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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A DEGENERATE NONLOCAL ELLIPTIC DIFFERENTIAL EQUATION

NGUYEN THANH CHUNG, HOANG QUOC TOAN

Abstract. Using variational arguments, we study the existence and multiplicity of solutions for the degenerate nonlocal differential equation

$$-M\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)\operatorname{div}\Big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\Big) = |x|^{-p(a+1)+c}f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ and the function M may be zero at zero.

1. Introduction

In this article, we study the boundary-value problem

$$-M\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)\operatorname{div}\Big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\Big) = |x|^{-p(a+1)+c}f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a smooth bounded domain, $0\in\Omega$, $0\leq a<\frac{N-p}{p}$, $1<p<N$, $0<c$, $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function, $\mathbb{R}^+=[0,\infty)$.

Since the first equation in (1.2) contains an integral over Ω, it is no longer a pointwise equation, and therefore it is often called nonlocal problem. It should be noticed that if $a=0$ and $c=p$ then problem (1.1) becomes

$$-M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\Delta_p u = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

This equation is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\big|\frac{\partial u}{\partial x}\big|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in 1883 [15]. This is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.3) have the following meanings: L is

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the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2, 4, 10, 11, 12, 13, 16, 17, 18, 19], in which the authors have used different methods to get the existence of solutions for (1.2). One of the important hypotheses in these papers is that the function M is non-degenerate; i.e.,

$$M(t) \geq m_0 > 0 \quad \text{for all } t \in \mathbb{R}^+. \tag{1.4}$$

We refer the readers to [3, 9] where the authors studied the existence of weak solutions for elliptic equations involving p-polyharmonic Kirchhoff operators.

Motivated by the ideas introduced in [7, 9, 14, 16, 20], the goal of this paper is to study the existence and multiplicity of solutions for (1.1) without condition (1.4). The approach is based on variational arguments. Our results complement the previous ones in the non-degenerate case. Moreover, we consider problem (1.1) in the general case $0 \leq a < \frac{N-p}{p}$, $1 < p < N$, $0 < c$. It should be noticed that in [8], we studied the existence of solutions for problem (1.1) in the sublinear case when $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,t)| \leq Ct^{\alpha p-1}, \quad 1 < \alpha < \min\big\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\big\}, \quad C > 0$$

for all $t \in [0, +\infty)$ and $x \in \Omega$.

We start by recalling some useful results in [5, 6, 20]. We have known that for all $u \in C_0^\infty(\mathbb{R}^N)$, there exists a constant $C_{a,b} > 0$ such that

$$\Big(\int_{\mathbb{R}^N} |x|^{-bq}|u|^q\,dx\Big)^{p/q} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx, \tag{1.5}$$

where

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = p^*(a,b) = \frac{Np}{N-dp}, \quad d = 1+a-b.$$

Let $W_0^{1,p}(\Omega, |x|^{-ap})$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{a,p} = \Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)^{1/p}.$$

Then $W_0^{1,p}(\Omega, |x|^{-ap})$ is reflexive and separable Banach space. From the boundedness of Ω and the standard approximation argument, it is easy to see that (1.5) holds for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ in the sense that

$$\Big(\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^l\,dx\Big)^{p/l} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx, \tag{1.6}$$

for $1 \leq l \leq p^* = \frac{Np}{N-p}$, $\alpha \leq (1+a)l + N\Big(1 - \frac{l}{p}\Big)$; that is, the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ is continuous, where $L^l(\Omega, |x|^{-\alpha})$ is the weighted $L^l(\Omega)$ space with the norm

$$|u|_{l,\alpha} := |u|_{L^l(\Omega,|x|^{-\alpha})} = \Big(\int_\Omega |x|^{-\alpha}|u|^l\,dx\Big)^{1/l}.$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem.

Lemma 1.1 (Compactness embedding theorem [20]). *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and that $0 \in \Omega$, where $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $1 \le l < \frac{Np}{N-p}$ and $\alpha < (1+a)l + N\big(1 - \frac{l}{p}\big)$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ is compact.*

2. Main results

In this section, will we discuss the existence of weak solutions for problem (1.1). For simplicity, we denote $X = W_0^{1,p}(\Omega, |x|^{-ap})$. In the following, when there is no misunderstanding, we always use c_i, C_i to denote positive constants.

