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EXISTENCE OF MULTIPLE SOLUTIONS FOR A $p(x)$-BIHARMONIC EQUATION

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Abstract. In this article, we show the existence of at least three solutions to a Navier boundary problem involving the $p(x)$-biharmonic operator. The technical approach is mainly base on a three critical points theorem by Ricceri.

1. Introduction and statement of the main result

In this article, we consider the fourth-order quasilinear elliptic equation

$$\begin{gathered} \Delta^2_{p(x)}u + |u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator of fourth order, λ, $\mu \in [0,\infty)$, $\Omega \subset \mathbb{R}^N (N > 1)$ is a nonempty bounded open set with a sufficient smooth boundary $\partial\Omega$. f, $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Next, let $F(x,u) = \int_0^u f(x,s)ds$ and $G(x,u) = \int_0^u g(x,s)ds$. For $p \in C(\overline{\Omega})$, denote $1 < p^- = \min_{x\in\overline{\Omega}} p(x) \le p^+ = \max_{x\in\overline{\Omega}} p(x) < +\infty$. Moreover,

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & p(x) < \frac{N}{2}, \\ \infty & p(x) \ge \frac{N}{2}, \end{cases}$$

is the critical exponent just as in many papers. Obviously, $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$. In the sequel, X will denote the Sobolev space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

The energy functional corresponding to problem (1.1) is defined on X as

$$H(u) = \Phi(u) + \lambda\Psi(u) + \mu J(u), \tag{1.2}$$

where

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})dx, \tag{1.3}$$

$$\Psi(u) = -\int_\Omega F(x,u)dx, \tag{1.4}$$

$$J(u) = -\int_\Omega G(x,u)dx. \tag{1.5}$$

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Let us recall that a weak solution of (1.1) is any $u \in X$ such that

$$\begin{aligned} &\int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v + |u|^{p(x)-2}uv)dx \\ &= \lambda \int_\Omega f(x,u)vdx + \mu \int_\Omega g(x,u)vdx \quad \text{for all } v \in X. \end{aligned}$$

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions has been an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [14], Zhikov [19], and the references therein. Moreover, we point out that elliptic equations involving the $p(x)$-biharmonic equations are not trivial generalizations of similar problems studied in the constant case since the $p(x)$-biharmonic operator is not homogeneous and, thus, some techniques which can be applied in the case of the p-biharmonic operators will fail in that new situation, such as the Lagrange Multiplier Theorem.

Ricceri's three critical points theorem is a powerful tool to study boundary problem of differential equation (see, for example, [1, 2, 3, 4]). Particularly, Mihailescu [10] use three critical points theorem of Ricceri [12] study a particular $p(x)$-Laplacian equation. He proved existence of three solutions for the problem. Liu [9] study the solutions of the general $p(x)$-Laplacian equations with Neumann or Dirichlet boundary condition on a bounded domain, and obtain three solutions under appropriate hypotheses. Shi [15] generalizes the corresponding result of [10]. To our best of knowledge, there no result of multiple solutions of $p(x)$-biharmonic equation under sublinear condition. The aim of this paper is to prove the following result

Theorem 1.1. *Assume that* $\sup_{(x,s)\in\Omega\times\mathbb{R}} \frac{|f(x,s)|}{1+|s|^{t(x)-1}} < +\infty$, *where* $t \in C(\overline{\Omega})$ *and* $t(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and there exist two positive constants* ϱ, ϑ *and a function* $\gamma(x) \in C(\overline{\Omega})$ *with* $1 < \gamma^- \leq \gamma^+ < p^-$, *such that*

- (I1) $F(x,s) > 0$ *for a.e.* $x \in \Omega$ *and all* $s \in]0, \varrho]$;
- (I2) *there exist* $p_1(x) \in C(\overline{\Omega})$ *and* $p^+ < p_1^- \leq p_1(x) < p^*(x)$, *such that*
$$\limsup_{s\to 0} \sup_{x\in\Omega} \frac{F(x,s)}{|s|^{p_1(x)}} < +\infty;$$
- (I3) $|F(x,s)| \leq \vartheta(1 + |s|^{\gamma(x)})$ *for a.e.* $x \in \Omega$ *and all* $s \in \mathbb{R}$.

