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EXISTENCE OF MULTIPLE SOLUTIONS TO ELLIPTIC EQUATIONS SATISFYING A GLOBAL EIGENVALUE-CROSSING CONDITION

DUONG MINH DUC, LOC HOANG NGUYEN, LUC NGUYEN

Abstract. We study the multiplicity of solutions to the elliptic equation $\Delta u + f(x,u) = 0$, under the assumption that $f(x,u)/u$ crosses globally but not pointwise any eigenvalue for every x in a part of the domain, when u varies from $-\infty$ to ∞. Also we relax the conditions on uniform convergence of $f(x,s)/s$, which are essential in many results on multiplicity for asymptotically linear problems.

1. Introduction

Let Ω be a bounded connected open subset with smooth boundary in $\mathbb{R}^N$ $(N \geq 3)$ and H be the usual Sobolev space $W_0^{1,2}(\Omega)$ with the inner product and norm

$$\langle u, v\rangle = \int_\Omega \nabla u.\nabla v\,dx \quad \forall u, v \in W_0^{1,2}(\Omega),$$
$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2} \quad \forall u \in W_0^{1,2}(\Omega).$$

Let f be a real-valued Caratheodory function on $\Omega \times \mathbb{R}$ such that the first order partial derivative in second variable $\frac{\partial f}{\partial t}(x,t)$ exists and is continuous at any t in $\mathbb{R}$ for every x in $\mathbb{R}$, and $f(x,0) = 0$ for all x in Ω. Assume that there exist measurable functions V_1, V_2, V_3 and V_4 on Ω such that

$$\liminf_{|t|\to 0} \frac{f(x,t)}{t} = V_1(x) \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{1.1}$$

$$|f(x,t) - f(x,s)| \leq V_2(x)|t-s| \quad \forall x \in \Omega,\ s,t \in \mathbb{R}, \tag{1.2}$$

$$\frac{f(x,t) - f(x,s)}{t-s} \leq V_3(x) \quad \forall x \in \Omega,\ s,t \in \mathbb{R}, s \neq t, \tag{1.3}$$

$$\liminf_{|t|\to\infty} \frac{f(x,t)}{t} = V_4(x) \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{1.4}$$

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We consider the problem (P),

$$\begin{aligned} \Delta u + f(x,u) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

This equation has been studied when f depends only on u in [2, 8, 9, 12, 21, 36]. The uniform convergence in (1.1) and (1.4) is essential in the articles [11, 10, 13, 15, 20, 19, 22, 23, 25, 26, 27, 28, 29, 30, 31, 37, 38, 39]. If $f(x,u)/u$ does not cross uniformly any λ_i, problem (1.5) may not have any solution (see [27]). In this paper we relax conditions on uniform convergence of $f(x,s)/s$. In the real world we can not estimate $f(x,u)/u$ pointwise, we have only its average values by integration. On the other hand we can neglect the behavior of $f(x,u)/u$ at every x in small parts of Ω. With these motivations, we introduce the concept of global eigenvalue-crossing defined by (1.7) and (1.8), below. Using this concept, we study problem (1.5), and illustrate our method by improving the results in [8]; see Theorem 1.1 below. It is interesting that the conditions in Theorem 1.1 are similar to the Landesman-Lazer conditions in [3, 19].

Let $\lambda_1 < \lambda_2 \leq \dots$ be the eigenvalues and $\varphi_1, \varphi_2, \dots$ be their corresponding eigenfunction of the Laplacian operator $-\Delta$ in H. Our result on multiplicity of solutions is stated in the following theorem.

Theorem 1.1. *Let Y be the subspace of H spanned by $\{\varphi_1, \varphi_2, \dots, \varphi_k\}$, and let Z be the orthogonal complement of Y in H. Let $W = |V_1| + V_2 + |V_3| + |V_4|$ and r be in the interval $(\frac{N}{2}, \infty)$. Suppose $W \in L^r(\Omega)$ and*

$$\int_\Omega \big(|\nabla u|^2 - V_1 u^2\big) dx \geq C_0 \|u\|^2 \quad \forall u \in W_0^{1,2}(\Omega), \tag{1.6}$$

$$\int_\Omega \big(|\nabla z|^2 - V_3 z^2\big) dx \geq C_1 \|z\|^2 \quad \forall z \in Z, \tag{1.7}$$

$$\int_\Omega \big(|\nabla y|^2 - V_4 y^2\big) dx \leq -C_2 \|y\|^2 \quad \forall y \in Y, \tag{1.8}$$

Then (i) Problem (1.5) *has at least five solutions. (ii) Moreover, one of the following cases occurs:*

- (a) *k is even and* (1.5) *has two solutions that change sign.*
- (b) *k is even and* (1.5) *has six solutions, three of which are of the same sign.*
- (c) *k is odd and* (1.5) *has two solutions that change sign.*
- (d) *k is odd and* (1.5) *has three solutions of the same sign.*

These results have been proved in [8] under the following conditions: f is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, such that $f(0) = 0$, $f'(0) < \lambda_1$, $\lim_{|t|\to\infty} \frac{f(t)}{t} \in (\lambda_k, \lambda_{k+1})$, and $f'(t) < \gamma < \lambda_{k+1}$ for all t in $\mathbb{R}$. If f' is continuous on $\mathbb{R}$ and $\sup\{|f'(t)| : t \in \mathbb{R}\} = M < \infty$, we can apply Theorem 1.1 to consider this case with $V_1(x) = f'(0)$, $V_2(x) = M$, $V_3(x) = \gamma$ and $V_4(x) = \frac{1}{2}(\lambda_{k+1} + \lim_{|t|\to\infty} \frac{f(t)}{t})$ for any x in Ω.

Let μ and ν be real numbers such that $\mu < \lambda_k < \nu$. We have to cross λ_k in order to go from μ to ν. Arguing as in [32, p. 26], we have

$$\int_\Omega (|\nabla z|^2 - \mu z^2) dx \geq (1 - \frac{\mu}{\lambda_{k+1}}) \|z\|^2 \quad \forall z \in Z. \tag{1.9}$$

$$\int_\Omega (|\nabla y|^2 - \nu y^2) dx \leq -(\frac{\nu}{\lambda_k} - 1) \|y\|^2 \quad \forall y \in Y. \tag{1.10}$$

These inequalities motivated us to introduce the global conditions (1.7) and (1.8).

Example 1.2. Let Ω be the unit sphere in $\mathbb{R}^N$, γ be in the interval $(\lambda_k, \lambda_{k+1})$, ε be a positive real number, and f be a real C^2-function on $\Omega \times \mathbb{R}$ such that

$$f(x,t) = \begin{cases} 0 & \text{if } |t| \le \frac{1}{2}, \\ (\gamma - \varepsilon(1-|x|^2)^{-1/N})t & \text{if } |t| \ge 1, \end{cases}$$

and

$$|\frac{\partial f(x,t)}{\partial t}| \le 4\gamma - 4\varepsilon(1-|x|^2)^{-1/N} \quad \text{if } 0 \le |t| \le 1.$$

Since $(1-|x|^2)^{-1/N}$ is in $L^{\frac{N}{2}}(\Omega)$, by inequalities of Sobolev and Poincaré there is a constant c_0 such that for any u in $W_0^{1,2}(\Omega)$,

$$\begin{aligned} \int_\Omega (1-|x|^2)^{-1/N} u^2 dx &\le \Big\{ \int_\Omega (1-|x|^2)^{-\frac{1}{2}} dx \Big\}^{2/N} \Big\{ \int_\Omega u^{\frac{2N}{N-2}} dx \Big\}^{\frac{N-2}{N}} \\ &\le c_0 \int_\Omega |\nabla u|^2 dx. \end{aligned}$$

Let ε be in the interval $(0, c_0^{-1}(\frac{\gamma}{\lambda_k} - 1))$. We see that $c_2 \equiv \frac{\gamma}{\lambda_k} - 1 - c_0\varepsilon$ is positive. Put $V_1(x) = 0$, $V_2(x) = 4\gamma - 4\varepsilon(1-|x|^2)^{-\frac{1}{N}}$, $V_3(x = V_4(x) = gamma - \varepsilon(1-|x|^2)^{-\frac{1}{N}}$ for any x in Ω. Then f satisfies (1.1), (1.2), (1.3), (1.4), (1.6), (1.7) and (1.8). Indeed, arguing as in [32, p. 26], we have

$$\begin{aligned} \int_\Omega [|\nabla z|^2 - V_3 z^2] dx &\ge \int_\Omega [|\nabla z|^2 - \gamma z^2] dx \ge (1 - \frac{\gamma}{\lambda_{k+1}}) \|z\|^2 \quad \forall z \in Z, \\ \int_\Omega [|\nabla y|^2 - V_4 y^2] dx &\le \int_\Omega [(1 + c_0\varepsilon)|\nabla y|^2 - \gamma y^2] dx \\ &= \sum_{j=1}^k [(1 + c_0\varepsilon)\lambda_j - \gamma] \int_\Omega \alpha_j^2 \varphi_j^2 dx \\ &= -\sum_{j=1}^k [\frac{\gamma}{\lambda_j} - 1 - c_0\varepsilon] \lambda_j \int_\Omega \alpha_j^2 \varphi_j^2 dx \\ &= -[\frac{\gamma}{\lambda_k} - 1 - c_0\varepsilon] \int_\Omega |\nabla y|^2 dx \quad \forall y = \sum_{j=1}^k \alpha_j \varphi_j \in Y. \end{aligned}$$

Remark 1.3. Note that the set $E = \{x \in \Omega : \gamma - \varepsilon(1-|x|^2)^{-\frac{1}{N}} < \lambda_1\}$ is a nonempty open subset of Ω. Then the Lebesgue measure of E is positive, and $\frac{f(x,t)}{t} < \lambda_1$ for any x in E and $|t| \ge 1$. Thus $\frac{f(x,t)}{t}$ does not cross any λ_i at any x in E.

2. Proof of main results

For any (x,ξ) in $\Omega \times \mathbb{R}$ and any u in H we define

$$\xi^+ = \max\{\xi, 0\}, \quad \xi^- = \min\{\xi, 0\}, \tag{2.1}$$

$$f_\pm(x,\xi) = f(x,\xi^\pm) \mp V_1(x)\xi^\mp, \tag{2.2}$$

$$F(x,\xi) = \int_0^1 f(x, s\xi)\xi ds, \quad F_\pm(x,\xi) = \int_0^1 f(x, s\xi^\pm)\xi^\pm ds + \frac{1}{2} V_1(x)|\xi^\mp|^2, \tag{2.3}$$

$$J(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - F(x,u(x))]dx, \quad J_\pm(u) = \int_\Omega [\frac{1}{2}|\nabla u(x)|^2 - F_\pm(x,u(x))]dx. \tag{2.4}$$

Some operators in this sections may not be compact vector fields but of class $(S)_+$, which has been introduced by Browder (see [5, 6]). We have the definitions and properties of the class $(S)_+$ as follows.

Definition 2.1. Let X be a subset of H and h be a mapping of X into H. We say:

- (i) h is demicontinuous if the sequence $\{h(x_m)\}$ converges weakly to $h(x)$ in H for any sequence $\{x_m\}$ converging strongly to x in H.
- (ii) h is of class $(S)_+$ if h is demicontinuous and has the following property : let $\{x_m\}$ be a sequence in X such that $\{x_m\}$ converges weakly to x in H. Then $\{x_m\}$ converges strongly to x in H if $\limsup_{n\to\infty}\langle h(x_m), x_m - x\rangle \le 0$.