Definition 2.1. We say that $u \in X$ is a weak solution of problem (1.1) if

$$M\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx - \int_\Omega |x|^{-p(a+1)+c} f(x,u)\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

Define

$$\Phi(u) = \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big), \quad \Psi(u) = \int_\Omega |x|^{-p(a+1)+c}F(x,u)\,dx, \tag{2.1}$$

where

$$\widehat{M}(t) = \int_0^t M(s)\,ds, \quad F(x,t) = \int_0^t f(x,s)\,ds.$$

By the condition (F0) (see Theorem 2.2 below), Lemma 1.1 implies that the energy functional $J(u) = \Phi(u) - \Psi(u) : X \to \mathbb{R}$ associated with problem (1.1) is well defined. Then it is easy to see that $J \in C^1(X, \mathbb{R})$ and $u \in X$ is a weak solution of (1.1) if and only if u is a critical point of J. Moreover, we have

$$\begin{aligned} J'(u)(\varphi) &= M\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\ &\quad - \int_\Omega |x|^{-p(a+1)+c}f(x,u)\varphi\,dx \\ &= \Phi'(u)(\varphi) - \Psi'(u)(\varphi) \end{aligned}$$

for all $\varphi \in X$.

For the next theorem, we use the following assumptions:

(M0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies
$$m_0 t^{\alpha-1} \le M(t) \quad \text{for all } t \in \mathbb{R}^+,$$
where $m_0 > 0$ and $\alpha > 1$;

(F0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that
$$|f(x,t)| \le C_1(1 + |t|^{q-1}) \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R},$$
where $C_1 > 0$ and $1 < q < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$;

(E0) $\alpha p > q$.

Theorem 2.2. *Under assumptions* (M0), (F0), (E0), *problem* (1.1) *has at least one weak solution.*

Proof. Let $\{u_m\}$ be a sequence that converges weakly to u in X. Then, by the weak lower semicontinuity of the norm, we have

$$\liminf_{m\to\infty} \int_\Omega |x|^{-ap} |\nabla u_m|^p \, dx \geq \int_\Omega |x|^{-ap} |\nabla u|^p \, dx.$$

Combining this with the continuity and monotonicity of the function $\psi : \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto \psi(t) = \frac{1}{p}\widehat{M}(t)$, we obtain

$$\begin{aligned}
\liminf_{m\to\infty} \Phi(u_m) &= \liminf_{m\to\infty} \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx\Big) \\
&= \liminf_{m\to\infty} \psi\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx\Big) \\
&\geq \psi\Big(\liminf_{m\to\infty} \int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx\Big) \\
&\geq \psi\Big(\int_\Omega |x|^{-ap}|\nabla u|^p \, dx\Big) \\
&= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p \, dx\Big) = \Phi(u).
\end{aligned} \tag{2.2}$$

Using (F0), Hölder's inequality, and Lemma 1.1, it follows that

$$\begin{aligned}
&\Big|\int_\Omega |x|^{-p(a+1)+c}[F(x,u_m) - F(x,u)] \, dx\Big| \\
&\leq \int_\Omega |x|^{-p(a+1)+c} |f(x, u + \theta_m(u_m - u))||u_m - u| \, dx \\
&\leq C_1 \int_\Omega |x|^{-p(a+1)+c} \left(1 + |u + \theta_m(u_m - u)|^{q-1}\right) |u_m - u| \, dx \\
&\leq C_1 \Big(\int_\Omega |x|^{-p(a+1)+c} \, dx\Big)^{\frac{q-1}{q}} \|u_m - u\|_{L^q(\Omega, |x|^{-p(a+1)+c})} \\
&\quad + C_1 \|u + \theta_m(u_m - u)\|^{q-1}_{L^q(\Omega, |x|^{-p(a+1)+c})} \|u_m - u\|_{L^q(\Omega, |x|^{-p(a+1)+c})},
\end{aligned} \tag{2.3}$$

which tends to 0 as $m \to \infty$, where $0 \leq \theta_m(x) \leq 1$ for all $x \in \Omega$. From (2.2) and (2.3), the functional J is weakly lower semi-continuous in X.