Then, there exist an open interval $\Lambda \subseteq (0, +\infty)$ *and a positive real number* ρ *with the following property: for each* $\lambda \in \Lambda$ *and each function* $g(x,s)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\sup_{(x,s)\in\Omega\times\mathbb{R}} \frac{|g(x,s)|}{1+|s|^{p_2(x)-1}} < +\infty,$$

where $p_2 \in C(\overline{\Omega})$ *and* $p_2(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *there exists* $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$, *problem* (1.1) *has at least three weak solutions whose norms in* X *are less than* ρ.

Remark 1.2. The conclusion of Theorem 1.1 gives a precise information about the $p(x)$-biharmonic equation (1.1) with parameter, namely, one can see that (1.1) is stable with respect to small perturbations.

This article is divided into four sections. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces. In the third section, we present some important properties of the $p(x)$-biharmonic operator. In section 4, we recall B. Ricceri's three critical points theorem at first, then prove our main result.

2. Preliminaries

To study $p(x)$-biharmonic problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of $p(x)$-biharmonic operator, which we will use later.

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\big\},$$

where $p(x) \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) := \big\{p \in C(\overline{\Omega}) : p(x) > 1\big\}, \text{ for any } x \in \overline{\Omega}.$$

Denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x),$$

and for any $x \in \overline{\Omega}$, $k \geq 1$,

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the norm

$$|u|_{p(x)} = \inf\big\{\alpha > 0 : \int_\Omega |\frac{u(x)}{\alpha}|^{p(x)}dx \leq 1\big\}.$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space.

Proposition 2.1 ([8]). *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_N^{\alpha_N}}u$ with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} := \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer the reader to [5, 6, 7, 8].

Proposition 2.2 ([8])**.** *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding*
$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

3. Properties of the $p(x)$-biharmonic operator

Note that the weak solutions of (1.1) are considered in the generalized Sobolev space
$$X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega),$$
equipped with the norm
$$\|u\| = \inf \Big\{ \alpha > 0 : \int_\Omega \Big(|\frac{\Delta u(x)}{\alpha}|^{p(x)} + |\frac{u(x)}{\alpha}|^{p(x)} \Big) dx \le 1 \Big\}.$$

Remark 3.1. (1) According to [17], the norm $\|\cdot\|_{2,p(x)}$, cited in the preliminaries, is equivalent to the norm $|\Delta \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta \cdot|_{p(x)}$ are equivalent.

(2) By the above remark and Proposition 2.2, there is a continuous and compact embedding of X into $L^{q(x)}(\Omega)$, where $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$.

We consider the functional
$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \big(|\Delta u|^{p(x)} + |u|^{p(x)} \big) dx,$$
It is well known that $\Phi(u)$ is well defined and continuous differentiable in X. Now we give the following fundamental proposition.

Proposition 3.2. *For $u \in X$ we have*

(1) $\|u\| < (=;>) 1 \Leftrightarrow \Phi(u) < (=;>) 1$,
(2) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \Phi(u) \le \|u\|^{p^-}$,
(3) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \Phi(u) \le \|u\|^{p^+}$, *for all $u_n \in X$ we have*
(4) $\|u_n\| \to 0 \Leftrightarrow \Phi(u_n) \to 0$,
(5) $\|u_n\| \to \infty \Leftrightarrow \Phi(u_n) \to \infty$.

The proof of this proposition is similar to the proof in [8, Theorem 1.3]. Moreover, the operator $T := \Phi' : X \to X'$ defined as
$$\langle T(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + |u|^{p(x)-2} uv) dx \quad \text{for any } u, v \in X,$$
satisfies the assertions of the following theorem.

Theorem 3.3. *The following statements hold:*

(1) *T is continuous, bounded and strictly monotone.*
(2) *T is of (S_+) type.*
(3) *T is a homeomorphism.*

Proof. (1) Since T is the Fréchet derivative of Φ, it follows that T is continuous and bounded. Let us define the sets

$$U_p = \{x \in \Omega : p(x) \geq 2\}, \quad V_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

Using the elementary inequalities [16]

$$|x-y|^\gamma \leq 2^\gamma (|x|^{\gamma-2}x - |y|^{\gamma-2}y)(x-y) \quad \text{if } \gamma \geq 2,$$
$$|x-y|^2 \leq \frac{1}{(\gamma-1)}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x - |y|^{\gamma-2}y)(x-y) \quad \text{if } 1 < \gamma < 2,$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, we obtain for all $u, v \in X$ such that $u \neq v$,

$$\langle T(u) - T(v), u - v\rangle > 0,$$

which means that T is strictly monotone.