Denote by $B_s(x_0)$ the open ball of radius s centered at x_0 for any x_0 in H. Let U be a bounded open subset of H and ∂U and $\overline{U}$ be the boundary and the closure of U in H respectively. Let f be a mapping of class $(S)_+$ on $\overline{U}$ and let p be in $H \setminus f(\partial U)$. By [5, Theorems 4 and 5], the topological degree of f on U at p is defined as a family of integers and is denoted by $\deg(f, U, p)$. In [34] Skrypnik showed that this topological degree is single-valued (see also [6]). The following result was proved in [6].

Proposition 2.2. *Let f be a mapping of class $(S)_+$ from $\overline{U}$ into H, and let y be in $H \setminus f(\partial U)$. Then we can define the degree $\deg(f, U, y)$ as an integer satisfying the following three conditions:*

- (a) *(Normalization) If $\deg(f, U, y) \neq 0$ then there exists $x \in U$ such that $f(x) = y$. If $y \in U$ then $\deg(Id, U, y) = 1$ where Id is the identity mapping.*
- (b) *(Additivity) If U_1 and U_2 are two disjoint open subsets of U and y does not belong to $f(\overline{U}\setminus(U_1 \cup U_2))$ then $\deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y)$.*
- (c) *(Invariance under homotopy). If $\{g_t : 0 \le t \le 1\}$ is a homotopy of class $(S)_+$ and $\{y_t : t \in [0, 1]\}$ is a continuous curve in H such that $y_t \notin g_t(\partial U)$ for all $t \in [0, 1]$, then $\deg(g_t, U, y_t)$ is constant in t on $[0, 1]$.*

Definition 2.3. If u_0 is an isolated zero of a map f of class $(S)_+$, then, by the additivity property of the degree, we can define

$$i(f, u_0) = \lim_{s\to 0} \deg(f, B_s(u_0), 0),$$

which is called the index of f at u_0.

Definition 2.4. Let j be a real-valued C^1-function on H. We say that j satisfies the Palais-Smale condition if for any sequence $\{x_m\}$ in H such that $\{j(x_m)\}$ is bounded and $\{\|Dj(x_m)\|\}$ converges to 0, there is a convergent subsequence $\{x_{m_k}\}$ of $\{x_m\}$.

Definition 2.5. Let j be a real C^1-function defined on an open subset U in H, and x be a critical point of j. Then x is said to be a critical point of mountain-pass type if there exists a neighborhood V of x contained in U such that $W \cap j^{-1}(-\infty, j(x))$ is nonempty and not path-connected whenever W is an open neighborhood of x contained in V.

Definition 2.6. Let j be a real-valued C^2-function on H and $x_0 \in H$. We say j satisfies the condition (Φ) at x_0 if: $Dj(x_0) = 0$ and 0 is a simple eigenvalue whenever it is the smallest eigenvalue of $D^2 j(x_0)$.

We shall extend the results in [1, 16] to operators of class $(S)_+$ in the appendix and use them in the present and next sections. The proof of Theorem 1.1 needs following lemmas.

Lemma 2.7. *Let W be as in Theorem 1.1 and u be in H. We have*

- (F1) *Wu is in $L^1(\Omega)$ for any u in H.*
- (F2) *There is a positive constant K such that*

$$\int_\Omega W u^2\, dx \le K\|u\|^2 \quad \forall u \in H. \tag{2.5}$$

- (F3) *For any sequence $\{v_m\}$ converging weakly to v in H, there are a measurable function g on Ω and a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ having the following properties: $|v_{m_k}| \le g$ a.e. on Ω, and for any k,*

$$\int_\Omega W|g|^2\, dx < \infty. \tag{2.6}$$

Proof. Let q be in $[1, \frac{N}{N-2})$ such that $\frac{1}{r} + \frac{1}{q} = 1$. By Hölder's and Sobolev's inequalities, there is constant c such that for any $u \in H$ and $s \in \{1, 2\}$

$$\int_\Omega W|u|^s dx \le \Big(\int_\Omega W^r dx\Big)^{1/r} \Big[\Big(\int_\Omega |u|^{sq} dx\Big)^{\frac{s}{q}}\Big]^s \le c^s \Big(\int_\Omega W^r dx\Big)^{1/r} \|u\|^s. \tag{2.7}$$

Therefore, $(F1)$ and $(F2)$ are satisfied.

Let $\{v_m\}$ be a sequence converging weakly to v in H. By [4, Theorem 4.9] and Rellich-Kondrachov's theorem, there exist g and v in $L^{2q}(\Omega)$, and a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ such that $\{v_{m_k}\}$ converges to v in $L^{2q}(\Omega)$, $\{v_{m_k}\}$ converges v a.e on Ω, and $|v_{m_k}| \le g$ a.e on Ω. Since g^2 is in $L^q(\Omega)$, Wg^2 is integrable on Ω. ☐

Lemma 2.8. *Let v and w be in H, such that w is nonnegative and not equal to* 0 *and*

$$\int_\Omega [\nabla w \nabla \varphi - \frac{\partial f}{\partial t}(x, v(x))\varphi] dx = 0 \quad \forall \varphi \in H.$$

Then $w > 0$ a.e. on Ω.

Proof. Let W and r be in Theorem 1.1, $B(x,t)$ be a ball in $\mathbb{R}^N$ with center x and radius t, and p be in $[2, \frac{N}{N-2})$ such that $\frac{1}{p} + \frac{1}{r} = 1$. Put $U(y) = \frac{\partial f}{\partial t}(y, v(y))$ for any y in Ω. By (1.2) and Hölder's inequality, there are positive contants M_1 and M_2 independing from x and t such that

$$\begin{aligned}
\int_{B(x,t)} \frac{|U(y)|}{|x-y|^{N-2}} \chi_\Omega(y) dy &\le \int_{B(x,t)} \frac{W(y)}{|x-y|^{N-2}} \chi_\Omega(y) dy \\
&\le (\int_\Omega W^r dy)^{1/r} (\int_{B(x,t)} |x-y|^{p(2-N)} dy)^{1/p} \\
&\le M_1 \int_0^t s^{p(2-N)+N-1} ds = M_2 t^\theta,
\end{aligned}$$

where $\theta = p(2-N) + N > \frac{N}{N-2}(2-N) + N = 0$.

Thus U is of Kato's class (see [33]). Let x_0 be in Ω and Ω' be an open set such that $w(x_0) > 0$, $x_0 \in \Omega'$ and $\overline{\Omega'} \subset \Omega$. By Harnack's inequality [33, Theorem 5.5]), $w(x) > 0$ for any x in Ω'. Since Ω is connected in $\mathbb{R}^N$, $w(z) > 0$ for any z in Ω. □

Lemma 2.9. *(i) The functional J is of class C^2, the functionals J_+ and J_- are of class C^1. For any u and v in H we have*

$$\langle DJ(u), v\rangle = \int_\Omega [\nabla u \nabla v - f(x,u)v]dx, \tag{2.8}$$

$$\langle DJ_\pm(u), v\rangle = \int_\Omega [\nabla u \nabla v - f_\pm(x,u)v]dx. \tag{2.9}$$

(ii)

$$\limsup_{y\in Y, \|y\|\to\infty} \frac{J(y)}{\|y\|^2} < 0. \tag{2.10}$$

Proof. (i) It is sufficient to prove that J is of class C^2. The prove for $J_\pm$ are similar. Let u, v be in H and x be in Ω. By (1.2), (2.3) and the mean value theorem, there is s_x in $[0,1]$ such that

$$\begin{aligned}
&|F(x,u(x)+v(x)) - F(x,u(x)) - f(x,u)v| \\
&= |f(x,u(x)+s_x v(x))v - f(x,u(x))v(x)| \\
&\le V(x)v^2(x).
\end{aligned}$$

Hence, by (2.4) and (2.7),

$$\begin{aligned}
&|J(u+v) - J(u) - \int_\Omega [\nabla u \nabla v - f(x,u)v]dx| \\
&= \||v\|^2 + \int_\Omega [F(x,u(x)+v(x)) - F(x,u(x)) - f(x,u)v]dx| \\
&\le \|v\|^2 + |\int_\Omega V_2 v^2 dx| \\
&\le \Big[1 + c^2\Big(\int_\Omega W^{\frac{N}{2}}dx\Big)^{2/N}\Big]\|v\|^2
\end{aligned}$$

Therefore, J is Fréchet-differentiable on H and

$$\langle DJ(u), v\rangle = \int_\Omega [\nabla u \nabla v - f(x,u)v]dx \quad \forall u, v \in H.$$

By $(2.5) and (1.2)$, we have that for any u, w and v in H,

$$\begin{aligned}
|\langle J(u) - J(w), v\rangle| &= \int_\Omega |\nabla(u-w)\nabla v - (f(x,u) - f(x,w))v|dx \\
&\le \|u-w\|\|v| + \int_\Omega |V_2(u-w)v|dx \\
&\le \|u-w\|\|v| + \Big\{\int_\Omega V_2(u-w)^2dx\Big\}^{1/2}\Big\{\int_\Omega V_2 v^2 dx\Big\}^{1/2} \\
&\le (1+K)\|u-w\|\|v\|.
\end{aligned} \tag{2.11}$$

Thus J is of class C^1. Similarly we see that DJ is Fréchet-differentiable and

$$D^2J(u)(v,w) = \int_\Omega [\nabla v \nabla w - \frac{\partial f}{\partial t}(x,u)vw]dx \quad \forall u, v, w \in H.$$

Let v and w be in H and $\{u_m\}$ be a sequence converging to u in H. We see that $\{u_m\}$ converges to u in $L^2(\Omega)$. Since V_2 is in $L^{N/2}(\Omega)$, by a result on page 30 in [17] and (1.2), we have

$$\lim_{m\to\infty}\Big\{\int_\Omega\big|\frac{\partial f}{\partial t}(x,u_m(x))-\frac{\partial f}{\partial t}(x,u(x))\big|^{N/2}dx\Big\}^{N/2}=0. \tag{2.12}$$

As in (2.7), we have

$$\begin{aligned}
&|[D^2J(u_m)-D^2J(u)](v,w)|\\
&=|\int_\Omega[\frac{\partial f}{\partial t}(x,u_m)-\frac{\partial f}{\partial t}(x,u)]vwdx|\\
&\le\Big\{\int_\Omega|\frac{\partial f}{\partial t}(x,u_m)-\frac{\partial f}{\partial t}(x,u)|v^2dx|\Big\}^{1/2}\\
&\quad\times\Big\{\int_\Omega|\frac{\partial f}{\partial t}(x,u_m)-\frac{\partial f}{\partial t}(x,u)|w^2dx|\Big\}^{N/2}\\
&\le c^2\{\int_\Omega|\frac{\partial f}{\partial t}(x,u_m(x))-\frac{\partial f}{\partial t}(x,u(x))|^{N/2}dx\}^{N/2}\|v\|\|w\|.
\end{aligned}\tag{2.13}$$

Combining (2.12) and (2.13), we obtain the continuity of D^2J.