On the other hand, by assumptions (M0) and (F0), we have

$$\begin{aligned}
J(u) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p \, dx\Big) - \int_\Omega |x|^{-p(a+1)+c} F(x,u) \, dx \\
&\geq \frac{m_0}{p} \int_0^{\|u\|^p_{a,p}} t^{\alpha-1} \, dt - c_1 \int_\Omega |x|^{-p(a+1)+c} \big(1 + |u|^q\big) \, dx \\
&\geq \frac{m_0}{\alpha p} \|u\|^{\alpha p}_{a,p} - c_2 \|u\|^q_{a,p} - c_3.
\end{aligned} \tag{2.4}$$

Since $1 < q < \alpha p$, it follows from (2.4) that the functional J is coercive. Therefore, using the minimum principle, we deduce that the functional J has at least one weak solution and thus problem (1.1) has at least one weak solution. □

For the next theroem, we sue the following conditions:

(M1) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies the condition

$$m_1 t^{\alpha_1 - 1} \leq M(t) \leq m_2 t^{\alpha_2 - 1} \quad \text{for all } t \in \mathbb{R}^+,$$

where $m_2 \geq m_1 > 0$ and $1 < \alpha_1 \leq \alpha_2$;

(M2) M satisfies
$$\widehat{M}(t) \ge M(t)t \text{ for all } t \in \mathbb{R}^+;$$
(F1) $f(x,t) = o\big(|t|^{\alpha_1 p-1}\big)$, $t \to 0$ uniformly for $x \in \Omega$;
(F2) There exists a positive constant $\mu > \alpha_2 p$ such that
$$0 < \mu F(x,t) := \int_0^t f(x,s)ds \le f(x,t)t$$
for all $x \in \Omega$ and $|t| \ge T > 0$;
(E1) $\alpha_1 p < q$.

Theorem 2.3. *Under assumptions* (F0)–(F2), (M1)–(M2), *problem* (1.1) *has at least one nontrivial weak solution.*

To prove the above theorem, we need to verify the following lemmas.

Lemma 2.4. *Assume that* (M1), (M2), (F0), (F2) *are satisfied. Then the functional J satisfies the (PS) condition.*

Proof. Let $\{u_m\} \subset X$ be a sequence such that
$$J(u_m) \to \bar{c} < \infty, \quad J'(u_m) \to 0 \quad \text{in } X^* \text{ as } m \to \infty, \tag{2.5}$$
where X^* is the dual space of X.

First, we will show that the sequence $\{u_m\}$ is bounded in X. Indeed, from (2.5), (M1), (M2) and (F2), we obtain that for all m large enough,
$$\begin{aligned}
&1 + \bar{c} + \|u_m\|_{a,p} \\
&\ge J(u_m) - \frac{1}{\mu} J'(u_m)(u_m) \\
&= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big) - \frac{1}{\mu} M\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx \\
&\quad - \int_\Omega |x|^{-p(a+1)+c} F(x,u_m)\,dx + \frac{1}{\mu}\int_\Omega |x|^{-p(a+1)+c} f(x,u_m)u_m\,dx \\
&\ge \Big(\frac{1}{p} - \frac{1}{\mu}\Big) M\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx \\
&\quad - \int_\Omega |x|^{-p(a+1)+c}\Big(\frac{1}{\mu} f(x,u_m)u_m - F(x,u_m)\Big)\,dx \\
&\ge m_1\big(\frac{1}{p} - \frac{1}{\mu}\big)\|u_m\|_{a,p}^{\alpha_1 p} - c_4.
\end{aligned} \tag{2.6}$$
Since $\alpha_1 p > 1$, it follows from (2.6) that $\{u_m\}$ is bounded. Passing to a subsequence if necessary, there exists $u \in X$, such that $\{u_m\}$ converges weakly to u in X. By (2.5), we obtain
$$\lim_{m\to\infty} J'(u_m)(u_m - u) = 0. \tag{2.7}$$

By (F0) and Lemma 1.1, we have

$$\begin{aligned}
&\Big|\int_\Omega |x|^{-p(a+1)+c} f(x,u_m)(u_m-u)\,dx\Big| \\
&\le \int_\Omega |x|^{-p(a+1)+c}|f(x,u_m)||u_m-u|\,dx \\
&\le C_1\int_\Omega |x|^{-p(a+1)+c}(1+|u_m|^{q-1})|u_m-u|\,dx \\
&\le C_1\Big(\int_\Omega |x|^{-p(a+1)+c}\,dx\Big)^{\frac{q-1}{q}}\|u_m-u\|_{L^q(\Omega,|x|^{-p(a+1)+c})} \\
&\quad + C_1\|u_m\|^{q-1}_{L^q(\Omega,|x|^{-p(a+1)+c})}\|u_m-u\|_{L^q(\Omega,|x|^{-p(a+1)+c})},
\end{aligned} \tag{2.8}$$

which tends to 0 as $m\to\infty$.

