Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 141, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO FRACTIONAL BOUNDARY-VALUE PROBLEMS WITH A PARAMETER

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Abstract. This article concerns the existence of solutions to the fractional boundary-value problem

$$-\frac{d}{dt}\big(\frac{1}{2}{}_0D_t^{-\beta}+\frac{1}{2}{}_tD_T^{-\beta}\big)u'(t)=\lambda u(t)+\nabla F(t,u(t)),\quad \text{a.e. } t\in[0,T],$$
$$u(0)=0,\quad u(T)=0.$$

First for the eigenvalue problem associated with it, we prove that there is a sequence of positive and increasing real eigenvalues; a characterization of the first eigenvalue is also given. Then under different assumptions on the nonlinearity $F(t,u)$, we show the existence of weak solutions of the problem when λ lies in various intervals. Our main tools are variational methods and critical point theorems.

1. Introduction

As a generalization of differentiation and integration to arbitrary non-integer order, fractional calculus, is a significant tool for solving complex problems from various fields such as engineering, science, viscoelasticity, diffusion and pure and applied mathematics. As the authors point out in [14], there is hardly a field of science or engineering that has remained untouched by this field. In the past few years, theory of fractional differential equation has been investigated extensively, see the monographs of Kilbas et al [12], Miller and Ross [14], and Podlubny [15], Samko [17], and the papers [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 18, 19, 20] and the reference therein.

In [7], Ervin and Loop investigated the steady state fractional advection dispersion equation

$$\begin{gathered}-\frac{d}{dt}\Big(p\,{}_0D_t^{-\beta}+q\,{}_tD_T^{-\beta}\Big)u'(t)+b(t)u'(t)+c(t)u(t)=\nabla F(t,u(t)),\quad \text{a.e. } t\in[0,T],\\ u(0)=0,\quad u(T)=0,\end{gathered}\tag{1.1}$$

by defining appropriate fractional derivative spaces, they established some existence and uniqueness results of the problem. Recently, there have been many papers

2000 *Mathematics Subject Classification.* 34A08, 34B09.
Key words and phrases. Fractional differential equation; eigenvalue; critical point theory; boundary value problem.

Submitted January 27, 2013. Published June 21, 2013.

dealing with the existence of solutions for this problem. Jiao and Zhou [10] showed the variational structure of the problem

$$
\begin{gathered}
-\frac{1}{2}\frac{d}{dt}\Big({}_0D_t^{-\beta}+{}_tD_T^{-\beta}\Big)u'(t)=\nabla F(t,u(t)),\quad \text{a.e. } t\in[0,T],\\
u(0)=0,\quad u(T)=0.
\end{gathered}
$$

By using the least action principle and Mountain Pass theorem, they obtained some sufficient conditions for the existence of one solution. The authors in [6, 8, 11, 18] further studied the existence and multiplicity of solutions for the above problem or related problems by critical point theory.

Inspired by the results in [6, 7, 8, 10, 11, 18], we consider the existence of weak solution to the fractional boundary-value problem

$$
\begin{gathered}
-\frac{1}{2}\frac{d}{dt}\Big({}_0D_t^{-\beta}+\ {}_tD_T^{-\beta}\Big)u'(t)=\lambda u(t)+\nabla F(t,u(t)),\quad \text{a.e. } t\in[0,T],\\
u(0)==0,\quad u(T)=0.
\end{gathered}\tag{1.2}
$$

where $0<\beta<1$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right fractional integrals of order β respectively, $\lambda\in\mathbb{R}$ is a parameter, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$, and $\nabla F(t,x)$ is the gradient of F with respect to x.

First, we consider the eigenvalue problem associates with (1.2),

$$
\begin{gathered}
-\frac{1}{2}\frac{d}{dt}\Big({}_0D_t^{-\beta}+\ {}_tD_T^{-\beta}\Big)u'(t)=\lambda u(t)\quad \text{a.e. } t\in[0,T],\\
u(0)=0,\quad u(T)=0.
\end{gathered}\tag{1.3}
$$

By Riesz-Schauder theory, we prove that (1.3) possesses a sequence of eigenvalues $\{\lambda_k\}$ with $0<\lambda_1\le\lambda_2\le\lambda_3\le\dots$ and $\lambda_k\to\infty$ as $k\to\infty$. Then under the assumption that $F(t,u)$ is superquadratic with respect to u, we show that (1.2) has at least one nontrivial weak solution when $\lambda<\lambda_1$ by using Mountain Pass theorem. In the special case $\lambda=0$ our results extend [10, Theorem 5.2]. When $\lambda\ge\lambda_1$, sufficient conditions for the existence of one solution is also given by applying Linking theorem. We obtain also the existence of at least two weak solutions for every real number λ via Brezis and Nirenberg's Linking theorem. Furthermore, for every positive integer k, the existence criteria of k pairs of weak solutions when $\lambda>\lambda_k$ are established by using Clark theorem. Our methods are different from those used in [6, 7, 8, 10, 11, 18].