(ii) Let $\{y_m\}$ be a sequence in Y with $a_m=\|y_m\|\to\infty$. We shall prove that

$$\limsup_{m\to\infty}\frac{J(y_m)}{a_m^2}<0.$$

Put $w_m=\frac{y_m}{a_m}$. Since $\|w_m\|=1$ and Y is of finite dimension we may assume that $\{w_m\}$ converges to w in Y with $\|w\|=1$. When m goes to ∞, by (1.2), (2.4), (2.5) and the mean value theorem, we have

$$\begin{aligned}
&|\frac{J(a_mw_m)-J(a_mw)}{a_m^2}|\\
&\le|\frac{\int_\Omega[\frac{1}{2}(|\nabla a_mw_m|^2-|a_m\nabla w|^2)-F(x,a_mw_m(x))+F(x,a_mw(x))]dx}{a_m^2}|\\
&\le\|w_m+w\|\|w_m-w\|+\int_\Omega V_2|w_m-w|^2dx\\
&\le(\|w_m+w\|+K\|w_m-w\|)\|w_m-w\|\to 0.
\end{aligned}\tag{2.14}$$

Thus

$$\limsup_{m\to\infty}\frac{J(a_mw_m)}{a_m^2}=\limsup_{m\to\infty}\frac{J(a_mw)}{a_m^2}.$$

Let s be in $(0,1]$ and x be in Ω such that $w(x)\neq 0$. Then $\lim_{m\to\infty}|sa_mw(x)|=\infty$ and by (1.4),

$$\liminf_{m\to\infty}\frac{f(x,sa_mw(x))}{sa_mw(x)}=V_4(x) \tag{2.15}$$

Put $D = \{x \in \Omega : w(x) \neq 0\}$. By (1.3), $\frac{f(x,t)}{t} + V_2(x) \geq 0$ for any (x,t) in $\Omega \times \mathbb{R}$. Therefore, by a general version of Fatou's lemma, (1.2) and (1.4), we have

$$\begin{aligned}
\liminf_{m\to\infty} a_m^{-2} \int_\Omega F(x, a_m w(x))dx &= \liminf_{m\to\infty} a_m^{-2} \int_\Omega \int_0^1 f(x, sa_m w(x)) a_m w(x)\, ds\, dx \\
&= \liminf_{m\to\infty} \int_D \int_0^1 \frac{f(x, sa_m w(x))}{sa_m w(x)} sw^2(x)\, ds\, dx \\
&\geq \int_D [\int_0^1 \liminf_{m\to\infty} \frac{f(x, sa_m w(x))}{sa_m w(x)} sw^2(x)\, ds\, dx \\
&= \int_D \int_0^1 V_4(x) sw^2(x)\, ds\, dx \\
&= \frac{1}{2} \int_D V_4(x) w^2(x) dx
\end{aligned} \tag{2.16}$$

Combining (2.14), (2.16) and (1.8), we obtain

$$\begin{aligned}
\limsup_{m\to\infty} \frac{J(a_m w_m)}{a_m^2} &= \limsup_{m\to\infty} \Big[\int_\Omega |\nabla w|^2 dx - a_m^{-2} \int_\Omega F(x, sa_m w(x)) dx \Big] \\
&= \frac{1}{2} \int_\Omega |\nabla w|^2 dx - \liminf_{m\to\infty} a_m^{-2} \int_\Omega F(x, sa_m w(x)) dx \\
&\leq \frac{1}{2} \int_\Omega [|\nabla w|^2 dx - V_4(x) w^2(x)] dx \\
&\leq -\frac{1}{2} C_2 \|w\|^2 < 0,
\end{aligned}$$

which completes the proof. □

Lemma 2.10. *(i) For every $y \in Y$ and $z_1, z \in Z$,*

$$\langle DJ(y + z_1) - DJ(y + z), z_1 - z\rangle \geq C_1 \|z_1 - z\|^2. \tag{2.17}$$

Moreover, if $\{u_m\}$ converges weakly to u_0 in H then: (ii)

$$\limsup_{m\to\infty} \langle DJ(u_m), u_m - u_0\rangle \geq C_1 \limsup_{m\to\infty} \|u_m - u_0\|^2, \tag{2.18}$$

(iii)

$$\limsup_{m\to\infty} \langle DJ_+(u_m), u_m - u_0\rangle \geq C_1 \limsup_{m\to\infty} \|u_m - u_0\|^2, \tag{2.19}$$

(iv)

$$\limsup_{m\to\infty} \langle DJ_-(u_m), u_m - u_0\rangle \geq C_1 \limsup_{m\to\infty} \|u_m - u_0\|^2. \tag{2.20}$$

Proof. (i) By (2.8), (1.3), (1.7) and the orthogonality between Y and Z in H, we have for all $y \in Y$ and $z_1, z \in Z$,

$$\begin{aligned}
&\langle DJ(y + z_1) - DJ(y + z), z_1 - z\rangle \\
&= \int_\Omega |\nabla(z_1 - z)|^2 dx - \int_\Omega (f(x, y + z_1) - f(x, y + z))(z_1 - z) dx \\
&\geq \int_\Omega |\nabla(z_1 - z)|^2 dx - \int_\Omega V_3(x)(z_1 - z)^2 dx \geq C_1 \|z_1 - z\|^2.
\end{aligned}$$

(ii) Write $u_m = y_m + z_m$ and $u_0 = y_0 + z_0$, where $y_m, y_0 \in Y$ and $z_m, z_0 \in Z$. Using the orthogonality between Y and Z in H, we obtain $y_m \rightharpoonup y_0$ and $z_m \rightharpoonup z_0$

in H. Since Y is finite dimensional, $\{y_m\}$ converges strongly to y_0 in H. By (2.11), DJ is Lipschitz continuous on H. Thus $DJ(A)$ is bounded for any bounded subset A of H. (i) implies that

$$
\begin{aligned}
&\limsup_{m\to\infty}\langle DJ(u_m), u_m-u_0\rangle\\
&=\limsup_{m\to\infty}\langle DJ(u_m), y_m-y_0+z_m-z_0\rangle\\
&=\limsup_{m\to\infty}\langle DJ(y_m+z_m), z_m-z_0\rangle\\
&=\limsup_{m\to\infty}\langle DJ(y_m+z_m)-DJ(y_m+z_0), z_m-z_0\rangle\\
&\quad+\lim_{m\to\infty}\langle DJ(y_m+z_0)-DJ(y_0+z_0), z_m-z_0\rangle+\lim_{m\to\infty}\langle DJ(y_0+z_0), z_m-z_0\rangle\\
&=\limsup_{m\to\infty}\langle DJ(y_m+z_m)-DJ(y_m+z_0), z_m-z_0\rangle\\
&\ge C_1\limsup_{m\to\infty}\|z_m-z_0\|^2\\
&=C_1\limsup_{m\to\infty}\|u_m-u_0\|^2.
\end{aligned}
$$

(iii) Arguing as in Lemma 2.9, by (2.2) and (2.4), we have

$$
\begin{aligned}
&\langle DJ_+(u_m), u_m-u_0\rangle\\
&=\int_\Omega \nabla u_m\nabla(u_m-u_0)dx-\int_\Omega[f(x,u_m^+(x))-V_1(x)u_m^-(x)](u_m(x)-u_0(x))dx\\
&=\langle DJ(u_m), u_m-u_0\rangle+\int_\Omega[f(x,u_m^-(x))+V_1(x)u_m^-(x)](u_m)-u_0)dx.
\end{aligned}
\tag{2.21}
$$

Let q be in $[1,\frac{N}{N-2})$ such that $\frac{1}{r}+\frac{1}{2q}=1$. By Rellich-Kondrachov's theorem, Hölder's theorem, (1.2) and (2.7), $\{u_m\}$ converges strongly to u_0 in $L^{2q}(\Omega)$ and

$$
\begin{aligned}
&|\int_\Omega[f(x,u_m^-(x)+V_1(x)u_m^-(x)](u_m(x)-u_0(x))dx\big|\\
&\le 2c^2\{\int_\Omega W^r dx\}^{1/r}\|u_m^-\|\|u_m-u_0\|\to 0\quad\text{as } m\to 0.
\end{aligned}
\tag{2.22}
$$

Thus by (ii), (2.21) and (2.22), we obtain

$$
\limsup_{m\to\infty}\langle DJ_+(u_m), u_m-u_0\rangle=\limsup_{m\to\infty}\langle DJ(u_m), u_m-u_0\rangle\ge C_1\|u_m-u_0\|^2.
$$

(iv) The proof of (iv) is similar to the proof of (2.20), and is omitted. □

Lemma 2.11. *The operators DJ and $DJ_\pm$ are of class $(S)_+$.*

The proof of the above lemma follows from Lemma 2.10 and Definition 2.1.

Lemma 2.12. *Let P be the orthogonal projection of H onto Y. Let $N(u)(x)=f(x,u(x))$ for all u in H and x in Ω. Then:*

(i) For any y in Y, there exists a unique $\psi(y)\in Z$ such that $\psi(y)|_{\partial\Omega}=0$, $J(y+\psi(y))=\min_{z\in Z}J(y+z)$ and

$$
(I-P)DJ(y+\psi(y))=-\Delta\psi(y)-(I-P)N(y+\psi(y))=0. \tag{2.23}
$$

(ii) The mapping ψ is continuous on Y.

(iii) The reduction mapping $\tilde{J} : Y \to \mathbb{R}$ determined by $\tilde{J}(y) = J(y + \psi(y))$ is of class C^1, and

$$D\tilde{J}(y) = PDJ(y + \psi(y)).$$

Moreover, y is a critical point of $\tilde{J}$ if and only if $y + \psi(y)$ is a critical point of J.

(iv) If $u_0 = y_0 + \psi(y_0)$ is an isolated critical point of mountain-pass type of J then y_0 is a critical point of mountain-pass type of $\tilde{J}$.

(v) If $y_0 \in Y$ such that $y_0 + \psi(y_0)$ is an isolated critical point of J, then

$$i(D\tilde{J}, y_0) = i(DJ, y_0 + \psi(y_0)). \tag{2.24}$$

Proof. The proofs of (i), (ii), (iii), (iv) are based on (2.17) and can be found in [7, Lemma 1] and [8, Lemma 2.1].

(v) Put $u_0 = y_0 + \psi(y_0)$. Because DJ and ψ are continuous and u_0 is an isolated critical point of J, we can choose $M > 0$ and $r > 0$ such that u_0 is the unique critical point of J in $\overline{B_r(u_0)}$ and

$$\|DJ(y + t\psi(y) + (1-t)z)\| \le M \quad \forall u = y + z \in \overline{B_r(u_0)},\ t \in [0,1].$$

We put

$$\begin{aligned} h_1(t,u) = PDJ(y + t\psi(y) + (1-t)z) + (1-t)(I-P)DJ(y+z) \\ + t(z - \psi(y)) \quad \forall t \in [0,1], u = y + z \in Y \oplus Z. \end{aligned} \tag{2.25}$$

First we show that u_0 is the unique zero of $h_1(t,\cdot)$ in $\overline{B_r(u_0)}$ for all $t \in [0,1]$. Indeed, let $(t,u) \in [0,1] \times \overline{B_r(u_0)}$ such that $u = y + z$ in $Y \oplus Z$ and $h_1(t,u) = 0$. By (i)

$$\langle DJ(y + \psi(y)), w\rangle = 0 \quad \forall w \in Z. \tag{2.26}$$

Thus by (2.17), we have

$$\begin{aligned} 0 &= \langle h_1(t,u), z - \psi(y)\rangle \\ &= (1-t)\langle DJ(y+z), z - \psi(y) > +t\|z - \psi(y)\|^2 \\ &= (1-t)\langle DJ(y+z) - DJ(y+\psi(y)), z - \psi(y)\rangle + t\|z - \psi(y)\|^2 \\ &\ge [(1-t)C_1 + t]\|z - \psi(y)\|^2, \end{aligned}$$

which implies $z = \psi(y)$. Therefore, by (i) and (2.25),

$$\begin{aligned} 0 = h_1(t,u) &= h_1(t, y + \psi(y)) \\ &= PDJ(y + \psi(y)) + (1-t)(I-P)DJ(y+\psi(y)) \\ &= DJ(y + \psi(y)) - t(I-P)DJ(y+\psi(y)) \\ &= DJ(y+\psi(y)) = DJ(u). \end{aligned}$$

By the choice of r, we obtain $u = u_0$.