By (2.7), (2.8) and the definition of the functional J, it follows that

$$\lim_{m\to\infty} M\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)\int_\Omega |x|^{-ap}|\nabla u_m|^{p-2}\nabla u_m\cdot(\nabla u_m-\nabla u)\,dx=0. \tag{2.9}$$

Since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\to t_0\ge 0 \quad \text{as } m\to\infty.$$

If $t_0=0$ then $\{u_m\}$ converges strongly to $u=0$ in X and the proof is finished. If $t_0>0$ then by (M1) and the continuity of M, we obtain

$$M\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)\to M(t_0)>0 \quad \text{as } m\to\infty.$$

Thus, for m sufficiently large, we have

$$0<c_5\le M\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)\le c_6. \tag{2.10}$$

From (2.9) and (2.10) and the condition (M1), we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u_m|^{p-2}\nabla u_m\cdot(\nabla u_m-\nabla u)\,dx=0. \tag{2.11}$$

On the other hand, since $\{u_m\}$ converges weakly to u in X, we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot(\nabla u_m-\nabla u)\,dx=0. \tag{2.12}$$

By (2.11) and (2.12),

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}\left(|\nabla u_m|^{p-2}\nabla u_m-|\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_m-\nabla u)\,dx=0.$$

or

$$\lim_{m\to\infty}\int_\Omega \left(|\nabla v_m|^{p-2}\nabla v_m-|\nabla v|^{p-2}\nabla v\right)\cdot(\nabla v_m-\nabla v)\,dx=0, \tag{2.13}$$

where $\nabla v_m=|x|^{-a}\nabla u_m$, $\nabla v=|x|^{-a}\nabla u\in L^p(\Omega)$.

We recall that the following inequalities hold

$$\begin{gathered}
\langle|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta\rangle\ge c_7\Big(|\xi|+|\eta|\Big)^{p-2}|\xi-\eta|^2 \quad \text{if } 1<p<2,\\
\langle|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta\rangle\ge c_8|\xi-\eta|^p \quad \text{if } p\ge 2,
\end{gathered} \tag{2.14}$$

for all $\xi,\eta\in\mathbb{R}^N$, where $\langle .,.\rangle$ denote the usual product in $\mathbb{R}^N$.

If $1 < p < 2$, using the Hölder inequality, by (2.13), we have

$$
\begin{aligned}
0 &\le \|u_m - u\|_{a,p}^p = \||\nabla v_m - \nabla v|\|_{L^p(\Omega)}^p \\
&\le \int_\Omega |\nabla v_m - \nabla v|^p \Big(|\nabla v_m| + |\nabla v|\Big)^{\frac{p(p-2)}{2}} \Big(|\nabla v_m| + |\nabla v|\Big)^{\frac{p(2-p)}{2}} dx \\
&\le \Big(\int_\Omega |\nabla v_m - \nabla v|^2 (|\nabla v_m| + |\nabla v|)^{p-2}\, dx\Big)^{p/2} \Big(\int_\Omega (|\nabla v_m| + |\nabla v|)^p\, dx\Big)^{\frac{2-p}{2}} \\
&\le c_9 \Big(\int_\Omega \langle |\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m - \nabla v\rangle\, dx\Big)^{\frac{p}{2}} \\
&\quad \times \Big(\int_\Omega (|\nabla v_m| + |\nabla v|)^p\, dx\Big)^{\frac{2-p}{2}} \\
&\le c_{10} \Big(\int_\Omega \langle |\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m - \nabla v\rangle\, dx\Big)^{p/2},
\end{aligned}
$$

which converges to 0 as $m \to \infty$. If $p \ge 2$, one has

$$
\begin{aligned}
0 &\le \|u_m - u\|_{a,p}^p = \||\nabla v_m - \nabla v|\|_{L^p(\Omega)}^p \\
&\le c_{11} \int_\Omega \langle |\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m - \nabla u\rangle\, dx,
\end{aligned}
$$

which converges to 0 as $m \to \infty$. So we deduce that $\{u_m\}$ converges strongly to u in X and the functional J satisfies the (PS) condition. □