This article is organized as follows. In Section 2, some preliminaries are presented. Section 3 presents the main result and its proof.

2. Preliminaries

To apply critical point theory for the existence of solutions for problem (1.2), we shall state some basic notation and results [11], which will be used in the proof of our main results.

Throughout this paper, we denote $\alpha=1-\frac{\beta}{2}$, and assume that the following condition is satisfied.

(H1) $F(t,x)$ is measurable in t for every $x\in\mathbb{R}^N$ and continuously differentiable in x for a.e $t\in[0,T]$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1(0,T;\mathbb{R}^+)$ such that

$$
|F(t,x)|\le a(|x|)b(t),\quad |\nabla F(t,x)|\le a(|x|)b(t)\tag{2.1}
$$

for all $x\in\mathbb{R}^N$ and $t\in[0,T]$.

The fractional derivative space E^α is defined by the completion of $C_0^\infty((0,T),\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int_0^T |u(t)|^2 dt + \int_0^T |{}_0D_t^\alpha u(t)|^2 dt\Big)^{1/2},$$

where ${}_0D_t^\alpha$ is the α-order left Riemann-Liouville fractional derivative. If $u \in E^\alpha$, then ${}_0D_t^\alpha u(t)$ exists a.e. in $[0,T]$. The set E^α is a reflexive and separable Hilbert space.

Lemma 2.1 ([11])**.** *For all $u \in E^\alpha$, we have*

$$\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha u\|_{L^2}, \tag{2.2}$$

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|{}_0D_t^\alpha u\|_{L^2}. \tag{2.3}$$

According to (2.2), one can consider E^α with respect to the equivalent norm

$$\|u\|_\alpha = \|{}_0D_t^\alpha u\|_{L^2}$$

Lemma 2.2 ([11])**.** *If the sequence $\{u_k\}$ converges weakly to u in E^α, i.e. $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, i.e. $\|u-u_k\|_\infty \to 0$ as $k\to\infty$.*

Similar to the proof of [10, Proposition 4.1], we have the following property.

Lemma 2.3. *For any $u \in E^\alpha$, we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T \big({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t)\big)dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.4}$$

To obtain a weak solution of (1.2), we assume that u is a sufficiently smooth solution of (1.2). Multiplying (1.2) by an arbitrary $v \in C_0^\infty(0,T)$, we have

$$\begin{aligned}&\int_0^T \Big(-\frac{1}{2}\frac{d}{dt}({}_0D_t^{-\beta} + {}_tD_T^{-\beta})u'(t), v(t)\Big) - \lambda\big(u(t),v(t)\big)dt\\ &= \int_0^T \big(\nabla F(t,u(t)),\ v(t)\big)dt.\end{aligned} \tag{2.5}$$

Observe that

$$\begin{aligned}&-\frac{1}{2}\int_0^T \Big(\frac{d}{dt}({}_0D_t^{-\beta}u'(t) + {}_tD_T^{-\beta}u'(t)), v(t)\Big)dt\\ &= \frac{1}{2}\int_0^T \Big(({}_0D_t^{-\beta}u'(t), v'(t)) + ({}_tD_T^{-\beta}u'(t), v'(t))\Big)dt\\ &= \frac{1}{2}\int_0^T \Big(({}_0D_t^{-\beta/2}u'(t), {}_tD_T^{-\beta/2}v'(t)) + ({}_tD_T^{-\beta/2}u'(t), {}_0D_t^{-\beta/2}v'(t))\Big)dt.\end{aligned}$$

As $u(0)=u(T)=v(0)=v(T)=0$, we have

$$\begin{gathered}{}_0D_t^{-\beta/2}u'(t) = {}_0D_t^{1-\frac{\beta}{2}}u(t), \quad {}_tD_T^{-\beta/2}u'(t) = -{}_tD_T^{1-\frac{\beta}{2}}u(t),\\ {}_0D_t^{-\beta/2}v'(t) = {}_0D_t^{1-\frac{\beta}{2}}v(t), \quad {}_tD_T^{-\beta/2}v'(t) = -{}_tD_T^{1-\frac{\beta}{2}}v(t).\end{gathered}$$

Then (2.5) is equivalent to

$$\begin{aligned}&\int_0^T -\frac{1}{2}[\big({}_0D_t^\alpha u(t),{}_t D_T^\alpha v(t)\big)+\big({}_tD_T^\alpha u(t),{}_0D_t^\alpha v(t)\big)]-\lambda\big(u(t),v(t)\big)dt\\&=\int_0^T \big(\nabla F(t,u(t)),v(t)\big)dt.\end{aligned}\tag{2.6}$$

Since (2.6) is well defined for $u, v \in E^\alpha$, the weak solution of (1.2) can be defined as follows.

