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GENERALIZED BOHL-PERRON PRINCIPLE FOR DIFFERENTIAL EQUATIONS WITH DELAY IN A BANACH SPACES

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Abstract. We consider a linear homogeneous functional differential equation with delay in a Banach space. It is proved that if the corresponding non-homogeneous equation, with an arbitrary free term bounded on the positive half-line and with the zero initial condition, has a bounded solution, then the considered homogeneous equation is exponentially stable.

1. Introduction and statement of the main result

Recall that the Bohl-Perron principle states that the homogeneous ordinary differential equation (ODE) $dy/dt = A(t)y$ $(t \geq 0)$ with a variable $n \times n$-matrix $A(t)$, bounded on $[0, \infty)$ is exponentially stable, provided the nonhomogeneous ODE $dx/dt = A(t)x + f(t)$ with the zero initial condition has a bounded solution for any bounded vector valued function f [7].

In [18, Theorem 4.15], the Bohl-Perron principle was generalized to a class of retarded systems with finite delays; also the asymptotic (not exponential) stability was proved. The result from [18] was a considerable development afterwards, cf. the book [3] and the very interesting papers [4, 5], in which the generalized Bohl-Perron principle was effectively used for the stability analysis of the first and second order scalar equations. In particular, in [4] the scalar non-autonomous linear functional differential equation $\dot{x}(t) + a(t)x(h(t)) = 0$ is considered. The authors give sharp conditions for exponential stability, which are suitable in the case that the coefficient function $a(t)$ is periodic, almost periodic or asymptotically almost periodic, as often encountered in applications. In [5], the authors provide sufficient conditions for the stability of rather general second-order delay differential equations. In [15, 16] a result similar to the Bohl-Perron principle has been derived in terms of the norm of the space L^p, which is called the L^p-version of the generalized Bohl-Perron principle.

In this article, we extend the Bohl-Perron principle to a class of functional differential equations with delay in a Banach space. In Section 3 below, we show that our results can be effectively used for the stability analysis. As it is well-known, the basic method for the stability analysis of functional differential equations is the

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direct Lyapunov method. By this method very strong results are obtained. But finding Lyapunov's type functionals for nonautonomous vector equations with delay is usually difficult. In Section 3 we suggest explicit sharp stability conditions, which supplement the well-known results on stability of equations with delay in a Banach space; see [1, 2, 9, 19, 20, 21] and references given therein.

Let X be a complex Banach space with a norm $\|\cdot\|_X$ and the unit operator I. Denote by $C(\omega) \equiv C(\omega, X)$ the space of continuous functions u defined on a set $\omega \subseteq \mathbb{R}$ with values in X and the finite sup-norm $\|\cdot\|_{C(\omega)}$. For a bounded linear operator T acting from X into a normed space Y we put $\|T\|_{X\to Y} = \sup_{u\in X} \|Tu\|_Y/\|u\|_X$.

Let $A(t)$ be a linear generally unbounded operator in X with a constant dense domain $\mathrm{Dom}(A)$. In X, for a positive constant $\eta < \infty$ consider the equation

$$\dot y(t) = A(t)y(t) + \int_0^\eta B(t,s)y(t-s)ds + \sum_1^m B_k(t)y(t-h_k(t)), \tag{1.1}$$

where $\dot y(t)$ is a strong derivative of y; $B_k(t)$ $(k = 1,\dots,m)$ are bounded continuous operator functions on $[0,\infty)$; $B(t,s)$ is an operator function defined and bounded on $[0,\infty)\times[0,\eta]$, which is continuous in t and integrable in s; $0 \le h_k(t) \le \eta$ are continuous functions. Let the initial condition be

$$y(t) = \phi(t)(-\eta \le t \le 0) \tag{1.2}$$

for a given $\phi \in C(-\eta,0) \cap \mathrm{Dom}(A)$. For $w \in C(-\eta,\infty)$, put

$$Ew = \int_0^\eta B(t,s)w(t-s)ds + \sum_1^m B_k(t)w(t-h_k(t))\,.$$

Then (1.1) takes the form

$$\dot y(t) = A(t)y(t) + Ey(t). \tag{1.3}$$

It is assumed that $A(t)$ generates a strongly continuous evolution family $\{U(t,s)\}$ $(t \ge s \ge 0)$ of bounded operators in X. That is, $U(t,s)$ is the evolution operator of the equation