Lemma 2.5. *Suppose that* (M1), (F0), (F1), (F2), (E1) *hold. Then we have:*

(i) *There exist two positive real numbers ρ and R such that $J(u) \ge R > 0$ for all $u \in X$ with $\|u\|_{a,p} = \rho$;*
(ii) *There exists $\widehat{u} \in X$ such that $\|\widehat{u}\|_{a,p} > \rho$ and $J(u) < 0$.*

Proof. (i) By (M1), we have

$$
\begin{aligned}
J(u) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\, dx\Big) - \int_\Omega |x|^{-p(a+1)+c} F(x,u)\, dx \\
&\ge \frac{m_1}{\alpha_1 p}\|u\|_{a,p}^{\alpha_1 p} - \int_\Omega |x|^{-p(a+1)+c} F(x,u)\, dx.
\end{aligned}
\tag{2.15}
$$

Since $\alpha_1 p < q < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$, the embeddings

$$
X \hookrightarrow L^{\alpha_1 p}(\Omega, |x|^{-p(a+1)+c}), \quad X \hookrightarrow L^q(\Omega, |x|^{-p(a+1)+c})
$$

are compact. Then there are constants $c_{12}, c_{13} > 0$ such that

$$
\|u\|_{L^{\alpha_1 p}(\Omega,|x|^{-p(a+1)+c})} \le c_{12}\|u\|_{a,p}, \tag{2.16}
$$

$$
\|u\|_{L^q(\Omega,|x|^{-p(a+1)+c})} \le c_{13}\|u\|_{a,p}. \tag{2.17}
$$

Let $\epsilon > 0$ be small enough such that $\epsilon < \frac{m_1}{\alpha_1 p c_{12}^{\alpha_1 p}}$. By (F0) and (F1), we obtain

$$
|F(x,t)| \le \epsilon |t|^{\alpha_1 p} + c_\epsilon |t|^q \text{ for all } x \in \Omega \text{ and } t \in \mathbb{R}. \tag{2.18}
$$

Therefore, by (2.15)-(2.18), we have

$$
\begin{aligned}
J(u) &\ge \frac{m_1}{\alpha_1 p}\|u\|_{a,p}^{\alpha_1 p} - \int_\Omega |x|^{-p(a+1)+c} F(x,u)\, dx \\
&\ge \frac{m_1}{\alpha_1 p}\|u\|_{a,p}^{\alpha_1 p} - \epsilon \int_\Omega |x|^{-p(a+1)+c}|u|^{\alpha_1 p}\, dx - c_\epsilon \int_\Omega |x|^{-p(a+1)+c}|u|^q\, dx
\end{aligned}
$$

$$\geq \Big(\frac{m_1}{\alpha_1 p} - \epsilon c_{12}^{\alpha_1 p}\Big)\|u\|_{a,p}^{\alpha_1 p} - c_\epsilon c_{13}^q \|u\|^q.$$

Since $\alpha_1 p < q$, there exist real numbers $\rho, R > 0$ such that $J(u) \geq R$ for all $u \in X$ with $\|u\|_{a,p} = \rho$.

(ii) By (F2), there exists $c_{14} > 0$ such that

$$F(x,t) \geq c_{14}|t|^\mu \text{ for all } x \in \Omega \text{ and } |t| \geq T. \tag{2.19}$$

For $w \in X\backslash\{0\}$ and $t > 0$, it follows from (2.19) that

$$\begin{aligned} J(tw) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla tw|^p\,dx\Big) - \int_\Omega |x|^{-p(a+1)+c}F(x,tw)\,dx \\ &\leq \frac{m_2 t^{\alpha_2 p}}{\alpha_2 p}\|w\|_{a,p}^{\alpha_2 p} - c_{14}t^\mu \int_\Omega |x|^{-p(a+1)+c}|w|^\mu\,dx - c_{15}, \end{aligned} \tag{2.20}$$

which tends to $-\infty$ as $t \to +\infty$ since $\alpha_2 p < \mu$. Then, there exists $t_0 > 0$ such that $J(t_0 w) < 0$ and $\|t_0 w\|_{a,p} > \rho$. We set $\widehat{u} = t_0 w$, then Lemma 2.5 is proved. □

