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EXISTENCE OF SOLUTIONS TO NON-LOCAL PROBLEMS FOR PARABOLIC-HYPERBOLIC EQUATIONS WITH THREE LINES OF TYPE CHANGING

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Abstract. In this work, we study a boundary problem with non-local conditions, by relating values of the unknown function with various characteristics. The parabolic-hyperbolic equation with three lines of type changing is equivalently reduced to a system of Volterra integral equations of the second kind.

1. Introduction

Consider an equation

$$\begin{gathered} u_{xx} - u_y = 0, \quad (x,y) \in \Omega_0, \\ u_{xx} - u_{yy}, \quad (x,y) \in \Omega_i \ i = 1,2,3 \end{gathered} \tag{1.1}$$

in the domain $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup AB \cup AA_0 \cup BB_0$; see Figure 1.

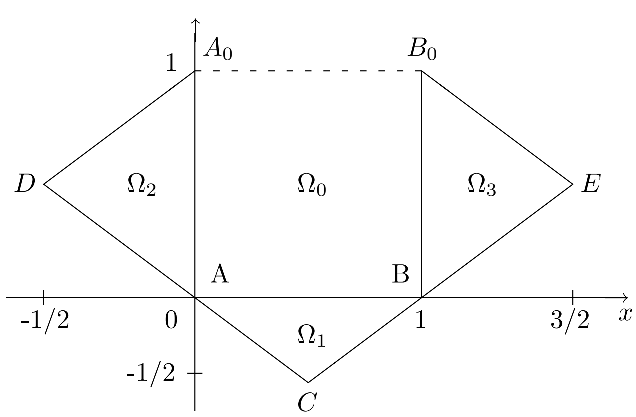

Figure 1. Domain Ω

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Problem AS. Find a regular solution of equation (1.1) in the domain Ω, satisfying the following conditions:

$$a_1(t)u(-t,t)+a_2(t)u(t,-t)=a_3(t), \quad 0\leqslant t\leqslant \frac{1}{2}, \tag{1.2}$$

$$b_1(t)u(t,t-1)+b_2(t)u(2-t,1-t)=b_3(t), \quad \frac{1}{2}\leqslant t\leqslant 1, \tag{1.3}$$

$$c_1(t)(u_x+u_y)(t-1,t)+c_2(t)(u_x-u_y)(2-t,t)=c_3(t), \quad \frac{1}{2}< t< 1. \tag{1.4}$$

Here $a_i(t), b_i(t), c_i(t)$ $(i=1,2,3)$ are given functions, such that

$$a_1(0)+a_2(0)\neq 0, \quad b_1(1)+b_2(1)\neq 0, \quad a_1^2(t)+a_2^2(t)>0,$$
$$b_1^2(t)+b_2^2(t)>0, \quad c_1^2(t)+c_2^2(t)>0, \quad a_1^2+b_2^2>0, \quad a_2^2+b_1^2>0.$$

Note that problem AS is a generalization of the following problems:

Case A $a_1\equiv 0$.
- (1) $a_2, b_1, b_2, c_1, c_2\neq 0$,
- (2) $b_1\equiv 0, a_2, b_2, c_1, c_2\neq 0$,
- (3) $c_2\equiv 0, a_2, b_1, b_2, c_1\neq 0$,
- (4) $b_1\equiv 0, c_2\equiv 0, a_2, b_2, c_1\neq 0$;

Case B $a_2\equiv 0$.
- (1) $a_1, b_1, b_2, c_1, c_2\neq 0$,
- (2) $b_2\equiv 0, a_1, b_1, c_1, c_2\neq 0$,
- (3) $c_1\equiv 0, a_1, b_1, b_2, c_2\neq 0$,
- (4) $b_2\equiv 0, c_1\equiv 0, a_1, b_1, c_2\neq 0$;

Case C $b_1\equiv 0$.
- (1) $a_1, a_2, b_2, c_1, c_2\neq 0$,
- (2) $c_2\equiv 0, a_1, a_2, b_2, c_1\neq 0$;

Case D $b_2\equiv 0$.
- (1) $a_1, a_2, b_1, c_1, c_2\neq 0$,
- (2) $c_1\equiv 0, a_1, a_2, b_1, c_2\neq 0$;

Case E $c_1\equiv 0$. $a_1, a_2, b_1, b_2, c_2\neq 0$;
Case F $c_2\equiv 0$. $a_1, a_2, b_1, b_2, c_1\neq 0$.