$$\dot\zeta(t) = A(t)\zeta(t) \tag{1.4}$$

cf. [6]. Following the Browder terminology [17], a continuous function y satisfying

$$y(t) = U(t,0)\phi(0) + \int_0^t U(t,t_1)Ey(t_1)dt_1 \tag{1.5}$$

and (1.2) we will be called a *mild solution* to (1.1), (1.2). Consider also the nonhomogeneous equation

$$\dot x(t) = A(t)x(t) + Ex(t) + f(t), \quad t > 0 \tag{1.6}$$

with a given function $f(t) \in C(0,\infty)$, and the zero initial condition

$$x(t) = 0, \quad -\eta \le t \le 0. \tag{1.7}$$

Then a continuous function x satisfying

$$x(t) = \int_0^t U(t,t_1)(Ex(t_1) + f(t_1))dt_1 \tag{1.8}$$

and (1.7) will be called a mild solution to (1.6), (1.7). Below we show that , problems (1.1), (1.2) and (1.6), (1.7) have unique mild solutions.

We will say that (1.1) is exponentially stable, if there are positive constants M_1, ϵ, such that $\|y(t)\| \le M_1 e^{-\epsilon t} \|\phi\|_{C(-\eta,0)}$ $(t \ge 0)$ for any mild solution $y(t)$ of (1.1), (1.2).

We assume that there are positive constants α_0 and M, such that

$$\|U(t,s)\|_X \le M e^{-\alpha_0(t-s)} \quad \forall t \ge s \ge 0, \tag{1.9}$$

$$A(t)z \in C(0,\infty) \text{ for any } z \in \mathrm{Dom}(A). \tag{1.10}$$

Theorem 1.1. *If conditions* (1.9) *and* (1.10) *hold, and for any* $f \in C(0,\infty)$*, problem* (1.6)*,* (1.7) *has a bounded mild solution on* $[0,\infty)$*, then* (1.1) *is exponentially stable.*

This theorem is proved in the next section.

Suppose $1 \le p < \infty$, then for an exponentially bounded and strongly continuous evolution family $U(t,s)$ of bounded linear operators acting in X, the following condition is equivalent to (1.9): there exists a constant $M_p > 0$, such that

$$\sup_{s\ge 0} \int_s^\infty \|U(t,s)z\|_X^p dt \le M_p \|z\|_X^p, \forall z \in X, \tag{1.11}$$

cf. [6, p. 75]. Other conditions equivalent to (1.9) can be found in [6, p. 77].

2. Proofs

It is not difficult to check that for all $\tau > 0$,

$$\|Ew\|_{C(0,\tau)} \le v_0 \|w\|_{C(-\eta,\tau)} \quad \text{for } w \in C(-\eta,\tau), \tag{2.1}$$

where

$$v_0 = \sup_t \Big(\int_0^\eta \|B(t,s)\|_X ds + \sum_1^m \|B_k(t)\|_X \Big).$$

For brevity, in this section, sometimes we use $\|\cdot\|_{C(0,\tau)} = |\cdot|_\tau$ for $\tau > 0$. Let us define the operator V by

$$Vw(t) = \int_0^t U(t,t_1)(Ew)(t_1)dt_1$$

for any integrable function $w(t)$ $(t \ge 0)$ with values in X. According to (1.9) and (2.1) it is easy to check that for any finite T and $u \in C(-\eta,T)$ with $u(t) = 0$ for $t \le 0$, V satisfies

$$\begin{aligned} |V^k u|_T &\le M v_0 \int_0^T |V^{k-1}u|_t dt \\ &\le (Mv_0)^2 \int_0^T \int_0^t |V^{k-2}u|_{t_1} dt_1 dt \\ &\le \dots \le \frac{(TMv_0)^k}{k!} |u|_T. \end{aligned}$$

Hence, it follows that

Corollary 2.1. *For any continuous* f*, problem* (1.6)*,* (1.7) *has a unique mild solution* $x(t)$*, which can be represented as*

$$x = \sum_1^\infty V^k f_1, \quad \textit{where } f_1(t) = \int_0^t U(t,t_1) f(t_1) dt_1. \tag{2.2}$$