(2) Let $(u_n)_n$ be a sequence of X such that

$$u_n \rightharpoonup u \text{ weakly in } X \quad \text{and} \quad \limsup_{n\to+\infty} \langle T(u_n), u_n - u\rangle \leq 0.$$

From Proposition 3.2, it suffices to shows that

$$\int_\Omega (|\Delta u_n - \Delta u|^{p(x)} + |u_n - u|^{p(x)})dx \to 0. \tag{3.1}$$

In view of the monotonicity of T, we have

$$\langle T(u_n) - T(u), u_n - u\rangle \geq 0,$$

and since $u_n \rightharpoonup u$ weakly in X, it follows that

$$\limsup_{n\to+\infty} \langle T(u_n) - T(u), u_n - u\rangle = 0. \tag{3.2}$$

Put

$$\varphi_n(x) = (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n - \Delta u),$$
$$\psi_n(x) = (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u).$$

By the compact embedding of X into $L^{p(x)}(\Omega)$, it follows that

$$u_n \to u \quad \text{in } L^{p(x)}(\Omega),$$
$$|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u \quad \text{in } L^{q(x)}(\Omega),$$

where $1/q(x) + 1/p(x) = 1$ for all $x \in \Omega$. It results that

$$\int_\Omega \psi_n(x)dx \to 0. \tag{3.3}$$

It follows by (3.2) and (3.3) that

$$\limsup_{n\to+\infty} \int_\Omega \varphi_n(x)dx = 0. \tag{3.4}$$

Thanks to the above inequalities,

$$\int_{U_p} |\Delta u_n - \Delta u_k|^{p(x)}dx \leq 2^{p^+} \int_{U_p} \varphi_n(x)dx,$$
$$\int_{U_p} |u_n - u_k|^{p(x)}dx \leq 2^{p^+} \int_{U_p} \psi_n(x)dx.$$

Then

$$\int_{U_p}\Big(|\Delta u_n-\Delta u|^{p(x)}+|u_n-u|^{p(x)}\Big)\,dx\to 0\quad\text{as } n\to+\infty. \tag{3.5}$$

On the other hand, in V_p, setting $\delta_n=|\Delta u_n|+|\Delta u|$, we have

$$\int_{V_p}|\Delta u_n-\Delta u|^{p(x)}dx\le\frac{1}{p^- -1}\int_{V_p}(\varphi_n)^{\frac{p(x)}{2}}(\delta_n)^{\frac{p(x)}{2}(2-p(x))}dx\,.$$

For $d>0$, by Young's inequality,

$$\begin{aligned} d\int_{V_p}|\Delta u_n-\Delta u|^{p(x)}dx&\le\int_{V_p}[d(\varphi_n)^{\frac{p(x)}{2}}](\delta_n)^{\frac{p(x)}{2}(2-p(x))}dx,\\ &\le\int_{V_p}\varphi_n(d)^{\frac{2}{p(x)}}dx+\int_{V_p}(\delta_n)^{p(x)}dx. \end{aligned}\tag{3.6}$$

From (3.4) and since $\varphi_n\ge 0$, one can consider that

$$0\le\int_{V_p}\varphi_n dx<1.$$

If $\int_{V_p}\varphi_n dx=0$ then $\int_{V_p}|\Delta u_n-\Delta u|^{p(x)}dx=0$. If $0<\int_{V_p}\varphi_n dx<1$, we choose

$$d=\Big(\int_{V_p}\varphi_n(x)dx\Big)^{-1/2}>1,$$

and the fact that $2/p(x)<2$, inequality (3.6) becomes

$$\begin{aligned} \int_{V_p}|\Delta u_n-\Delta u|^{p(x)}dx&\le\frac{1}{d}\Big(\int_{V_p}\varphi_n d^2dx+\int_\Omega\delta_n^{p(x)}dx\Big),\\ &\le\Big(\int_{V_p}\varphi_n dx\Big)^{1/2}\Big(1+\int_\Omega\delta_n^{p(x)}dx\Big). \end{aligned}$$