We will prove that h_1 is a homotopy of class $(S)_+$ on $\overline{B_r(u_0)}$. Let $\{(t_m, u_m)\}$ be a sequence in $[0,1] \times \overline{B_r(u_0)}$ such that $\{t_m\}$ converges to t in $[0,1]$ and $\{u_m\}$ converges weakly to u in $\overline{B_r(u_0)}$ and

$$\limsup_{m\to\infty} \langle h_1(t_m, u_m), u_m - u\rangle \le 0. \tag{2.27}$$

We will show that $\{u_m\}$ converges strongly to u in H. We write $u_m = y_m + z_m$ and $u = y + z$, where (y,z) and (y_m, z_m) are in $Y \times Z$ for any integer n.

Since Y is finite-dimensional, $\{y_m\}$ converges strongly to y. Using the continuity of ψ and the boundedness of $\{DJ(y_m + t_m\psi(y_m) + (1-t_m)z_m)\}$ and $\{u_m\}$, we can assume that $\{PDJ(y_m+t_m\psi(y_m)+(1-t_m)z_m)\}$ and $\{PDJ(u_m)\}$ converge strongly in H. By (2.25) we have

$$\begin{aligned}
&\langle h_1(t_m,u_m), u_m-u\rangle\\
&= \langle PDJ(y_m+t_m\psi(y_m)+(1-t_m)z_m), u_m-u\rangle\\
&\quad -(1-t_m)\langle PDJ(u_m), u_m-u\rangle + (1-t_m)\langle DJ(u_m)-DJ(u), u_m-u\rangle\\
&\quad +(1-t_m)\langle DJ(u), u_m-u\rangle + t_m\rangle z_m, y_m-y\rangle\\
&\quad + t_m\langle z_m-z, z_m-z\rangle + t_m\langle z, z_m-z\rangle - t_m\langle \psi(y_m), u_m-u\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{m\to\infty}(1-t_m)\langle DJ(u_m)-DJ(u), u_m-u\rangle\\
&=\limsup_{m\to\infty}(1-t_m)\langle DJ(y+z_m)-DJ(y+z), z_m-z\rangle.
\end{aligned}$$

Thus, by (2.17), (2.27) is equivalent to

$$\begin{aligned}
0 &\ge \limsup_{m\to\infty}\{(1-t_m)\langle DJ(y+z_m)-DJ(y+z), z_m-z\rangle + t_m\|z_m-z\|^2\}\\
&\ge \limsup_{m\to\infty}\{(1-t_m)C_1\|z_m-z\|^2 + t_m\|z_m-z\|^2\}\\
&\ge \min\{C_1,1\}\lim_{m\to\infty}\|z_m-z\|^2,
\end{aligned}$$

which gives the strong convergence of $\{u_m\}$ to u. Hence, h_1 is a homotopy of class $(S)_+$ on $\overline{B_r(u_0)}$. Since $h_1(0,u) = DJ(u)$, by Proposition 2.2 we have

$$i(DJ, y_0+\psi(y_0)) = i(h_1(1,\cdot), y_0+\psi(y_0)). \tag{2.28}$$

For (t,u) in $[0,1]\times\overline{B_r(u_0)}$, put

$$\begin{aligned}
h_2(t,u) = u &+ t[PDJ(Pu+\psi(Pu)) - Pu - \psi(Pu)]\\
&+ (1-t)[PDJ(Pu+\psi(Pu)) - Pu - \psi(y_0)]
\end{aligned}\tag{2.29}$$

We write $u = y+z$ with $y = P(u)$ and $z = u-P(u)$, we have $u = P(u)+u-P(u)$ and

$$\begin{gathered}
h_2(t,u) = PDJ(y+\psi(y)) + t(z-\psi(y)) + (1-t)(z-\psi(y_0)),\\
h_2(0,u) = PDJ(y+\psi(y)) + z - \psi(y_0),\\
h_2(1,u) = PDJ(y+\psi(y)) + z - \psi(y).
\end{gathered}$$

If $h_2(t,u) = 0$ for some (t,u) in $[0,1]\times\overline{B_r(u_0)}$, then it is implied that

$$PDJ(y+\psi(y)) = Ph_2(t,u) = 0.$$

Thus by (2.23), we see that $y+\psi(y)$ is a critical point of J. Since u_0 is the unique critical point of J in $\overline{B_r(u_0)}$, we have $y = y_0$. Hence

$$0 = (I-P)h_2(t,u) = z - t\psi(y) - (1-t)\psi(y_0) = z - \psi(y_0).$$

Thus $z = \psi(y_0)$ and $u = u_0$. Therefore $h_2(t,.)$ has a unique zero $u = u_0$ in $\overline{B_r(u_0)}$ for all $t\in[0,1]$, and h_2 is a homotopy on $[0,1]\times\overline{B_r(u_0)}$ of the compact vector fields $h_2(0,.)$ and $h_2(1,.)$. By the product formula and the homotopy invariance of topological degree for compact vector fields, we have

$$i(D\tilde{J}, y_0) = i(h_2(0,.), y_0+\psi(y_0)) = i(h_2(1,.), y_0+\psi(y_0)). \tag{2.30}$$

Combining (2.28), (2.30) and the fact that $h_1(1,\cdot) = h_2(1,\cdot)$ we obtain (v). □

Lemma 2.13. *There exist positive real numbers r and C such that*

$$C\|u\|^2 \leq \min\{J(u), J_+(u), J_-(u)\} \forall u \in B_r(0).$$

Proof. By (2.2), (2.3), (2.4) and (1.6), we have

$$J_+(u) = J(u^+) + \frac{1}{2}\int_\Omega [|\nabla u^-|^2 - V_1|u^-|^2]dx \geq J(u^+) + C_0\|u^-\|^2,$$
$$J_-(u) = J(u^-) + \frac{1}{2}\int_\Omega [|\nabla u^+|^2 - V_1|u^+|^2]dx \geq J(u^-) + C_0\|u^+\|^2.$$

Thus, since $\|u\|^2 = \|u^+\|^2 + \|u^-\|^2$, it suffices to show that there exist positive constants C and r such that $J(u) \geq C\|u\|^2$ for all u in $B_r(0)$. Assume by contradiction that there exist sequences $\{u_m\} \subset H$ and $\{s_m\} \subset \mathbb{R}$ such that $0 < a_m = \|u_m\| \to 0$, $s_m \to 0$ and $J(u_m) \leq s_m\|u_m\|^2$; i.e.,

$$\frac{1}{2}\int_\Omega |\nabla v_m|^2 dx - \int_\Omega \frac{F(x, a_m v_m)}{a_m^2} dx \leq s_m, \tag{2.31}$$

where $v_m = a_m^{-1}u_m$. Since $\|v_m\| = 1$ we can assume that $v_m \rightharpoonup v_0$ in H. By Hospital's rule,

$$\lim_{t\to 0} \frac{F(x,t)}{t^2} = \frac{1}{2}V_1(x). \tag{2.32}$$

By (1.2), we have

$$|\frac{F(x, a_m v_0(x))}{a_m^2}| = a_m^{-2}|\int_0^1 f(x, ta_m v_0(x))a_m v_0(x)dt| \leq \frac{1}{2}V_2(x)v_0^2(x)$$

Since $V_2 v_0^2$ is integrable, by Lebesgue's dominated convergence theorem, (1.6) and (2.32),

$$\lim_{m\to\infty}\int_\Omega (\frac{1}{2}|\nabla v_0|^2 - \frac{F(x, a_m v_0(x))}{a_m^2})dx = \frac{1}{2}\int_\Omega (|\nabla v_0|^2 - V_1 v_0^2)dx \geq 0. \tag{2.33}$$

Combining (2.31) and (2.33), one has

$$\limsup_{m\to\infty}\int_\Omega [\frac{1}{2}(|\nabla v_m|^2 - |\nabla v_0|^2) - \frac{F(x, a_m v_m(x)) - F(x, a_m v_0(x))}{a_m^2}]dx \leq 0. \tag{2.34}$$

On the other hand, by (2.3) and (1.2), and replacing $\{v_m\}$ by its subsequence converging pointwise to v_0 in Ω as in (F3), which is also denoted by $\{v_m\}$, we have

$$\begin{aligned}
&|\int_\Omega \frac{F(x, a_m v_m(x)) - F(x, a_m v_0(x))}{a_m^2}dx| \\
&= \Big|\int_\Omega\int_0^1 \frac{f(x, sa_m v_m(x) - f(x, sa_m v_0(x))}{a_m}(v_m(x) - v_0(x))\, ds\, dx\Big| \\
&\leq \frac{1}{2}\int_\Omega V_2(x)(v_m(x) - v_0(x))^2 dx,
\end{aligned}$$

which converges to 0 as $n \to \infty$. Thus, by (2.34) we obtain $\lim_{m\to\infty}\|v_m\| = \|v_0\|$. Since $v_m \rightharpoonup v_0$ in H, we see that $v_m \to v_0$ in H and $\|v_0\| = 1$. By (2.31), (2.33) and (1.6) we have

$$0 \geq \limsup_{m\to\infty}\frac{1}{2}\int_\Omega [|\nabla v_0|^2 - \frac{F(x, a_m v_0(x))}{a_m^2}]dx = \frac{1}{2}\int_\Omega (|\nabla v_0|^2 - V_1 v_0^2)dx \geq C_0 > 0,$$

This contradiction completes the proof. □

Lemma 2.14. *Let u_0 be a critical point of J in H. Then J satisfies condition (Φ) at u_0.*

Proof. This proof is based on the ideas by Manes and Micheletti in [24] and the regularity and strong Harnack's inequality for Schrodinger operators in [33] (see also [35]). Let B be the second derivative of J at u_0. We have

$$\langle Bw, \varphi\rangle = \int_\Omega [\nabla w \nabla \varphi - \frac{\partial f}{\partial t}(x, u_0(x)) w\varphi] dx \quad \forall w, \varphi \in H. \tag{2.35}$$

Suppose that 0 is the smallest eigenvalue of B, we must show that it is simple. Let E_1 be its corresponding eigenspace. The proof consists of four steps.

Step 1. Firstly, we show that B is of class $(S)_+$. Let $\{w_m\}$ be a sequence converging weakly to w_0 in H and

$$\limsup_{m\to\infty} < B(w_m), w_m - w_0 > \leq 0. \tag{2.36}$$

By (F3), we can suppose that $\{w_m\}$ convergent pointwise to w_0 and there is a measurable function g having the following properties: $|w_m| \leq g$ for every integer m and $|\frac{\partial f}{\partial t}(x, u_0(x))|g^2$ is integrable on Ω. Thus by Lebesgue's Dominated convergence theorem, we have

$$\lim_{m\to\infty} \int_\Omega \frac{\partial f}{\partial t}(x, u_0(x))(w_m - w_0)^2 dx = 0. \tag{2.37}$$

By (2.35), we have

$$\begin{aligned} &\langle B(w_m) - B(w_0), w_m - w_0\rangle \\ &= \int_\Omega [|\nabla(w_m - w_0)|^2 - \frac{\partial f}{\partial t}(x, u_0(x))(w_m - w_0)^2] dx \\ &= \|w_m - w_0\|^2 - \int_\Omega \frac{\partial f}{\partial t}(x, u_0(x))(w_m - w_0)^2 dx. \end{aligned} \tag{2.38}$$

Since $\{w_m\}$ converges weakly to w_0, By (2.35) and (2.37), we obtain

$$\begin{aligned} &\lim_{m\to\infty} |\langle B(w_0), w_m - w_0\rangle| \\ &\leq \lim_{m\to\infty} \| \int_\Omega [\nabla w_0 \nabla(w_m - w_0)| + \lim_{m\to\infty} |\frac{\partial f}{\partial t}(x, u_0(x)) w_0 (w_m - w_0) = 0 \end{aligned} \tag{2.39}$$

Combining (2.36), (2.37), (2.38) and (2.39), we see that $\{w_m\}$ converges strongly to w_0. Hence B is of class $(S)_+$. Since B is of class $(S)_+$ and 0 is the smallest eigenvalue of B, by Lemma 3.2, we have

$$\langle Bw, w\rangle \geq 0 \quad \forall w \in H, \tag{2.40}$$

and the equality holds if and only if $Bw = 0$.