Lemma 2.2. *Under condition* (1.10)*, if for any* $f \in C(0,\infty)$*, problem* (1.6)*,* (1.7) *has a bounded mild solution on* $[0,\infty)$*, then for any* $\phi \in C(-\eta,0)\cap \mathrm{Dom}(A)$ *problem* (1.1)*,* (1.2) *has a unique mild solution bounded on* $(0,\infty)$*.*

Proof. Put

$$\hat\phi(t) = \begin{cases} \phi(0) & \text{if } t \ge 0, \\ \phi(t) & \text{if } -\eta \le t < 0\,. \end{cases}$$

Then $d\hat\phi(t)/dt = 0$ for $t \ge 0$. Consider the equation

$$dx(t)/dt = A(t)(x(t) + \phi(0)) + E(x(t) + \hat\phi(t))\ (t > 0),$$

with condition (1.7). According to (1.10) and (2.1), $A(t)\phi(0) + E\hat\phi(t) \in C(-\eta,\infty)$. Due to the hypotheses of this lemma, the latter equation has a solution $x \in C(0,\infty)$. Then the function $y(t) = x(t) + \hat\phi(t) \in C(-\eta,\infty)$ and satisfies problem (1.1), (1.2). As claimed. □

Proof of Theorem 1.1. Substituting

$$y(t) = y_\epsilon(t)e^{-\epsilon t} \tag{2.3}$$

with an $\epsilon > 0$ in (1.1), we obtain

$$dy_\epsilon(t)/dt = (A(t) + \epsilon)y_\epsilon(t) + E_\epsilon y_\epsilon(t) \quad (t > 0), \tag{2.4}$$

where

$$E_\epsilon w(t) = \int_0^\eta B(t,s)e^{\epsilon s}w(t-s)ds + \sum_1^m e^{\epsilon h_k(t)}B_k(t)w(t-h_k(t))$$

for a continuous w. It is easy to check that $E_\epsilon \to E$ in the operator norm of $C(0,\infty)$ as $\epsilon \to 0$.

Furthermore, due to (2.2) we obtain $x = \hat G f$, where

$$\hat G := (I-V)^{-1}W = \sum_1^\infty V^k W, \quad \text{with } Wf(t) = \int_0^t U(t,t_1)f(t_1)dt_1.$$

By the hypothesis of the theorem, we have

$$x = \hat G f \in C(0,\infty) \quad \text{for any } f \in C(0,\infty).$$

So $\hat G$ is defined on the whole space $C(0,\infty)$. It is closed, since problem (1.6), (1.7) has a unique solution. Therefore, $\hat G$ is bounded according to the Closed Graph theorem [8].

Consider now the equation

$$\dot x_\epsilon(t) = (A(t) + \epsilon I)x_\epsilon(t) + E_\epsilon x_\epsilon(t) + f(t) \tag{2.5}$$

with the zero initial condition. Its mild solution is defined by

$$x_\epsilon(t) = \int_0^t U(t,t_1)(\epsilon x_\epsilon(t_1) + E_\epsilon x_\epsilon(t_1)dt_1) + f_1 \tag{2.6}$$

where f_1 is defined as in (2.2). For solutions x and x_ϵ of (1.8) and (2.6), respectively, we obtain

$$\begin{aligned} x_\epsilon(t) - x(t) &= \int_0^t U(t,t_1)(\epsilon x_\epsilon(t_1) + E_\epsilon x_\epsilon(t_1) - Ex(t_1))dt_1 \\ &= V(x_\epsilon(t) - x(t)) + f_\epsilon(t), \end{aligned}$$

where

$$f_\epsilon(t) = \int_0^t U(t,t_1)(\epsilon x_\epsilon(t_1) + (E_\epsilon - E)x_\epsilon(t_1))dt_1.$$

Consequently,

$$x - x_\epsilon = \hat{G} f_\epsilon. \tag{2.7}$$

However $|\hat{G}|_T \le \|\hat{G}\|_{C(0,\infty)}$, and

$$\begin{aligned} |(E_\epsilon - E)w|_T &\le \sup_{t\ge 0}\Big(\int_0^\eta \|B(t,s)\|_X |e^{\epsilon s} - 1|\, \|w(t-s)\|_X ds \\ &\quad + \sum_1^m |e^{\epsilon h_k(t)} - 1| \|B_k(t) w(t - h_k(t))\|_X\Big) \\ &\le v_0(e^{\epsilon\eta} - 1)|w|_T. \end{aligned}$$