Definition 2.4. A weak solution of (1.2) is a function $u \in E^\alpha$ such that

$$\begin{aligned}&\int_0^T -\frac{1}{2}\big[\big({}_0D_t^\alpha u(t),{}_t D_T^\alpha v(t)\big)+\big({}_tD_T^\alpha u(t),{}_0D_t^\alpha v(t)\big)\big]\\&-\lambda\big(u(t),v(t)\big)-\big(\nabla F(t,u(t)),\ v(t)\big)dt=0\end{aligned}$$

for every $v \in E^\alpha$.

We consider the functional $\varphi : E^\alpha \to \mathbb{R}$, defined by

$$\varphi(u)=\int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),{}_t D_T^\alpha u(t)\big)-\frac{\lambda}{2}\big(u(t),u(t)\big)-F(t,u(t))\Big]dt.\tag{2.7}$$

Then φ is continuously differentiable under assumption (H1), and

$$\begin{aligned}\langle\varphi'(u),v\rangle=&-\int_0^T \frac{1}{2}\left[\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha v(t)\big)+\big({}_tD_T^\alpha u(t),\ {}_0D_t^\alpha v(t)\big)\right]dt\\&-\int_0^T \lambda\big(u(t),v(t)\big)dt-\int_0^T \big(\nabla F(t,u(t)),\ v(t)\big)dt\end{aligned}\tag{2.8}$$

for $u, v \in E^\alpha$. Hence a critical point of φ is a weak solution of (1.2).

For our proofs, we need the following results in critical point theory.

Definition 2.5. Let E be a real Banach space and $\varphi \in C^1(E,\mathbb{R})$. We say that φ satisfies the (PS) condition if any sequence $\{u_m\} \subset E$ for which $\varphi(u_m)$ is bounded and $\varphi'(u_m) \to 0$, as $m \to \infty$, posses a convergent subsequence.

Lemma 2.6 (Mountain Pass theorem [16, Theorem 2.2])**.** *Let E be a real Banach space and $\varphi \in C^1(E,\mathbb{R})$ satisfying (PS). Suppose $\varphi(0) = 0$ and*

- (C1) *there are constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho} \geq \alpha$, where $B_\rho = \{x \in E : \|x\| < \rho\}$,*
- (C2) *there is an $e \in E \setminus \overline{B}_\rho$ such that $\varphi(e) \leq 0$.*

Then φ possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c=\inf_{g\in\Gamma}\max_{u\in g([0,1])}\varphi(u),$$

where $\Gamma = \{g \in C([0,1],E) | g(0) = 0, g(1) = e\}$.

Lemma 2.7 (Linking theorem [16, Theorem 5.3])**.** *Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $\varphi \in C^1(E,\mathbb{R})$, satisfies (PS), and*

- (C1') *there are constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho \cap X} \geq \alpha$, where $B_\rho = \{x \in E : \|x\| < \rho\}$,*
- (C3) *there is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\overline{B}_R \cap V) \oplus \{re | 0 < r < R\}$, then $\varphi|_{\partial Q} \leq 0$.*

Then φ possesses a critical value $c \geq \alpha$, which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q} \varphi(h(u)),$$

where $\Gamma = \{h \in C(\overline{Q}, E) : h = \mathrm{Id}$ on $\partial Q\}$.

Remark 2.8. It is easy to obtain the following conclusion. Suppose that $\varphi|_V \leq 0$ and there are an $e \in \partial B_1 \cap X$ and an $\overline{R} \geq \rho$ such that $\varphi(u) \leq 0$ for $u \in V \oplus \operatorname{span}\{e\}$ and $\|u\| \geq \overline{R}$. Then for any large R, Q as defined in (C3) satisfies $\varphi|_{\partial Q} \leq 0$.

Lemma 2.9 (Clark theorem [16, Theorem 9.1])**.** *Let E be a real Banach space, $\varphi \in C^1(E, \mathbb{R})$, with φ even, bounded from below, and satisfying (PS). Suppose $\varphi(0) = 0$, there is a set $E' \subset E$ such that E' is homeomorphic to S^{j-1} ($j-1$ dimension unit sphere) by an odd map, and $\sup_{E'} \varphi < 0$. Then φ possesses at least j distinct pairs of critical points.*

Next we have the Brezis and Nirenberg's linking theorem.