Also note that cases A4 and B4 were studied in [9]. Other cases were not investigated, and the main result of this paper is true for these particular cases.

Boundary problems for parabolic-hyperbolic equations with two lines of type changing were investigated in [1, 6, 7, 8], and with three lines of type changing in [2, 3]. The main point in this present work is the non-local condition, which relates values of the unknown function with various characteristics. It makes very difficult the reduction of the considered problem to a system of integral equations, we need a special algorithm for solving this problem.

2. Main results

In the domain Ω_1 solution of the Cauchy problem with initial data $u(x,0)=\tau_1(x)$, $u_y(x,0)=\nu_1(x)$ can be represented, as in [4], by

$$2u(x,y)=\tau_1(x+y)+\tau_1(x-y)+\int_{x-y}^{x+y}\nu_1(z)dz. \tag{2.1}$$

Assuming that in condition (1.2),

$$u(-t,t)=\varphi_1(t), \quad 0\leqslant t\leqslant \frac{1}{2}, \tag{2.2}$$

from (2.1), we find that

$$\tau_1'(t)=\nu_1(t)+\Big(\frac{2[a_3(\frac{t}{2})-a_1(\frac{t}{2})\varphi_1(\frac{t}{2})]}{a_2(\frac{t}{2})}\Big)', \quad 0<t<1. \tag{2.3}$$

In condition (1.3) introduce

$$u(2-t,1-t)=\varphi_2(t), \quad \frac{1}{2}\leqslant t\leqslant 1, \tag{2.4}$$

and from (2.1), we obtain

$$\tau_1'(t)=-\nu_1(t)+\Big(\frac{2[b_3(\frac{t+1}{2})-b_2(\frac{t+1}{2})\varphi_2(\frac{t+}{2})]}{b_1(\frac{t+1}{2})}\Big)', \quad 0<t<1. \tag{2.5}$$

From (2.3) and (2.5), it follows that

$$\tau_1'(t)=\Big(\frac{a_3(\frac{t}{2})-a_1(\frac{t}{2})\varphi_1(\frac{t}{2})}{a_2(\frac{t}{2})}\Big)'+\Big(\frac{b_3(\frac{t+1}{2})-b_2(\frac{t+1}{2})\varphi_2(\frac{t+1}{2})}{b_1(\frac{t+1}{2})}\Big)', \quad 0<t<1. \tag{2.6}$$

The solution of the Cauchy problem in the domain Ω_2, with given data $u(0,y)=\tau_2(y)$, $u_x(0,y)=\nu_2(y)$, is written as follows [4],

$$2u(x,y)=\tau_2(y+x)+\tau_2(y-x)+\int_{y-x}^{y+x}\nu_2(z)dz. \tag{2.7}$$

Considering (2.2) from (2.7) we obtain

$$\tau_2'(t)=\nu_2(t)+\varphi_1'(\frac{t}{2}), \quad 0<t<1. \tag{2.8}$$

In condition (1.4) introduce another designation

$$(u_x-u_y)(2-t,t)=\varphi_3(t), \quad \frac{1}{2}<t<1. \tag{2.9}$$

Then from (2.7) we obtain

$$\frac{c_3(\frac{t+1}{2})-c_2(\frac{t+1}{2})\varphi_3(\frac{t+1}{2})}{c_1(\frac{t+1}{2})}=\tau_2'(t)+\nu_2(t),\ 0<t<1. \tag{2.10}$$

From (2.8) and (2.10) we deduce

$$2\tau_2'(t)=\varphi_1'(\frac{t}{2})+\frac{c_3(\frac{t+1}{2})-c_2(\frac{t+1}{2})\varphi_3(\frac{t+1}{2})}{c_1(\frac{t+1}{2})}, \quad 0<t<1. \tag{2.11}$$

The solution of the Cauchy problem with data $u(1,y)=\tau_3(y)$, $u_x(1,y)=\nu_3(y)$ in the domain Ω_3 has a form [4]

$$2u(x,y)=\tau_3(y+x-1)+\tau_2(y-x+1)+\int_{y-x+1}^{y+x-1}\nu_3(z)dz. \tag{2.12}$$

Using (2.4) and (2.9) from (2.12), after some evaluations one can get

$$2\tau_3'(t)=-\varphi_2'(\frac{2-t}{2})-\varphi_3(\frac{t+1}{2}), \quad 0<t<1. \tag{2.13}$$