Note that, $\int_\Omega\delta_n^{p(x)}dx$ is bounded, which implies

$$\int_{V_p}|\Delta u_n-\Delta u|^{p(x)}dx\to 0\quad\text{as } n\to+\infty.$$

A similar method gives

$$\int_{V_p}|u_n-u|^{p(x)}dx\to 0\quad\text{as } n\to+\infty.$$

Hence, it result that

$$\int_{V_p}(|\Delta u_n-\Delta u|^{p(x)}+|u_n-u|^{p(x)})dx\to 0\quad\text{as }\ n\to+\infty. \tag{3.7}$$

Finally, (3.1) is given by combining (3.5) and (3.7).

(3) Note that the strict monotonicity of T implies its injectivity. Moreover, T is a coercive operator. Indeed, since $p^- -1>0$, for each $u\in X$ such that $\|u\|\ge 1$ we have

$$\frac{\langle T(u),u\rangle}{\|u\|}=\frac{\Phi(u)}{\|u\|}\ge\|u\|^{p^- -1}\to\infty\quad\text{as }\|u\|\to\infty.$$

Consequently, thanks to Minty-Browder theorem [18], the operator T is an surjection and admits an inverse mapping. It suffices then to show the continuity of T^{-1}.

Let $(f_n)_n$ be a sequence of X' such that $f_n \to f$ in X'. Let u_n and u in X such that

$$T^{-1}(f_n) = u_n \quad \text{and} \quad T^{-1}(f) = u.$$

By the coercivity of T, one deducts that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ in X, which implies

$$\lim_{n\to+\infty} \langle T(u_n) - T(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty} \langle f_n - f, u_n - \widehat{u}\rangle = 0.$$

It follows by the second assertion and the continuity of T that

$$u_n \to \widehat{u} \quad \text{in } X \quad \text{and} \quad T(u_n) \to T(\widehat{u}) = T(u) \quad \text{in } X'.$$

Moreover, since T is an injection, we conclude that $u = \widehat{u}$. $\square$

4. Proof of main theorem

For the reader's convenience, we recall the revised form of Ricceri's three critical points theorem [13, Theorem 1] and [11, Proposition 3.1].

Theorem 4.1 ([13, Theorem 1]). *Let X be a reflexive real Banach space. $\Phi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X' and Φ is bounded on each bounded subset of X; $\Psi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Assume that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty \tag{4.1}$$

for all $\lambda \in I$, and that there exists $h \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x\in X} \sup_{\lambda\in I} (\Phi(x) + \lambda(\Psi(x) + h)). \tag{4.2}$$

Then, there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $J\colon X \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

Proposition 4.2 ([11, Proposition 3.1]). *Let X be a non-empty set and Φ, Ψ two real functions on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = -\Psi(x_0) = 0, \quad \Phi(x_1) > r, \quad \sup_{x\in\Phi^{-1}(]-\infty,r])} -\Psi(x) < r\frac{-\Psi(x_1)}{\Phi(x_1)}.$$

Then, for each h satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0} \inf_{x\in X} (\Phi(x) + \lambda(h + \Psi(x))) < \inf_{x\in X} \sup_{\lambda\geq 0} (\Phi(x) + \lambda(h + \Psi(x))).$$

Now we can give the proof of our main result.

Proof Theorem 1.1. Set $\Phi(u)$, $\Psi(u)$ and $J(u)$ as (1.3), (1.4) and (1.5). So, for each u, $v \in X$, one has

$$\begin{gathered}\langle \Phi'(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v + |u|^{p(x)-2}uv)\,dx,\\ \langle \Psi'(u), v\rangle = -\int_\Omega f(x,u)v\,dx,\\ \langle J'(u), v\rangle = -\int_\Omega g(x,u)v\,dx.\end{gathered}$$

From Theorem 3.3, of course, Φ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X', moreover, Ψ and J are continuously Gâteaux differentiable functionals whose Gâteaux derivative is compact. Obviously, Φ is bounded on each bounded subset of X under our assumptions.