Lemma 2.10 ([5])**.** *Let E have a direct sum decomposition $E = X \oplus Y$, where $\dim X < \infty$, and φ be a C^1 functional on E with $\varphi(0) = 0$, satisfying (PS) and assume that, for some $r > 0$,*

$$\varphi(x) \leq 0,\ \forall x \in X, \quad \|x\| \leq r,\ \varphi(y) \geq 0,\ \forall y \in Y,\ \|y\| \leq r.$$

Assume also that φ is bounded below and $\inf_E \varphi < 0$. Then φ has at least two nonzero critical points.

3. Main results

First we consider the eigenvalue problem

$$\begin{gathered} -\frac{1}{2}\frac{d}{dt}\Big({}_0D_t^{-\beta} + {}_tD_T^{-\beta}\Big)u'(t) = \lambda u, \quad \text{a.e. } t \in [0,T], \\ u(0) = 0, \quad u(T) = 0\,. \end{gathered} \tag{3.1}$$

Its weak solution $u \in E^\alpha$ satisfies

$$-\int_0^T \frac{1}{2}\left[\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha v(t)\big) + \big({}_tD_T^\alpha u(t),\ {}_0D_t^\alpha v(t)\big)\right] dt = \int_0^T \lambda(u(t), v(t)) dt \tag{3.2}$$

for every $v \in E^\alpha$.

Theorem 3.1. *Each eigenvalue of (3.1) is real and if we repeat each eigenvalue according to its multiplicity, we h $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ and $\lambda_k \to \infty$ as $k \to \infty$. λ_1 can be characterized as*

$$\lambda_1 = \inf_{u\in E^\alpha\setminus\{0\}} \frac{-\int_0^T \big({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t)\big) dt}{\int_0^T (u(t), u(t)) dt}. \tag{3.3}$$

Furthermore, there exists an orthogonal basis $\{w_k\}_{k=1}^\infty$ of E^α, where $w_k \in E^\alpha$ is an eigenfunction corresponding to λ_k for $k = 1, 2, \dots$.

Proof. For $u \in E^\alpha$, let

$$\|u\|_1 = \Big(-\int_0^T \big({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t)\big) dt\Big)^{1/2}.$$

From (2.4), we have

$$|\cos \pi\alpha|^{1/2}\|u\|_\alpha \le \|u\|_1 \le |\cos \pi\alpha|^{-\frac{1}{2}}\|u\|_\alpha.$$

So $\|\cdot\|_1$ is an equivalent norm on E^α, while E^α is a Banach space with this new norm, and there is an inner product induced by $\|\cdot\|_1$, we denote

$$(u,v)_1 = -\int_0^T \frac{1}{2}\big[\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha v(t)\big) + \big({}_tD_T^\alpha u(t),\ {}_0D_t^\alpha v(t)\big)\big]dt, \quad u,v \in E^\alpha.$$

Then E^α is a Hilbert space with this inner product.

Next, we will transform (3.2) into a problem about symmetric compact operator. From Hölder inequality and (2.2), for given $u \in L^2(0,T)$ and any $v \in E^\alpha$,

$$\begin{aligned}
\Big|\int_0^T (u,v)dt\Big| &\le \|u\|_{L^2}\|v\|_{L^2} \\
&\le \frac{T^\alpha}{\Gamma(\alpha+1)}\|u\|_{L^2}\|v\|_\alpha \\
&\le \frac{T^\alpha}{\Gamma(\alpha+1)|\cos\pi\alpha|^{1/2}}\|u\|_{L^2}\|v\|_1.
\end{aligned}$$

In view of the Riesz theorem, there exists a unique $w \in E^\alpha$ such that

$$\int_0^T (u,v)dt = (w,v)_1, \quad \forall v \in E^\alpha.$$

If we define the operator $K : L^2(0,T) \to E^\alpha$ as $Ku = w$, then

$$\|Ku\|_\alpha \le \frac{T^\alpha}{\Gamma(\alpha+1)|\cos\pi\alpha|^{1/2}}\|u\|_{L^2}$$

and K is a bounded linear operator from $L^2(0,T)$ to E^α. Let $S : E^\alpha \to L^2(0,T)$ be an embedding operator, by Lemma 2.2, S is compact. Thus (3.2) is equivalent to

$$(u,v)_1 = (\lambda w, v)_1 = (\lambda KSu, v)_1, \quad \forall v \in E^\alpha.$$

That is,

$$(I - \lambda KS)u = 0.$$

Since E^α is separable and KS is symmetric and compact, by Riesz-Schauder theory, we know that all eigenvalue $\{\lambda_k\}$ of KS are positive real numbers and there are corresponding eigenfunctions which make up an orthogonal basis of E^α and (3.3) holds. □

Lemma 3.2. *Suppose the following condition holds*

(H2) *there are constants $\mu > 2$ and $R > 0$ such that, for $|x| \ge R$,*

$$0 < \mu F(t,x) \le (x,\ \nabla F(t,x)). \tag{3.4}$$

Then φ satisfies the (PS) condition.