Step 2. We show that w_0^+ and w_0^- are in E_1 for any w_0 in E_1. Indeed, let w_0 be in E_1. By (2.35), we have

$$0 = \langle Bw_0, w_0\rangle = \langle Bw_0^+, w_0^+\rangle + \langle Bw_0^-, w_0^-\rangle.$$

Hence, (2.40) implies

$$\langle Bw_0^+, w_0^+\rangle = \langle Bw_0^-, w_0^-\rangle = 0. \tag{2.41}$$

By (2.40) and (2.41) we obtain

$$Bw_0^+ = Bw_0^- = 0. \tag{2.42}$$

Step 3. We show that if $w_0 \in E_1$ then w_0 is continuous and either positive, negative or identically vanishing in Ω. As in Lemma 2.8, $\frac{\partial f}{\partial t}(x, u_0(x))$ is of Kato's class. Using (2.42) and results of [33, sections 2 and 3], we see that w_0^+ and w_0^- are continuous functions. If w_0 vanishes in Ω then this step is done. Now suppose that $w_0(x_1) \neq 0$ for some $x_1 \in \Omega$. If $w_0(x_1) > 0$, by Lemma 2.8, we see that w_0^+ is positive. Thus $w_0(x) = w_0^+(x) > 0$ for all $x \in \Omega$. If $w_0(x_1) < 0$, a similar argument shows that $w_0(x) < 0$ for all x in Ω.
Step 4. Finally, we show that 0 is a simple eigenvalue of B. Indeed, let w_1 and w_2 be two distinct elements of E such that $w_2(x_0) \neq 0$ for some $x_0 \in \Omega$. Put $\lambda = \frac{w_1(x_0)}{w_2(x_0)}$ and $w_3 = w_1 - \lambda w_2$. Then $Bw_3 = 0$ and $w_3(x_0) = 0$. Thus $w_3 \equiv 0$ and therefore $w_1 = \lambda w_2$ or 0 is simple. □

Lemma 2.15. *J_+ and J_- satisfy the Palais-Smale condition.*

Proof. We only give the proof for J_+, because the case of J_- is similar. Let $\{u_m\}$ be a sequence in H such that $\|DJ_+(u_m)\| \leq \frac{1}{m}$ for any n. We prove that $\{u_m\}$ has a converging subsequence. Because DJ_+ is of class $(S)_+$ it suffices to show that $\{u_m\}$ is bounded in H. For any m in $\mathbb{N}$ and any φ in H

$$|\langle DJ_+(u_m), \varphi\rangle| = \Big|\int_\Omega [\nabla u_m \nabla\varphi - (f(x, u_m^+) + V_1(x)u_m^-)\varphi]dx\Big| \leq \frac{1}{m}\|\varphi\|. \tag{2.43}$$

Using $\varphi = u_m^-$ in (2.43), by (1.6), we obtain

$$C_0\|u_m^-\|^2 \leq \int_\Omega [|\nabla u_m^-|^2 - V_1(x)|u_m^-|^2]dx \leq \frac{1}{m}\|u_m^-\|.$$

Thus $\lim_{m\to\infty} u_m^- = 0$ in H. Thus there is a sequence of positive real numbers $\{\varepsilon_m\}$ converging to 0 such that

$$|\int_\Omega [\nabla u_m^+ \nabla\varphi - f(x, u_m^+)\varphi]dx| \leq \varepsilon_m\|\varphi\| \quad \forall\varphi \in H. \tag{2.44}$$

Let v_m and w_m be in Y and Z respectively such that $u_m^+ = v_m + w_m$ for any integer m. Put $a_m = \|u_m^+\|$ for every positive integer n. Suppose by contradiction that

$$\lim_{m\to\infty} \|u_m^+\|^2 \equiv \lim_{m\to\infty} (\|v_m\|^2 + \|w_m\|^2) = \infty.$$

Replacing $\{u_m^+\}$ by its subsequence, by (F3), we can assume that $\{\frac{u_m^+}{a_m}\}$, $\{\frac{v_m}{a_m}\}$ and $\{\frac{w_m}{a_m}\}$ converge almost everywhere on Ω, and there is a measurable function g such that $\frac{|u_m^+|}{a_m} + \frac{|v_m|}{a_m} + |\frac{w_m|}{a_m} \leq g$ and V_2g^2, V_3g^2 and V_4g^2 are integrable on Ω. Put $D = \{x \in \Omega : \sup_m u_m^+(x) < \infty\}$. We have

$$\lim_{m\to\infty} \frac{u_m^+}{a_m}(x) = 0 \quad \forall x \in D, \tag{2.45}$$

$$\lim_{m\to\infty} \frac{|v_m^2 - w_m^2|}{a_m^2}(x) = \lim_{m\to\infty} \frac{u_m^+}{a_m}\frac{|v_m - w_m|}{a_m}(x) \leq \lim_{m\to\infty} 2g(x)\frac{u_m^+}{a_m}(x) = 0 \quad \forall x \in D. \tag{2.46}$$

$$\frac{|v_m^2 - w_m^2|}{a_m^2}(x) \leq 2g^2(x) \quad \forall x \in D. \tag{2.47}$$

Using $\varphi = w_m$ and v_m in (2.44), by (1.3), (1.7) and (1.8), we have

$$\begin{aligned}
&\varepsilon_m\|w_m\| \\
&\geq \int_\Omega [|\nabla w_m|^2 - f(x,u_m^+)w_m]dx \\
&= \|w_m\|^2 - \int_\Omega \frac{f(x,u_m^+(x))}{u_m^+(x)} u_m^+ w_m dx \\
&= [\|w_m\|^2 - \int_\Omega V_3 w_m^2 dx] + \int_\Omega V_3 w_m^2 dx - \int_\Omega \frac{f(x,u_m^+(x))}{u_m^+(x)} u_m^+ w_m dx \\
&\geq C_1\|w_m\|^2 + \int_\Omega [V_3 - \frac{f(x,u_m^+(x))}{u_m^+(x)}] w_m^2 dx - \int_\Omega \frac{f(x,u_m^+(x))}{u_m^+(x)} v_m w_m dx,
\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}
&\varepsilon_m\|v_m\| \\
&\geq -\int_\Omega [|\nabla v_m|^2 - f(x,u_m^+)v_m]dx \\
&= -[\|v_m\|^2 - \int_\Omega V_4 v_m^2 dx] + \int_\Omega V_4 v_m^2 dx - \frac{f(x,u_m^+(x))}{u_m^+(x)} u_m^+ v_m dx \\
&\geq C_2\|v_m\|^2 + \int_\Omega [\frac{f(x,u_m^+(x))}{u_m^+(x)} - V_4] v_m^2 dx + \int_\Omega \frac{f(x,u_m^+(x))}{u_m^+(x)} w_m v_m dx.
\end{aligned} \tag{2.49}$$

Put $\gamma = \min\{C_1, C_2\} > 0$. Using the orthogonality of v_m and w_m in H, by (1.4), (2.48) and (2.49), we obtain

$$\begin{aligned}
&\frac{2\varepsilon_m}{a_m} - \gamma \\
&\geq \int_\Omega V_3 \frac{w_m^2}{a_m^2} dx - \int_\Omega \frac{f(x,u_m^+(x))}{u_m^+(x)} \frac{w_m^2}{a_m^2} dx - \int_\Omega V_4 \frac{v_m^2}{a_m^2} dx + \int_\Omega \frac{f(x,u_m^+(x))}{u_m^+(x)} \frac{v_m^2}{a_m^2} dx \\
&= \int_D [V_3 - V_4] \frac{w_m^2}{a_m^2} dx + \int_D V_4 [\frac{w_m^2}{a_m^2} - \frac{v_m^2}{a_m^2}] dx - \int_D \frac{f(x,u_m^+(x))}{u_m^+(x)} [\frac{w_m^2}{a_m^2} - \frac{v_m^2}{a_m^2}] dx \\
&\quad + \int_{\Omega\setminus D} [V_3 - \frac{f(x,u_m^+(x))}{u_m^+(x)}] \frac{w_m^2}{a_m^2} dx - \int_{\Omega\setminus D} [V_4 - \frac{f(x,u_m^+(x))}{u_m^+(x)}] \frac{v_m^2}{a_m^2} dx \\
&\geq \int_D [V_4 - \frac{f(x,u_m^+(x))}{u_m^+(x)}][\frac{w_m^2}{a_m^2} - \frac{v_m^2}{a_m^2}] dx \\
&\quad + \int_{\Omega\setminus D} [\frac{f(x,u_m^+(x))}{u_m^+(x)} - V_4] \frac{v_m^2}{a_m^2} dx
\end{aligned} \tag{2.50}$$

By (2.46), (2.47), (1.4), Lebesgue's dominated convergence theorem and Fatou's Lemma, as in (2.16), we have

$$\lim_{m\to\infty} \int_D [V_4 - \frac{f(x,u_m^+(x))}{u_m^+(x)}][\frac{w_m^2}{a_m^2} - \frac{v_m^2}{a_m^2}] dx = 0, \tag{2.51}$$

$$\begin{aligned}
&\liminf_{m\to\infty} \int_{\Omega\setminus D} [\frac{f(x,u_m^+(x))}{u_m^+(x)} - V_4] \frac{v_m^2}{a_m^2} dx \\
&\geq \int_{\Omega\setminus D} \liminf_{m\to\infty} [\frac{f(x,u_m^+(x))}{u_m^+(x)} - V_4] \frac{v_m^2}{a_m^2} dx \geq 0.
\end{aligned} \tag{2.52}$$

Combining (2.50), (2.51) and (2.52), we obtain a contradiction, which implies the boundedness of $\{u_m^+\}$. Therefore, $\{u_m\}$ is bounded in H and we have the conclusion. $\square$

Lemma 2.16. *(i) If R is sufficiently large then DJ_+ and DJ_- have no solution outside $B_R(0)$, and (ii)* $\deg(DJ_+, B_R(0), 0) = \deg(DJ_-, B_R(0), 0) = 0$.