From Proposition 3.2, we have: if $\|u\| \geq 1$, then

$$\frac{1}{p^+}\|u\|^{p^-} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^+}. \tag{4.3}$$

Meanwhile, for each $\lambda \in \Lambda$,

$$\begin{aligned}\lambda\Psi(u) &= -\lambda\int_\Omega F(x,u)dx\\ &\geq -\lambda\int_\Omega \vartheta(1+|u|^{\gamma(x)})dx\\ &\geq -\lambda\vartheta(|\Omega| + |u|_{\gamma(x)}^{\gamma^+})\\ &\geq -C_2(1+|u|_{\gamma(x)}^{\gamma^+})\\ &\geq -C_3(1+\|u\|^{\gamma^+})\end{aligned}$$

for any $u \in X$, where C_2 and C_3 are positive constants. Here, we use condition (I3) and (ii) of Proposition 2.1. Combining the two inequalities above, we obtain

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - C_3(1+\|u\|^{\gamma^+}),$$

because of $\gamma^+ < p^-$, it follows that

$$\lim_{\|u\|\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty \quad \forall u \in X, \quad \lambda \in [0,+\infty).$$

Then assumption (4.1) of Theorem 4.1 is satisfied.

Next, we will prove that assumption (4.2) is also satisfied. It suffices to verify the conditions of Proposition 4.2. Let $u_0 = 0$, we can easily have

$$\Phi(u_0) = -\Psi(u_0) = 0.$$

Now we claim that (4.2) is satisfied.

From (I2), exist $\eta \in [0,1]$, $C_4 > 0$, such that

$$F(x,s) < C_4|s|^{p_1(x)} < C_4|s|^{p_1^-} \quad \forall s \in [-\eta,\eta], \text{ a.e. } x \in \Omega.$$

Then, from (I3), we can find a constant M such that

$$F(x,s) < M|s|^{p_1^-}$$

for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Consequently, by the Sobolev embedding theorem ($X \hookrightarrow L^{p_1^-}(\Omega)$ is continuous), we have (for suitable positive constant C_5, C_6)

$$-\Psi(u) = \int_\Omega F(x,u)dx < M \int_\Omega |u|^{p_1^-} dx \leq C_5 \|u\|^{p_1^-} \leq C_6 r^{p_1^-/p^+},$$

when $\|u\|^{p^+}/p^+ \leq r$. Hence, being $p_1^- > p^+$, it follows that

$$\lim_{r \to 0^+} \frac{\sup_{\|u\|^{p^+}/p^+ \leq r} -\Psi(u)}{r} = 0. \tag{4.4}$$

Let $u_1 \in C^2(\Omega)$ be a function positive in Ω, with $u_1|_{\partial\Omega} = 0$ and $\max_{\overline{\Omega}} u_1 \leq d$. Then, of course, $u_1 \in X$ and $\Phi(u_1) > 0$. In view of (i_1) we also have $-\Psi(u_1) = \int_\Omega F(x, u_1(x))dx > 0$. Therefore, from (4.4), we can find $r \in \big(0, \min\{\Phi(u_1), \frac{1}{p^+}\}\big)$ such that

$$\sup_{\|u\|^{p^+}/p^+ \leq r} (-\Psi(u)) < r \frac{-\Psi(u_1)}{\Phi(u_1)}.$$

Now, let $u \in \Phi^{-1}((-\infty, r])$. Then, $\int_\Omega (|\Delta u|^{p(x)} + |u|^{p(x)})dx \leq rp^+ < 1$ which, by Proposition 3.2, implies $\|u\| < 1$. Consequently,

$$\frac{1}{p^+}\|u\|^{p^+} \leq \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})dx < r.$$

Therefore, we infer that $\Phi^{-1}((-\infty, r]) \subset \Big\{u \in X : \frac{1}{p^+}\|u\|^{p^+} < r\Big\}$, and so

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} -\Psi(u) < r \frac{-\Psi(u_1)}{\Phi(u_1)}.$$

At this point, conclusion follows from Proposition 4.2 and Theorem 4.1. $\square$

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