Proof. Let $\{u_n\} \subset E^\alpha$, $\{\varphi(u_n)\}$ be bounded and $\varphi'(u_n) \to 0$. First we show that $\{u_n\}$ is bounded. From (3.4), we know that there exist constants $a_1, a_2 > 0$ such that

$$F(t,x) \ge a_1|x|^\mu - a_2, \quad t \in [0,T],\ x \in \mathbb{R}^N. \tag{3.5}$$

Since $\mu > 2$, then for $\varepsilon > 0$, $u \in E^\alpha$ and by Young's inequality, we have

$$\|u\|_{L^2}^2 \le T^{\frac{\mu-2}{\mu}}\|u\|_{L^\mu}^2 \le C(\varepsilon) + \varepsilon\|u\|_{L^\mu}^\mu$$

where $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Choose $2 < \mu_1 < \mu$, and denote $\tilde{\lambda} = \lambda$ for $\lambda > 0$, and $\tilde{\lambda} = 0$ otherwise. Then for large n and choose ε small enough,

$$
\begin{aligned}
&\mu_1\varphi(u_n) - (\varphi'(u_n), u_n) \\
&= (1 - \frac{\mu_1}{2}) \int_0^T \big({}_0D_t^\alpha u_n(t),\ {}_tD_T^\alpha u_n(t)\big)dt + \lambda(1 - \frac{\mu_1}{2})\|u_n\|_{L^2}^2 \\
&\quad + \int_{|u_n| \geq R} ((u_n(t), \nabla F(t, u_n(t))) - \mu_1 F(t, u_n(t)))dt \\
&\quad + \int_{|u_n| < R} ((u_n(t), \nabla F(t, u_n(t))) - \mu_1 F(t, u_n(t)))dt \\
&\geq (\frac{\mu_1}{2} - 1)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \tilde{\lambda}(1 - \frac{\mu_1}{2})\|u_n\|_{L^2}^2 + (\mu - \mu_1)a_1\|u_n\|_{L^\mu}^\mu \\
&\quad - (\mu - \mu_1)Ta_2 + c \\
&\geq (\frac{\mu_1}{2} - 1)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \tilde{\lambda}(1 - \frac{\mu_1}{2})(C(\varepsilon) + \varepsilon\|u_n\|_{L^\mu}^\mu) + (\mu - \mu_1)a_1\|u_n\|_{L^\mu}^\mu \\
&\quad - (\mu - \mu_1)Ta_2 + c.
\end{aligned}
$$

where c is a constant. So this implies that $\{u_n\}$ is bounded since ε is small enough.

From the reflexivity of E^α, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_n\}$, $u_n \rightharpoonup u$, then $\|u_n - u\|_\infty \to 0$. Next, we prove that $\{u_n\}$ strongly converges to u. By (H1), we know that

$$
\int_0^T \big(\ u_n(t) - u(t), \nabla F(t, u_n(t)) - \nabla F(t, u(t))\big)dt \to 0 \quad \text{as } n \to \infty. \tag{3.6}
$$

From (2.7), we have

$$
\begin{aligned}
&(\varphi'(u_n) - \varphi'(u), u_n - u) \\
&= -\int_0^T \big({}_0D_t^\alpha(u_n(t) - u(t)), {}_tD_T^\alpha(u_n(t) - u(t))\big)dt \\
&\quad - \lambda\int_0^T ((u_n(t) - u(t)), (u_n(t) - u(t)))dt \\
&\quad - \int_0^T \big(\ u_n(t) - u(t), \nabla F(t, u_n(t)) - \nabla F(t, u(t))\big)dt \\
&\geq |\cos(\pi\alpha)|\|u_n - u\|_\alpha^2 - \tilde{\lambda}T\|u_n - u\|_\infty^2 \\
&\quad - \int_0^T \big(\ u_n(t) - u(t), \nabla F(t, u_n(t)) - \nabla F(t, u(t))\big)dt.
\end{aligned} \tag{3.7}
$$