Proof. Using Lemma 2.15 we obtain (i). It suffices to prove (ii) for J_+. Let η_1 and η_2 be two positive numbers such that $\eta_1 < \lambda_1 < \lambda_k < \eta_2 < \lambda_{k+1}$. For any u in H, by the Riesz representation theorem, there is a unique $\pi(u)$ in H such that

$$\langle \pi(u), \varphi\rangle = \int_\Omega [\nabla u \nabla \varphi - \eta_2 u^+ \varphi + \eta_1 u^- \varphi] dx \quad \forall \varphi \in H.$$

It is easy to prove that π is a compact-vector field on H. Arguing as in [8, Lemma 3.1] we see that 0 is the unique zero of π and $\deg(\pi, B, 0) = 0$ if B is a ball in H containing zero. Put

$$h(s, u) = sDJ_+(u) + (1 - s)\pi(u) \quad \forall (s, u) \in [0, 1] \times H.$$

By the homotopy invariance of topological degree of S_+ operators, it is sufficient to show that there exists a sufficiently large R such that $h(s, u) \neq 0$ for all $sin[0, 1]$ and $u \in H \setminus B_R(0)$. Suppose by contradiction that there exist a sequence $\{u_m\}$ in H and a sequence $\{s_m\}$ in $[0, 1]$ such that $\{s_m\}$ converges to s in $[0, 1]$, $\|u_m\| \geq n$ and

$$s_m DJ_+(u_m) + (1 - s_m)\pi(u_m) = 0$$

or

$$\int_\Omega \nabla u_m \nabla \varphi dx - \int_\Omega \{[s_m f(x, u_m^+) + (1 - s_m)\eta_2 u_m^+] - [s_m V_1(x) + (1 - s_m)\eta_1] u_m^-\} \varphi dx = 0$$

for any $m \in \mathbb{N}$ and $\varphi \in H$.

Arguing as in the proof of Lemma 2.15 with $[s_m f(x, u_m^+) + (1 - s_m)\eta_2 u^+]$ and $[s_m V_1(x) + (1 - s_m)\eta_1]$ instead of $f(x, u_m^+)$ and $V_1(x)$ respectively , we obtain a contradiction. $\square$

Lemma 2.17. *(i) u_0 is a critical point of J_+ (respectively J_-) if and only if u_0 is a nonnegative (respectively non-positive) critical point of J.*

(ii) Moreover if u_0 is a common isolated critical point of both J and J_+ (respectively J_-), then $i(DJ, u_0) = i(DJ_+, u_0)$ (respectively $= i(DJ_-, u_0)$).

Proof. (i) Suppose that u_0 is a critical point of J_+; i.e.,

$$\int_\Omega [\nabla u_0 \nabla \varphi - f(x, u_0^+)\varphi - V_1(x) u_0^- \varphi] dx = 0 \forall \varphi \in H.$$

Choosing $\varphi = u_0^-$, we have

$$\int_\Omega [|\nabla u_0^-|^2 - V_1 |u_0^-|^2] dx = 0.$$

By (1.6), we have $u_0^- = 0$ and thus $u_0 \geq 0$.

(ii) Let u_0 be a common isolated critical point of J and J_+. Choose $r > 0$ such that J and J_+ have no any other critical point inside $B_r(u_0)$. By the homotopy invariance property of topological degree for operators of class $(S)_+$, it is sufficient to show that: there exists $r_1 < r$ such that $sDJ(u) + (1 - s)DJ_+(u) \neq 0$, for all $s \in [0, 1]$ for any u in $B_{r_1}(u_0) \setminus \{u_0\}$. Assume by contradiction that there exists a sequence $\{(u_m, s_m)\}$ in $B_r(u_0) \times [0,1]$ such that $u_m \neq u_0$, $u_m \to u_0$ in H and $s_m DJ(u_m) + (1-s_m)DJ_+(u_m) = 0$. For any φ in H and m in $\mathbb{N}$ we have

$$\int_\Omega (\nabla u_m \nabla \varphi - [f(x,u_m^+) + s_m f(x,u_m^-) + (1-s_m)V_1 u_m^-]\varphi)dx = 0. \tag{2.53}$$

Choosing $\varphi = u_m^-$, we obtain

$$\int_\Omega \{|\nabla u_m^-|^2 - s_m f(x,u_m^-)u_m^- - (1-s_m)V_1(x)|u_m^-|^2\}dx = 0 \quad \forall n \in \mathbb{N}. \tag{2.54}$$

Since $\{u_m\}$ converges to u_0 in H and $u_0 \geq 0$, $\{u_m^-\}$ converges to 0. Using Rellich-Kondrachov's theorem and Egorov's theorem , we can suppose: for any positive real number ε, there is a measurable subset D_ε such that the Lebesgue measure of $\Omega \setminus D_\varepsilon$ is less than ε and $\{u_m^-\}$ converges uniformly to 0 on D_ε.

We claim that $u_m^- \not\equiv 0$ for every n in $\mathbb{N}$. If $u_m^- \equiv 0$ for some n, then u_m is nonnegative and $DJ(u_m) = DJ_+(u_m)$. Thus $DJ(u_m) = s_m DJ(u_m) + (1-s_m)DJ_+(u_m) = 0$, which contradicts the choice of r.

Put $a_m = \|u_m^-\| > 0$ and $v_m = a_m^{-1}u_m^-$. We can assume that $v_m \rightharpoonup v_0$ in H and then $s_m \to s$ in $[0,1]$. By (1.6) and (2.54) we have

$$\begin{aligned} 0 &= \int_\Omega \{|\nabla v_m|^2 - s_m \frac{f(x,a_m v_m)}{a_m v_m} v_m^2 - (1-s_m)V_1 v_m^2\}dx \\ &= \int_\Omega (|\nabla v_m|^2 - V_1 v_m^2)dx - s_m \int_\Omega (\frac{f(x,a_m v_m)}{a_m v_m} - V_1)v_m^2 dx \\ &\geq C_0 + s_m \int_{D_\varepsilon} (\frac{f(x,u_m^-)}{u_m^-} - V_1)v_m^2 dx - s_m \int_{\Omega\setminus D_\varepsilon} (\frac{f(x,a_m v_m)}{a_m v_m} - V_1)v_m^2 dx. \end{aligned} \tag{2.55}$$

Since $\{u_m^-\}$ converges uniformly to 0 on D_ε, we see that $\{\frac{f(x,u_m^-)}{u_m^-} - V_1\}$ converges uniformly to 0 on D_ε and

$$\lim_{m\to\infty} \int_{D_\varepsilon} (\frac{f(x,u_m^-)}{u_m^-} - V_1)v_m^2 dx = 0. \tag{2.56}$$

Since $r > \frac{N}{2}$, there is q in the interval $(1,\infty)$ such that $\frac{1}{q} + \frac{1}{r} + \frac{N-2}{N} = 1$. By (1.2) and Hölder's inequality, we have

$$\begin{aligned} &\Big|\int_{\Omega\setminus D_\varepsilon} (\frac{f(x,a_m v_m)}{a_m v_m} - V_1)v_m^2 dx\Big| \\ &\leq 2\int_{\Omega\setminus D_\varepsilon} W v_m^2 dx \\ &\leq 2(m(\Omega\setminus D_\varepsilon))^{1/q}\Big\{\int_{\Omega\setminus D_\varepsilon} W^r dx\Big\}^{1/r}\Big\{\int_{\Omega\setminus D_\varepsilon} |v_m|^{\frac{2N}{N-2}}dx\Big\}^{\frac{N-2}{N}} \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{2.57}$$

Combining (2.55), (2.56) and (2.57), we find a contradiction, which completes the proof of the lemma. □

Lemma 2.18. *Let u_0 be an isolated critical point of mountain-pass type of J_+ (respectively J_-). Then it is also an isolated critical point of mountain-pass type of J.*

Proof. It suffices to prove the lemma for the case u_0 is an isolated critical point of mountain-pass type of J_+. We shall find a neighborhood U of u_0 such that for all open neighborhood $V_2 \subset U$ of u_0, the set $V \cap J^{-1}(-\infty, J(u_0))$ is nonempty and not path-connected. By calculations, we have

$$J_+(u) = J(u^+) + \frac{1}{2}\int_\Omega [|\nabla u^-(x)|^2 - V_1(x)|u^-(x)|^2]dx \quad \forall u \in H. \tag{2.58}$$

Since u_0 is a critical point of J_+, u_0 is nonnegative by Lemma 2.17. Therefore, $u_0 = u_0^+$ and

$$J(u_0) = J(u_0^+) = J_+(u_0).$$

Put $\mu = J(u_0) = J_+(u_0)$. By definition, there exists a neighborhood E_0 of u_0 such that the set $E \cap {J_+}^{-1}(-\infty, \mu)$ is nonempty and not path-connected for all open neighborhood $E \subset E_0$ of u_0. By Lemma 2.13, we can choose an r such that $J(u^-) \geq 0$ if $\|u^-\| \leq r$. Since

$$\|u - u_0\| = \|(u^+ - u_0) + u^-\| \leq \|u^+ - u_0\| + \|u^-\|,$$

there exist $\delta \in (0, r)$ such that

$$U = \{u : \|u^+ - u_0\| < \delta, \|u^-\| < \delta\} \subset E_0. \tag{2.59}$$

We see that U is an open neighborhood of u_0 and

$$u^+ + tu^- \in U \forall t \in [0, 1], \tag{2.60}$$

$$J(u) = J(u^+) + J(u^-) \geq J(u^+) = J_+(u^+) \quad \forall u \in U. \tag{2.61}$$

Let V be an open neighborhood of u_0 in U. As in (2.59), there is a $\delta_1 \in (0, \delta)$ such that

$$U_1 = \{u : \|u^+ - u_0\| < \delta_1, \|u^-\| < \delta_1\} \subset V. \tag{2.62}$$

We accomplish the proof by following steps

Step 1. We show that $V \cap J^{-1}(-\infty, \mu)$ is nonempty. If $s \geq t \geq 0$, by (2.58) and (1.6) we have

$$J_+(u^+ + su^-) - J_+(u^+ + tu^-) = (s^2 - t^2)\int_\Omega [|\nabla u^-(x)|^2 - V_1(x)|u^-(x)|^2]dx \geq 0. \tag{2.63}$$

Since $u_0 \in U_1 \subset E_0$ and U_1 is open, the set $U_1 \cap {J_+}^{-1}(-\infty, \mu)$ is nonempty. Pick an element v in this set. By (2.63), we have $J(v^+) = J_+(v^+) \leq J^+(v^+ + v^-) = J_+(v) < \mu$, and hence $v^+ \in J^{-1}(-\infty, \mu)$. Furthermore, by (2.62), $v^+ \in U_1$. It follows that $U_1 \cap J^{-1}(-\infty, \mu)$ is nonempty, then $V \cap J^{-1}(-\infty, \mu)$ is nonempty.

Step 2. We show that $S \equiv V \cap J^{-1}(-\infty, \mu)$ is not path-connected. Assume by contradiction that it is path-connected. Put

$$W_1 = \{u^+ + tu^- : u \in U, u^+ \in V \cap J^{-1}(-\infty, \mu), t \in [0, 1]\},$$
$$W_2 = \{u^+ + tu^- : u \in V \cap J^{-1}(-\infty, \mu), t \in (0, 1]\},$$
$$W_3 = \{u : \|u^+ - u_0\| < \delta_1, \|u^-\| < \delta_1\},$$
$$W_0 = W_1 \cup W_2 \cup W_3.$$

It is clear that W_1, W_2, W_3 and W_0 are open sets in E_0, $W_3 = U_1$ and $u_0 \in W_3 \subseteq W_0$. We will show that $G = W_0 \cap {J_+}^{-1}(-\infty, \mu)$ is path-connected, which yields a contradiction.

For any v and w in G, we say $v \sim w$ if and only if there exists a continuous mapping φ from $[1, 2]$ into G such that $\varphi(1) = v$ and $\varphi(2) = w$.