Further, from the equation (1.1) we pass to the limit at $y \to +0$ and considering (2.3) we find

$$\tau_1''(t) - \tau_1'(t) = -\Big(\frac{2\left[a_3(\frac{t}{2}) - a_1(\frac{t}{2})\varphi_1(\frac{t}{2})\right]}{a_2(\frac{t}{2})}\Big)'. \tag{2.14}$$

The solution of (2.14) with the conditions

$$\tau_1(0) = \frac{a_3(0)}{a_1(0)+a_2(0)}, \quad \tau_1(1) = \frac{b_3(1)}{b_1(1)+b_2(1)}, \tag{2.15}$$

which is deduced from (1.2) and (1.3), can be represented as [5]

$$\begin{aligned} \tau_1(x) &= \frac{a_3(0)}{a_1(0)+a_2(0)} + x\Big[\frac{b_3(1)}{b_1(1)+b_2(1)} - \frac{a_3(0)}{a_1(0)+a_2(0)}\Big] \\ &\quad + \int_0^1 G(x,t)\Big[\frac{b_3(1)}{b_1(1)+b_2(1)} - \frac{a_3(0)}{a_1(0)+a_2(0)}\Big]dt \\ &\quad - \int_0^1 G(x,t)\Big(\frac{2[a_3(\frac{t}{2}) - a_1(\frac{t}{2})\varphi_1(\frac{t}{2})]}{a_2(\frac{t}{2})}\Big)' dt, \quad 0 \leqslant x \leqslant 1, \end{aligned} \tag{2.16}$$

where $G(x,t)$ is Green's function of problem (2.14)-(2.15).

Continuing to assume that the function φ_1 is known, using the formula (2.6) we represent function φ_2 via φ_1. Then using the solution of the first boundary problem for equation (1.1) in the domain Ω_0 (see [5]) and functional relations between functions τ_j and ν_j $(j = 2,3)$, we obtain

$$\begin{gathered} {\tau'}_2(y) = \int_0^y {\tau'}_3(\eta)N(0,y,1,\eta)d\eta - \int_0^y {\tau'}_2(\eta)N(0,y,0,\eta)d\eta + F_1(y), \\ {\tau'}_3(y) = \int_0^y {\tau'}_3(\eta)N(1,y,1,\eta)d\eta - \int_0^y {\tau'}_2(\eta)N(1,y,0,\eta)d\eta + F_2(y), \end{gathered} \tag{2.17}$$

where

$$\begin{aligned} F_1(y) &= \int_0^1 \tau_1(\xi)\overline{G}_x(o,y,\xi,0)d\xi - \frac{a_3(0)}{a_1(0)+a_2(0)}N(0,y,0,0) \\ &\quad + \frac{b_3(1)}{b_1(1)+b_2(1)}N(0,y,1,0) + {\varphi'}_1(\frac{y}{2}), \\ F_2(y) &= \int_0^1 \tau_1(\xi)\overline{G}_x(1,y,\xi,0)d\xi - \frac{a_3(0)}{a_1(0)+a_2(0)}N(1,y,0,0) \\ &\quad + \frac{b_3(1)}{b_1(1)+b_2(1)}N(1,y,1,0) - \varphi_3(\frac{y+1}{2}), \end{aligned}$$

and

$$\overline{G}(x,y,\xi,\eta) = \frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{\infty}\Big[e^{-\frac{(x-\xi+2n)^2}{4(y-\eta)}} - e^{-\frac{(x+\xi+2n)^2}{4(y-\eta)}}\Big]$$

is the Green's function of the first boundary problem; see [5],

$$N(x,y,\xi,\eta) = \frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{\infty}\Big[e^{-\frac{(x-\xi+2n)^2}{4(y-\eta)}} + e^{-\frac{(x+\xi+2n)^2}{4(y-\eta)}}\Big].$$

From the first equation in (2.17), we represent function φ_3 via φ_1 and further, from the second equation of (2.17), we find the function φ_1.

After the finding function φ_1, using appropriate formulas, we find functions φ_2, φ_3, τ_i, ν_i, $(i = 1,2,3)$. Solution of the problem AS can be established in the domain

Ω_0 as a solution of the first boundary problem, and in the domains Ω_i $(i = 1, 2, 3)$ as a solution of the Cauchy problem.

Theorem 2.1. *If the functions a_i, b_i, c_i are continuously differentiable on a segment, and have continuous second-order derivatives on an interval, then problem AS has a unique regular solution.*

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