By (1.9) and (2.1), $|V|_T \le M v_0/\alpha_0$. So $|f_\epsilon|_T \le |x_\epsilon|_T(\epsilon + M v_0 \alpha_0^{-1}(e^{\epsilon\eta} - 1))$ and

$$|x_\epsilon|_T \le |x|_T + \|\hat{G}\|_{C(0,\infty)} |x_\epsilon|_T (\epsilon + M v_0 \alpha_0^{-1}(e^{\epsilon\eta} - 1)).$$

Thus, for a sufficiently small ϵ,

$$|x_\epsilon|_T \le \frac{|x|_T}{1 - \|\hat{G}\|(\epsilon + M\|\hat{G}\|_{C(0,\infty)} v_0 \alpha_0^{-1}(e^{\epsilon\eta} - 1))}.$$

Letting $T \to \infty$, we obtain $x_\epsilon \in C(0,\infty)$. Hence, by Lemma 2.2, a solution y_ϵ of (2.4) is bounded. Now (2.3) proves the exponential stability, as claimed. □

3. Equations in a Hilbert space

In this section we illustrate Theorem 1.1 in a Hilbert space. Let $X = H$ be a Hilbert space with a scalar product $(.,.)$, and the norm $\|\cdot\|_H = \sqrt{(.,.)}$. Let $A(t)$ map $\mathrm{Dom}(A)$ into itself and

$$\sup_{z \in \mathrm{Dom}(A)} \frac{Re(A(t)z,z)}{(z,z)} \le -\alpha(t) \le -\alpha_0 \quad \forall t \ge 0, \tag{3.1}$$

where $\alpha(t)$ is a positive continuous function and α_0 is a positive constant. From (1.4) it follows that

$$\frac{d}{dt}(\zeta(t),\zeta(t)) = (\dot{\zeta}(t),\zeta(t)) + (\zeta(t),\dot{\zeta}(t)) = 2Re(\dot{\zeta}(t),\zeta(t)) = 2Re(A(t)\zeta(t),\zeta(t)).$$

Thus

$$\frac{d}{dt}(\zeta(t),\zeta(t)) = 2\|\zeta(t)\|_H \frac{d}{dt}\|\zeta(t)\|_H \le -2\alpha(t)(\zeta(t),\zeta(t)),$$

or

$$\frac{d}{dt}\|\zeta(t)\|_H \le -\alpha(t)\|\zeta(t)\|_H.$$

Solving this inequality with $\zeta(s) \in \mathrm{Dom}(A)$, we obtain

$$\|U(t,s)\|_H \le e^{-\int_s^t \alpha(\tau)d\tau} \le e^{-\alpha_0(t-s)}, \quad \text{for } t \ge s \ge 0.$$

Hence,

$$\sup_t \int_0^t \|U(t,t_1)\|_H dt_1 \le J,$$

where

$$J := \sup_t \int_0^t e^{-\int_{t_1}^t \alpha(\tau)d\tau} dt_1.$$

From (1.8) and (2.1) it follows that

$$\|x\|_{C(0,\infty)} \le v_0 J \|x\|_{C(0,\infty)} + \|f_1\|_{C(0,\infty)}.$$

Consequently, if

$$v_0 J < 1, \tag{3.2}$$

then

$$\|x\|_{C(0,\infty)} \le \frac{\|f_1\|_{C(0,\infty)}}{1 - v_0 J}.$$

Using Theorem 1.1 we arrive at the following result.

Corollary 3.1. *Suppose E maps $\operatorname{Dom}(A)$ into itself, and conditions* (1.10), (3.1), *and* (3.2) *hold. Then* (1.1) *is exponentially stable.*

Some additional stability criteria can be found, for instance, in [19, 14, 11]. In particular, in [19], the authors prove important results on the asymptotic behavior of solutions for semilinear autonomous functional differential equations with infinite delay. In [14], the authors considered equations with unbounded history response. Article [11] is devoted to the stability of linear time-variant functional differential equations in a Hilbert space. The generalized Aizerman-Myshkis problem for abstract differential-delay equations is considered in [12, 13]. A criterion for global stability of parabolic systems with delay is suggested in [10].

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