Let w_1 and w_2 be in G. If w_1 and w_2 are in W_1, then by definition we see that $w_1 \sim w_1^+$, $w_2 \sim w_2^+$, and w_1^+ and w_2^+ are in S. Since S is path-connected, there exists a continuous mapping ϕ from $[1,2]$ into G such that $\phi(1) = w_1^+$ and $\phi(2) = w_2^+$. By definition $\varphi(t,\epsilon) = (\phi(t))^+ + \epsilon(\phi(t))^- \in W_2$ for all (t,ϵ) in $[1,2] \times (0,1]$. By (2.61), we obtain $(\phi(t))^+ \in {J_+}^{-1}(-\infty,\mu)$. Hence, by the continuity of J^+ and the compactness of $\phi([1,2])$, $\varphi(t,\epsilon)$ is in G if ϵ is sufficiently small for any t in $[1,2]$. Note that $\varphi(1,\epsilon) = w_1^+$ and $\varphi(2,\epsilon) = w_2^+$. Thus, $w_1^+ \sim w_2^+$ which gives $w_1 \sim w_2$.

If w_1 and w_2 belong to W_3 then $w_1 \sim w_1^+$, $w_2 \sim w_2^+$, and w_1^+ and w_2^+ are in S. Arguing as above we have $w_1 \sim w_2$.

If w_1 and w_2 are in W_2 then $w_1 = u_1^+ + t_1 u_1^-$ and $w_2 = u_2^+ + t_2 u_2^-$ where $t_1, t_2 > 0$, and u_1 and u_2 in S. As in the first case, there is a positive real number ε_0 such that $(u_1^+ + \varepsilon u_1^-) \sim (u_2^+ + \varepsilon u_2^-)$ for any ε in $[0,\varepsilon_0]$. On the other hand we have $w_1 \sim (u_1^+ + \varepsilon u_1^-)$ and $w_2 \sim (u_2^+ + \varepsilon u_2^-)$ for any ε in $(0, \min\{\varepsilon_0, t_1, t_2\}]$. Therefore $w_1 \sim w_2$. Similarly $w_1 \sim w_2$ for other cases. Thus we have shown that $W_0 \cap {J_+}^{-1}(-\infty,\mu)$ is path-connected, contradicting the way we choose U^+. Thus $V \cap J^{-1}(-\infty,\mu)$ is not path-connected. This completes the proof. □

Proof of Theorem 1.1. We use the following steps

Step 1. Note that a weak solution $u \in H$ of (1.5) is a critical point of J and vice-versa. Moreover, it suffices to consider the case in which the set of solutions of (1.5) is finite. In this case, all critical points of J are isolated. Since $f(x,0) = 0$, by Lemma 2.9, $u_1 \equiv 0$ is a solution of (1.5). On the other hand, since φ_1 is positive in Ω, by (ii) of Lemma 2.9, we have

$$\lim_{m\to\infty} J^+(n\varphi_1) = \lim_{m\to\infty} J(n\varphi_1) = -\infty.$$

Thus by [16, Theorem 1] and Lemmas 2.13, 2.15, 2.17 and 2.18, J has a nonnegative critical point of mountain-pass type u_2. Similarly J has a non-positive critical point of mountain-pass type u_3. Let $\widetilde{J}$ be the reduction function of J as in Lemma 2.12. Then it follows from (ii) of Lemma 2.9 that $\lim_{y\in Y, \|y\|\to\infty} \widetilde{J}(y) = -\infty$. Hence $\widetilde{J}$ has a global maximum at y_0 and $u_4 = y_0 + \psi(y_0)$ is a critical point of J. By (iv) of Lemma 2.12 and Definition 2.5, we see that $u_4 \notin \{u_2, u_3\}$. By Lemma 2.13, J has a strictly local minimum at 0, which shows that $\widetilde{J}$ has a strictly local minimum at 0. Hence u_4 is not equal to 0. Thus we have found four distinct solutions u_1, u_2, u_3 and u_4 of (1.5).

Step 2. Choose a sufficiently large real number R such that all critical points of J, J_+, J_- and $\widetilde{J}$ lie in $B_R(0)$. We can find four open subsets U_1, U_2, U_3 and U_4 in $B_R(0)$ such that $U_1 \cap DJ^{-1}(0) = \{u_1\}$, $U_2 \cap DJ^{-1}(0) = \{u \geq 0, u \not\equiv 0\}$, $U_3 \cap DJ^{-1}(0) = \{u \leq 0, u \not\equiv 0\}$, and $U_4 \cap DJ^{-1}(0) = \{u_4\}$. Moreover, we can assume that U_1, U_2 and U_3 are disjoint.

Now we consider the case in which k is even. By Corollaries 2.1 and 2.2 in [14], Lemma 2.13, Lemma 2.12 and Lemma 2.16, we have

$$\begin{gathered}
\deg(DJ, U_1, 0) = 1, \\
\deg(DJ, B_R(0), 0) = \deg(\nabla\widetilde{J}, B_R(0), 0) = (-1)^k = 1, \\
\deg(DJ_+, B_R(0), 0) = \deg(DJ_-, B_R(0), 0) = 0.
\end{gathered}$$

Thus by Lemma 2.17 and the excision property of the degree,

$$\deg(DJ, U_2, 0) = \deg(DJ^+, U_2, 0) = \deg(DJ_+, B_R(0), 0) - \deg(DJ_+, U_1, 0) = -1,$$

$$\deg(DJ, U_3, 0) = \deg(DJ_-, U_3, 0) = \deg(DJ_-, B_R(0), 0) - \deg(DJ_-, U_2, 0) = -1.$$

Since y_0 is a global maximum point of $\widetilde{J}$ and hence a global minimum point of $-\widetilde{J}$, it follows from [14, Corollary 2.2] that $i(\nabla\widetilde{J}, y_0) = (-1)^k = 1$. Thus, by Lemma 2.12(v),

$$\deg(DJ, U_4, 0) = 1.$$

If u_4 is not in $U_2 \cup U_3$, we may assume $U_4 \cap (\cup_{i=1}^3 U_i) = \emptyset$. Thus, by Proposition 2.2,

$$\deg(DJ, B_R(0) \setminus \overline{\cup_{i=1}^4 U_i}, 0) = \deg(DJ, B_R(0), 0) - \sum_{i=1}^4 deg(DJ, U_i, 0) = 1,$$

which implies that J has a sign-changing critical point $u_5 \notin \{u_1, u_2, u_3, u_4\}$. Therefore (1.5) has two sign-changing solutions u_4 and u_5. If u_4 is in $U_2 \cup U_3$, we can assume that $u_4 \in U_2$. By Lemma 2.14, J satisfies (Φ) at u_2, and by Proposition 3.1, we have $i(DJ, u_2) = -1$. Let U_5 be an open neighborhood of u_2 in H containing no other critical point of J then $deg(DJ, U_5, 0) = i(DJ, u_2) = -1$. Thus, by the additivity property of the degree,

$$\deg(DJ, U_2 \setminus \overline{U_4 \cup U_5}, 0) = \deg(DJ, U_2, 0) - \deg(DJ, U_4, 0) - \deg(DJ, U_5, 0) = -1.$$

By the normalization property of the degree, there is a solution u_5 of (1.5) in $U_2 \setminus \overline{U_4 \cup U_5}$. Hence (1.5) has three solutions u_2, u_4, u_5 of the same sign. Moreover, by Proposition 2.2 and the excision property of the degree, we have

$$\deg(DJ, B_R(0) \setminus \overline{\cup_{i=1}^3 U_i}, 0) = \deg(DJ, B_R(0), 0) - \sum_{i=1}^3 deg(DJ, U_i, 0) = 2,$$

implying that (1.5) has a sign-changing solution $u_6 \notin \{u_1, u_2, u_3, u_4, u_5\}$.

Next, suppose that k is odd. If $u_4 \notin U_2 \cup U_3$ then the proof is similar to that of the case k is even. It remains to consider the case $u_4 \in U_2 \cup U_3$. We can assume $u_4 \in U_2$. Let U_5 be as above. Arguing as above, we have

$$\deg(DJ, U_4, 0) = i(\nabla\widetilde{J}, y_0) = (-1)^k = -1,$$
$$\deg(DJ, U_5, 0) = i(DJ, u_2) = -1,$$
$$\deg(DJ, U_2 \setminus \overline{U_4 \cup U_5}, 0) = \deg(DJ, U_2, 0) - \deg(DJ, U_4, 0) - \deg(DJ, U_5, 0) = 1.$$

Thus, by the normalization property of the degree, there exists $u_5 \in U_2 \setminus \overline{U_4 \cup U_5}$ with $DJ(u_5) = 0$. Thus, (1.5) has five solutions u_1, u_2, u_3, u_4 and u_5, where u_2, u_4 and u_5 are of the same sign. The proof is complete. □

3. Appendix

In this section, we extend the results of Hofer [16] on the index at a critical point of mountain-pass type of a functional whose gradient is a compact vector field to the case where the gradient is an operator of class $(S)_+$. Throughout this section, the dual space H^* is identified with H. Our main result of the appendix is the following theorem.

Theorem 3.1. *Let x_0 be in H, and let j be a C^2-real function on H such that Dj is of class $(S)_+$ on H and x_0 is an isolated critical point of mountain-pass type of j. Assume that $D^2 j(x_0)$ is of class $(S)_+$ and j satisfies (Φ) at x_0. Then*

$$i(Dj, x_0) = -1.$$

To prove the above theorem, we need the following lemmas.

Lemma 3.2. *Let A be a bounded self-adjoint linear operator of class $(S)_+$ on H and Y be $A^{-1}(\{0\})$. Then Y is finite dimensional and there exist a positive number C and vector subspaces X and Z of H such that X is finite dimensional and*

- (i) *$X \oplus Y \oplus Z$ is an orthogonal decomposition of H,*
- (ii) *X, Y and Z are invariant under A,*
- (iii) *the restriction of A on $X \oplus Z$ is a one-to-one mapping from $X \oplus Z$ onto itself,*
- (iv) *$\langle Ax, x\rangle \le -C\|x\|^2$ for all $x \in X$,*
- (v) *$\langle Az, z\rangle \ge C\|z\|^2$ for all $z \in Z$.*

Proof. Let $\{y_m\}$ is a sequence in Y and weakly converges to y in H. Since $\limsup_{n\to\infty} < A(y_m), y_m - y >= 0$ and A is of class $(S)_+$, $\{y_m\}$ converges strongly to y. Thus Y is locally compact. It implies Y is finite dimensional.

Put $E = Y^{\perp}$. We see that $\langle Au, v\rangle = \langle u, Av\rangle = 0$ for all $u \in E$, $v \in Y$. Therefore, $A(E) \subset E$. Denote by B the restriction of A on E. We see that B is a bounded self-adjoint linear operator on E. It is clear that B is one-to-one.
We shall prove that $B(E)$ is a closed subspace of E. Let $\{x_m\}$ be a sequence in E such that $\{B(x_m)\}$ converges to y in E, we will prove that $y \in B(E)$. First, we show that $\{x_m\}$ is bounded. Suppose by contradiction that $\{\|x_m\|\}$ tends to ∞. Put $v_m = (\|x_m\| + 1)^{-1}x_m$ for any integer n, then $\{\|v_m\|\}$ converges to 1 and $\{B(v_m)\}$ converges to 0. Without loss of generality, we can (and shall) suppose that $\{v_m\}$ converges weakly to a vector v_0 in E. Since A is of class $(S)_+$, and

$$\limsup_{m\to\infty}\langle A(v_m), v_m - v_0\rangle = \limsup_{m\to\infty}\langle B(v_m), v_m - v_0\rangle = 0,$$

the sequence $\{v_m\}$ converges to v_0. Thus, $\|v_0\| = 1$ and $A(v_0) = 0$, which is a contradiction. Therefore $\{x_m\}$ is bounded and we can suppose that it converges weakly to a vector x_0 in E. Since $\{A(x_m)\}$ converges to y, by the definition of class $(S)_+$, the sequence $\{x_m\}$ converges to x_0 in E. Therefore $A(x_0) = y$ and $B(E)$ is closed.

