\frac12\big)^2+\frac49, & t\in[\frac13,\frac23[\\ \frac{1-t}{3}, & t\in[\frac23,1], \end{cases}$$

$$f(x) = \begin{cases} |x|(3+2\cos(\ln(|x|))-\sin(\ln(|x|))), & x\neq 0 \\ 0, & x=0, \end{cases}$$

$$g(t,x) = \begin{cases} \frac45 + t\sqrt[3]{x}, & x\le 1 \\ t+\frac{x}{5}(3+2\sin(\ln x)+\cos(\ln x)), & x>1. \end{cases}$$

Then, for every $\lambda\in]5.4503, 5.4911[$ and $\mu\in[0, 0.8132(1-01821\lambda)[$, BVP (3.21) admits an unbounded sequence of solutions in $E_0^{0.8}$. In fact, we have

$$F(x) = \begin{cases} x^2(\frac32+\cos(\ln x)), & x>0 \\ 0, & x=0 \\ -x^2(\frac32+\cos(\ln(|x|))), & x<0, \end{cases}$$

and

$$G(t,x) = \begin{cases} \frac45 x + \frac34 t x^{\frac43}, & x\le 1 \\ \frac12 - \frac{t}{4} + tx + \frac{x^2}{5}(\frac32+\sin(\ln x)), & x>1. \end{cases}$$

It is easy to verify that

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} = \frac12, \quad \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} = \frac52,$$

$$G_\infty = \limsup_{\xi\to+\infty}\frac{\int_0^1 \max_{|x|\le\xi} G(t,x)dt}{\xi^2} = \frac12.$$

Moreover, it is easy to calculate that $M = 1.1089$, $A(\alpha, h) = A(0.8, 1/3) = 1.2762$ (here $h = 1/3$) and

$$\frac{\liminf_{\xi\to+\infty} F(\xi)/\xi^2}{\limsup_{\xi\to+\infty} F(\xi)/\xi^2} = 0.2 < 0.2015 = \frac{\Gamma^2(1.2)\int_{1/3}^{2/3} a(t)dt}{2M^2 A(0.8,1/3)\int_0^1 a(t)dt} = K,$$

which implies that condition (3.8) holds. Obviously, condition (3.9) holds. Thus, by Theorem 3.1, for each $\lambda \in]\lambda_1, \lambda_2[=]5.4503, 5.4911[$ and $\mu \in [0, 0.8132(1 - 0.1821\lambda)[$, the problem (3.21) has an unbounded sequence of solutions in $E_0^{0.8}$.

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