From $\varphi'(u_n) \to 0$ and $u_n \rightharpoonup u$, we obtain that

$$
(\varphi'(u_n) - \varphi'(u), u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{3.8}
$$

In view of (3.6), (3.7) and (3.8), it is easy to see that $\|u_n - u\|_\alpha \to 0$ as $n \to \infty$. Therefore φ satisfies the (PS) condition. $\square$

Theorem 3.3. *If* (H2) *holds and*

(H3)

$$
\limsup_{|x| \to 0} \frac{F(t, x)}{|x|^2} \leq \frac{(\Gamma(\alpha + 1))^2|\cos(\pi\alpha)|}{4T^{2\alpha}}(1 - \frac{\tilde{\lambda}}{\lambda_1})
$$

uniformly for $t \in [0,T]$, where $\tilde{\lambda} = \lambda$ for $\lambda > 0$, and $\tilde{\lambda} = 0$ otherwise.

Then for $\lambda < \lambda_1$, (1.2) has at least one nontrivial weak solution.

Proof. The proof relies on the Mountain Pass theorem. It is clear that $\varphi \in C^1(E^\alpha, R)$, $\varphi(0) = 0$, and φ satisfies the (PS) condition from Lemma 3.2.

From (H3), for

$$\varepsilon_1 = \frac{(\Gamma(\alpha+1))^2|\cos(\pi\alpha)|}{4T^{2\alpha}}(1 - \frac{\tilde{\lambda}}{\lambda_1}),$$

there exists a constant $\delta > 0$, such that

$$F(t,x) \le \varepsilon_1 |x|^2, \quad t \in [0,T],\ |x| < \delta.$$

Let $u \in E^\alpha$ with $\|u\|_\alpha \le \frac{\Gamma(\alpha)(2\alpha-1)^{1/2}\delta}{T^{\alpha-\frac{1}{2}}}$, then by (2.3), $\|u\|_\infty \le \delta$, and from (3.3) and (2.2), we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t), u(t)) - F(t, u(t))\Big] dt \\
&\ge \int_0^T -\frac{1}{2}\big({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t)\big) dt + \frac{\tilde{\lambda}}{2\lambda_1}\int_0^T \big({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t)\big) dt - \varepsilon_1 \|u\|_{L^2}^2 \\
&\ge (1 - \frac{\tilde{\lambda}}{\lambda_1})\frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{\varepsilon_1 T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2 \\
&= (1 - \frac{\tilde{\lambda}}{\lambda_1})\frac{|\cos(\pi\alpha)|}{4}\|u\|_\alpha^2.
\end{aligned}$$

If we choose $\rho = \frac{\Gamma(\alpha)(2\alpha-1)^{1/2}\delta}{T^{\alpha-\frac{1}{2}}}$ and $\varrho = (1 - \frac{\tilde{\lambda}}{\lambda_1})\frac{|\cos(\pi\alpha)|\rho^2}{4}$, then $\varphi|_{\partial B_\rho} \ge \varrho$.

Let $w_1 \in E^\alpha$ be an eigenfunction corresponding to λ_1 in (3.3), and choose $r > 0$, it follows from (3.5) that

$$\begin{aligned}
\varphi(rw_1) &= \int_0^T \Big[-\frac{r^2}{2}\big({}_0D_t^\alpha w_1(t), {}_tD_T^\alpha w_1(t)\big) - \frac{r^2\lambda}{2}(w_1(t), w_1(t)) - F(t, rw_1(t))\Big] dt \\
&\le \frac{\lambda_1 r^2}{2}\|w_1\|_{L^2}^2 - \frac{\lambda r^2}{2}\|w_1\|_{L^2}^2 - a_1 r^\mu \|w_1\|_{L^\mu}^\mu + a_2 T,
\end{aligned}$$

which implies that $\varphi(rw_1) \to -\infty$ as $r \to \infty$.

The above discussions show that φ has at least one nontrivial critical point, thus (1.2) has at least one nontrivial weak solution for $\lambda < \lambda_1$. □

Note that when $\lambda = 0$, Theorem 3.3 extends the results in [10, Theorem 5.2].

Theorem 3.4. *Suppose* (H2) *holds and*

- (H4) $F(t,x) \ge 0$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$.
- (H5) $F(t,x) = o(|x|^2)$ *as* $x \to 0$.

Then the problem (1.2) *possesses a nontrivial weak solution for* $\lambda \ge \lambda_1$.

Proof. We will show that the functional φ satisfies the hypotheses in Lemma 2.7 when $\lambda \ge \lambda_1$.