Next we show that $B(E) = E$. Otherwise, there is a vector x in $E \setminus \{0\}$ such that

$$\langle B(z), x\rangle = 0 \quad \text{or} \quad \langle z, A(x)\rangle = 0 \quad \forall z \in E.$$

Thus, $A(x)$ is in Y. It implies that $A(A(x))$ is also in Y and

$$\langle A(x), A(x)\rangle = \langle x, A(A(x))\rangle = 0.$$

It follows that x is in $Y \cap E$, then $x = 0$, which is impossible. This contradiction shows that $B(E) = E$.

We have proved that B is an one-to-one mapping from E onto E. Thus, by the open mapping theorem, B is an invertible self-adjoint bounded operator on E. By a result on self-adjoint operator (see [18, p. 172]), there exist a positive real number C and an orthogonal decomposition $X \oplus Z$ of E such that X and Z are A-invariant closed subspaces of E and

$$\langle A(x), x\rangle \le -C\|x\|^2 \quad \forall x \in X, \tag{3.1}$$

$$\langle A(x), x\rangle \ge C\|x\|^2 \quad \forall x \in Z. \tag{3.2}$$

Finally we prove that X is finite dimensional. It is sufficient to show that X is locally compact. Let $\{x_m\}$ be a sequence weakly converging to x in X. We see that

$$\lim_{m\to\infty} < A(x), x_m - x >= 0. \tag{3.3}$$

On the other hand, by (3.1),

$$\langle A(x_m - x), x_m - x\rangle \le 0 \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Combining (3.3) and (3.4), we have $\limsup_{m\to\infty}\langle A(x_m), x_m - x\rangle \le 0$. Since A is of class $(S)_+$, $\{x_m\}$ converges to x. Therefore X is locally compact and finite dimensional. The proof is complete. □

Lemma 3.3. *Let U be an open subset of a Hilbert space H, and let j be a C^2-real function on U. Suppose that $j(0) = 0$, 0 is an isolated critical of j, Dj and $D^2j(0)$ are of class $(S)_+$ on H. Let $X \oplus Y \oplus Z$ be the decomposition of H for $A = D^2j(0)$ as in Lemma* 3.2*. Then there exist a homeomorphism G defined on a neighborhood of 0 in H into H and a C^1-map β defined on a neighborhood V of 0 in Y into $X \oplus Z$ with $G(0) = \beta(0) = 0$ such that*

(a) *for all $u = x + y + z \in X \oplus Y \oplus Z$ with $\|u\|$ sufficiently small,*

$$j(G(u)) = -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + j(y + \beta(y)), \tag{3.5}$$

(b) *and for all $y \in Y$,*

$$P(Dj(y + \beta(y))) = 0, \tag{3.6}$$

where P is the orthogonal projection of H onto $X \oplus Z$.

Moreover,

$$i(Dj, 0) = (-1)^{\dim(X)} i(D\psi, 0), \tag{3.7}$$

where $\psi(y) = j(y + \beta(y))$.

Proof. Using lemma 3.2 and arguing as in the proof of [16, Theorem 3] we obtain the existence of functions G and β satisfying (3.5) and (3.6). As in the proof of the cited Theorem, we obtain (3.7) by using the following homotopy in sense of class $(S)_+$,

$$h(t,u) = \begin{cases} P(Dj(u)) + Q(Dj(t\beta(y) + (1-t)(x+z) + y) & t \in [0,1] \\ P(Dj(x + z + (2-t)y)) + Q(Dj(y + \beta(y))) & t \in [1,2] \\ (3-t)(P(Dj(x+z)) + (t-2)(-x+z) + Q(Dj(y + \beta(y))) & t \in [2,3], \end{cases}$$

where $u = x + y + z \in X \oplus Y \oplus Z$, and Q is the projection of H onto Y. □

Proof of Theorem 3.1. Using Proposition 3.3 and arguing as in the proofs of [16, Theorems 2 and 3], we obtain the theorem. □

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References

[1] H. Amman; *A note on degree theory for gradient mappings*, Proc. Amer. Math. Soc., **85** (1982), pp. 591–595.

[2] H. Amman, E. Zehnder; *Nontrivial solutions for a class of non-resonance problems and applications to nonlinear differential equations*, Annali Scoula Norm. Sup. Pisa, **7** (1980), pp. 539–603.

[3] D. Arcoya, L. Orsina; *Landesman-Laser conditions and quasi-linear elliptic equations*, Nonlinear Analysis, **28** (1997), pp. 1623–1632.

[4] H. Brezis; *Functional analysis,Sobolev spaces and partial differential equations*, Springer, Berlin, (2011).

[5] F. E. Browder; *Nonlinear elliptic boundary value problems and the generalized topological degree*, Bull. Amer. Math. Soc., **76** (1970), pp. 999–1005.

[6] F. E.Browder; *Fixed point theory and nonlinear problems*, Proc. Sym. Pure. Math., **39** (1983), Part 2, pp. 49–88.

[7] A. Castro; *Reduction methods via minimax*, Primer Simposio Colombiano de Analisis Funcional, Medellin, Colombia (1981).

[8] A. Castro, J.Cossio; *Multiple solutions for a nonlinear Dirichlet problem,* Siam. J. Math. Anal., (6) **25** (1994), pp. 1554–1561.

[9] A. Castro, P. Drabek, J. Neuberger; *A sign-changing solution for a super-linear Dirichlet problem II,* Fifth Mississippi State Conference on Differential Equations and Computational Simulations, Electronic Journal of Differential Equations, Conference 10, (2003), pp 101–107.

[10] B. Cheng, X. Wu, J. Liu; *Multiple solutions for a class of Kirchhoff type problems with concave nonlinearity* , Nonlinear Differ. Equ. Appl. **19**, (2012), pp. 521537.

[11] F. Colasuonno, P. Pucci, C. Varga; *Multiple solutions for an eigenvalue problem involving p-Laplacian type operators,* Nonlinear Analysis, **75** (2012), pp. 4496–4512.

[12] J. Cossio, S. Herron, C. Velez; *Existence of solutions for an asymptotically linear Dirichlet problem via Lazer-Solimini results,* Nonlinear Analysis, **71** (2009), pp. 66–71.

[13] E. Dancer, Z. Zhang; *Fucik Spectrum, Sign-Changing, and Multiple Solutions for Semilinear Elliptic Boundary Value Problems with Resonance at Infinity,* Journal of Mathematical Analysis and Applications **250**, (2000), pp 449–464.

[14] D. M. Duc, N. L. Luc, L. Q. Nam, T. T. Tuyen; *On topological degree for potential operators of class* $(S)_+$, Nonlinear Analysis, **55** (2003), pp. 951–968.

[15] A. Fonda, R. Toader; *Radially symmetric systems with a singularity and asymptotically linear growth*, Nonlinear Analysis, **74** (2011), pp. 2485–2496.

[16] H. Hofer; *The topological degree at a critical point of mountain-pass type*, Proc. Sym. Pure. Math., **45** (1986), Part 1, pp. 501–509.

[17] M.A. Krasnosel'kii; *Topological methods in the theory of nonlinear integral equations*, Pergamon, Oxford, (1964).

[18] S. Lang; *Analysis II*, Addison- Wileys, Reading, (1969).

[19] E. M. Landesman, A. C. Lazer; *Nonlinear perturbations of linear elliptic problems at resonance*, J. Math. Mech., **19** (1970), pp. 609–623.

[20] A. C. Lazer, J. P. McKenna; *Multiplicity results for a class of semilinear elliptic and parabolic boundary value problems*, J. Math. Anal. Appl., **107** (1985), pp. 371–395.

[21] S. Li, K. Perera; *Computation of critical groups in resonance problems where the nonlinearity may not be sublinear*, Nonlinear Analysis, **46** (2001), pp. 777–787.

[22] S. Li, M. Willem; *Multiple solutions for asymptotically linear boundary value problems in which the nonlinearity crosses at least one eigenvalue*, Nonlinear Differential. Equ. Appl., **5** (1998), pp. 479–490.

[23] S. Li, S. P. Wu, H. S. Zhou; *Solutions to semilinear elliptic problems with combined nonlinearities*, Journal of Differential Equations 185 (2002), 200–224.

[24] A. Manes, A. M. Micheletti; *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Bollettino U.M.I, **7** (1973), pp. 285–301.

[25] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou; *Positive solutions and multiple solutions at non-resonance, resonance and near resonance for hemivariational inequalities with p-Laplacian*, Trans. American Mathematical Society, Vol.360, 2008, 2527–2545.

[26] D. Motreanu, M. Tanaka; *Existence of solutions for quasilinear elliptic equations with jumping nonlinearities under the Neumann boundary condition* , Calc. Var. (**43**, (2012), pp. 231-264.

[27] Y. Naito, S. Tanaka; *On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations*, Nonlinear Analysis, **56** (2004), pp. 919–935.

[28] F. de Paiva; *Multiple solutions for elliptic problems with asymmetric nonlinearity* , J. Math. Anal. Appl., **292** (2004), pp. 317–327.

[29] N.S. Papageorgiou, A. I. S. C. Rodrigues, V. Staicu; *On resonant Neumann problems : Multiplicity of solutions*, Nonlinear Analysis, **74** (2011), pp. 6487–6498.

[30] K. Perera, M. Schechter; *A generalization of the Amann-Zehnder theorem to non-resonance problems with jumping nonlinearities*, Nonlinear Diff. Equ. Appl., **7** (2000), pp. 361–367.

[31] A. Qian, S. Li; *Multiple nodal solutions for elliptic equations*, Nonlinear Analysis, **57** (2004), pp. 615–632.

[32] P.H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*,Conference board of the Mathematical sciences, America Mathematics Society, Providence, (1986).

[33] C. G. Simader; *An elementary proof of Harnack's inequality for Schrodinger operators and related topics*, Math. Z., **203** (1990), pp. 129–152.

[34] I. V. Skrypnik; *Nonlinear Higher Order Elliptic Equations* (in Russian), Noukova Dumka. Kiev, (1973).

[35] G. Stampacchia; *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinuous*, Annales de l'institut Fourier, **15**, (1965), pp. 189–257.

[36] J. Su; *Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues*, Nonlinear Analysis, **48**, (2002), pp. 881–895.

[37] F. Zhao, L. Zhao, Y. Ding; *Multiple solutions for asymptotically linear elliptic systems*, Nonlinear differ. equ. appl. **15**, (2008), pp. 673–688.

[38] W. Zou; *Multiple solutions for elliptic equations with resonance*, Nonlinear Anal, **48**,(2002), pp. 363–376.

[39] W. Zou, J. Q. Liu; *Multiple solutions for resonant elliptic equations via local linking theory and Morse theory*, Journal of Differential Equations, **170**,(2001), pp. 68–95.

Duong Minh Duc
University of Sciences - Hochiminh City, Vietnam
E-mail address: dmduc@hcmuns.edu.vn

Loc Hoang Nguyen
Department of Mathematics, Ecole Normale Suprieure, Paris, France
E-mail address: lnguyen@dma.ens.fr

lu Nguyen
Department of Mathematics, Princeton University, Princeton, New Jersey 08544, USA
E-mail address: llnguyen@princeton.edu