Lemma 3.2 tell us that φ satisfies the (PS) condition. Since $\lambda \ge \lambda_1$, we can assume $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Set $V = \operatorname{span}\{w_1, \dots, w_k\}$ and $X = V^\perp$, where $\{w_j\}$ are eigenfunctions of (3.1) corresponding to the eigenvalues $\{\lambda_j\}$.

From (H5) and (2.2), for a small positive number ε_2, there exists a constant $\delta_1 > 0$, such that, for $u \in E^\alpha$ with $\|u\|_\infty < \delta_1$, we have

$$\int_0^T F(t,u)dt \le \varepsilon_2 \|u\|_{L^2}^2 \le \frac{\varepsilon_2 T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2.$$

Hence for $u \in X$, with $\|u\|_\infty \le \delta_1$, we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t),u(t)) - F(t,u(t))\Big]dt \\
&\ge -\frac{1}{2}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt + \frac{\lambda}{2\lambda_{k+1}}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt \\
&\quad - \frac{\varepsilon_2 T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2 \\
&\ge -\frac{1}{2}(1-\frac{\lambda}{\lambda_{k+1}})\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt - \frac{\varepsilon_2 T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2 \\
&\ge \frac{|\cos(\pi\alpha)|}{2}(1-\frac{\lambda}{\lambda_{k+1}})\|u\|_\alpha^2 - \frac{\varepsilon_2 T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2.
\end{aligned}$$

If we choose ε_2 small enough, we can get $\rho, \varrho > 0$ such that $\varphi|_{\partial B_\rho \cap X} \ge \varrho$, and φ satisfies (C_1') in Lemma 2.7.

To check (C3) in Lemma 2.7, it suffices to verify the conditions in Remark 2.8. In fact, for $u \in V$, by (H4), we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t),u(t)) - F(t,u(t))\Big]dt \\
&\le -\frac{1}{2}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt + \frac{\lambda}{2\lambda_k}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt \\
&\le -\frac{1}{2}(1-\frac{\lambda}{\lambda_k})\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt \\
&\le \frac{|\cos(\pi\alpha)|(\lambda_k-\lambda)}{2\lambda_k}\|u\|_\alpha^2 < 0.
\end{aligned} \tag{3.9}$$

Let $u_0 = \frac{w_{k+1}}{\|w_{k+1}\|_\alpha}$, then for $u \in V \oplus span\{u_0\}$, we obtain

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u_0(t)\big) - \frac{\lambda}{2}(u(t),u(t)) - F(t,u(t))\Big]dt \\
&\le \frac{\|u\|_\alpha^2}{2|\cos(\pi\alpha)|} - \frac{\lambda}{2}\|u\|_{L^2}^2 - a_1\|u\|_{L^\mu}^\mu + a_2T.
\end{aligned}$$

Since $\mu > 2$, and $V \oplus span\{u_0\}$ is a finite dimensional space on which all norms are equivalent. So we obtain $\varphi(u) \to -\infty$ as $\|u\|_\alpha \to \infty, u \in V \oplus span\{u_0\}$. This implies that for any large R, Q as defined in (C3), $\varphi|_{\partial Q} \le 0$.

By Lemma 2.7, φ has at least a nontrivial critical point, so (1.2) possesses a nontrivial weak solution for $\lambda \ge \lambda_1$. □

Remark 3.5. In fact, (H5) implies (H3), so when $\lambda < \lambda_1$, Theorem 3.3 gives the conclusion, that is, under the assumptions of (H2), (H4) and (H5), Equation (1.2) possesses at least one nontrivial weak solution for $\lambda \in \mathbb{R}$.

Theorem 3.6. *If* (H1) *holds and*

(H6) *There exist* $b_1, b_2 > 0$, *and* $\eta \in (0,2)$ *such that*

$$F(t,x) \le -\frac{\lambda}{2}|x|^2 + b_1|x|^\eta + b_2, \quad x \in \mathbb{R}^N,\ t \in [0,T].$$

(H7) *There are* $k \in \mathbb{N}$ *and* $r_1 > 0$ *such that, for* $|x| \le r_1$

$$\frac{\lambda_k - \lambda}{2}|x|^2 \le F(t,x) \le \frac{\lambda_{k+1} - \lambda}{2}|x|^2, \quad t \in [0,T]. \tag{3.10}$$

Then (1.2) *possesses at least two nontrivial weak solutions for* $\lambda \in \mathbb{R}$.

Proof. First we show that φ is bounded from below. Since $\eta \in (0,2)$, for $u \in E^\alpha$, by (H6) and (2.3), we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t),u(t)) - F(t,u(t))\Big]dt \\
&\ge -\frac{1}{2}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt - b_1 T\|u\|_\infty^\eta - b_2 T \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{b_1 T^{\eta(\alpha-\frac{1}{2})+1}}{(\Gamma(\alpha))^\eta(2\alpha-1)^{\eta/2}}\|u\|_\alpha^\eta - b_2 T.
\end{aligned} \tag{3.11}$$

This implies φ is bounded from below. If $\{u_n\}$ is a (PS) sequence, then $\{u_n\}$ is bounded from (3.11). Similar to the later part proof of Lemma 3.2, we can get that φ satisfies the (PS) condition.

Set $V = \operatorname{span}\{w_1, \dots, w_k\}$ and $X = V^\perp$, where $\{w_j\}$ are eigenfunctions of (3.1). From (H7), for $u \in V$ with $\|u\|_\alpha \le \frac{\Gamma(\alpha)(2\alpha-1)^{1/2}r_1}{T^{\alpha-\frac{1}{2}}}$, then $\|u\|_\infty \le r_1$, and

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t),u(t)) - F(t,u(t))\Big]dt \\
&\le -\frac{1}{2}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt - \frac{\lambda_k}{2}\|u\|_{L^2}^2 \le 0.
\end{aligned} \tag{3.12}$$

For $u \in X$ with $\|u\|_\infty \le r_1$, by (H7), we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t),u(t)) - F(t,u(t))\Big]dt \\
&\ge -\frac{1}{2}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big)dt - \frac{\lambda_{k+1}}{2}\|u\|_{L^2}^2 \ge 0.
\end{aligned} \tag{3.13}$$

If $\inf_{u\in E^\alpha}\varphi(u) \ge 0$, then $\varphi(u) = 0$ for all $u \in V$ with $\|u\|_\alpha \le \frac{\Gamma(\alpha)(2\alpha-1)^{1/2}r_1}{T^{\alpha-\frac{1}{2}}}$, which implies that all $u \in V$ with $\|u\|_\alpha \le \frac{\Gamma(\alpha)(2\alpha-1)^{1/2}r_1}{T^{\alpha-\frac{1}{2}}}$ are solutions of (1.2). If $\inf_{u\in E^\alpha}\varphi(u) < 0$, by Lemma 2.10, we get that φ has at least two nontrivial weak solutions for $\lambda \in (\lambda_k, \lambda_{k+1})$. □

Theorem 3.7. *Suppose* (H6) *holds and*

(H8) *There exist* $\varepsilon_3, r_2 > 0$, *such that* $F(t,x) \ge \varepsilon_3$ *for* $|x| \le r_2$.
(H9) $F(t,x) = F(t,-x)$.

Then for $k = 1,2,\dots$, *problem* (1.2) *possesses at least* k *distinct pairs of weak solutions for* $\lambda > \lambda_k$.

Proof. It is clear that $\varphi(0) = 0$ and from (H9), $\varphi(u)$ is even. (H6) and (3.11) show that φ is bounded from below and satisfies the (PS) condition.

Let $\{w_j\}$ be the eigenfunctions of (3.1) corresponding to $\{\lambda_j\}$. Choose

$$E' = \{u|u = \sum_{j=1}^{k} \alpha_j w_j, \sum_{j=1}^{k} \alpha_j^2 = \frac{\Gamma(\alpha)(2\alpha-1)^{1/2} r_2}{T^{\alpha-\frac{1}{2}}}\},$$

then E' is homeomorphic to the $k-1$ dimension unit sphere S^{k-1} by an odd map.

Assume $u \in E'$, then $\|u\|_\alpha = \frac{\Gamma(\alpha)(2\alpha-1)^{1/2} r_2}{T^{\alpha-\frac{1}{2}}}$, so $\|u\|_\infty \le r_2$, via (H8), we have

$$\begin{aligned} \varphi(u) &= \int_0^T \Big[-\frac{1}{2}\big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) - \frac{\lambda}{2}(u(t), u(t)) - F(t, u(t))\Big] dt \\ &\le -\frac{1}{2}\int_0^T \big({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\big) dt - \frac{\lambda_k}{2}\|u\|_{L^2}^2 - \varepsilon_3 \\ &\le -\varepsilon_3. \end{aligned}$$

This implies that $\sup_{E'} \varphi < 0$. And by Clark theorem, φ possesses at least k distinct pairs of critical points which correspond to the weak solutions of (1.2). $\square$

Acknowledgments. This research was supported grant NECT-12-0246 and grant lzujbky-2013-k02 from the program for New Century Excellent Talent in Universities.

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