Proof of Theorem 2.3. By Lemmas 2.4 and 2.5, all assumptions of the mountain pass theorem in [1] are satisfied. Then the functional J has a nontrivial critical point in X and thus problem (1.1) has a nontrivial weak solution. □

Next, we will use the Fountain theorem and the Dual fountain theorem in order to study the existence of infinitely many solution for (1.1). More exactly, we will prove the following theorems.

Theorem 2.6. *Assume that* (M1), (M2), (F0), (F2), (E1) *are satisfied. Moreover, we assume that*

(F3) $f(x,-t) = -f(x,t)$ *for all* $x \in \Omega$ *and* $t \in \mathbb{R}$.

Then problem (1.1) *has a sequence of weak solutions* $\{\pm u_k\}_{k=1}^\infty$ *such that* $J(\pm u_k) \to +\infty$ *as* $k \to +\infty$.

Theorem 2.7. *Assume that* (M1), (M2), (F0)–(F2) *are satisfied. Moreover, we assume that*

(F4) $f(x,t) \geq C_2|t|^{r-1}$, $t \to 0$, *where* $\alpha_2 p < r < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$ *for all* $x \in \Omega$ *and* $t \in \mathbb{R}$.

Then problem (1.1) *has a sequence of weak solutions* $\{\pm v_k\}_{k=1}^\infty$ *such that* $J(\pm v_k) < 0$ *and* $J(\pm v_k) \to 0$ *as* $k \to +\infty$.

Because X is a reflexive and separable Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1, 2, \dots, \}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, \dots, \}},$$

and

$$\langle e_i, e_j^*\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

For convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \oplus_{j=k}^\infty X_j$.

Lemma 2.8. *If* $1 < l < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$, *denote*

$$\beta_k = \sup\{\|u\|_{L^l(\Omega,|x|^{-p(a+1)+c})} : \|u\|_{a,p} = 1, u \in Z_k\},$$

then $\lim_{k\to\infty}\beta_k = 0$.

Proof. Obviously, for any k, $0 < \beta_{k+1} \le \beta_k$, so $\beta_k \to \beta \ge 0$ as $k \to \infty$. Let $u_k \in Z_k$, $k = 1, 2, \dots$ satisfy

$$\|u_k\|_{a,p} = 1, \quad 0 \le \beta_k - \|u_k\|_{L^l(\Omega, |x|^{-p(a+1)+c})} < \frac{1}{k}.$$

Then there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$ such that $\{u_k\}$ converges weakly to u in X and

$$\langle e_j^*, u\rangle = \lim_{k\to\infty} \langle e_j^*, u_k\rangle, \quad j = 1, 2, \dots,$$

which implies that $u = 0$ and so $\{u_k\}$ converges weakly to 0 in X as $k \to \infty$. Since $1 < l < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$, the embedding $X \hookrightarrow L^l(\Omega, |x|^{-p(a+1)+c})$ is compact (see Lemma 1.1), then $\{u_k\}$ converges strongly to 0 in $L^l(\Omega, |x|^{-p(a+1)+c})$. Hence, $\lim_{k\to\infty} \beta_k = 0$. □

Lemma 2.9 (Fountain theorem [21]). *Assume that $(X, \|\cdot\|)$ is a separable Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional satisfying the (PS) condition. Moreover, for each $k = 1, 2, \dots$, there exist $\rho_k > r_k > 0$ such that*

- (A1) $\inf_{\{u\in Z_k: \|u\| = r_k\}} J(u) \to +\infty$ *as* $k \to \infty$;
- (A2) $\max_{\{u\in Y_k: \|u\| = \rho_k\}} J(u) \le 0$.

Then J has a sequence of critical values which tends to $+\infty$.

Definition 2.10. We say that J satisfies the $(\mathrm{PS})_c^*$ condition (with respect to (Y_n)) if any sequence $\{u_{n_j}\} \subset X$ such that $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$ as $n_j \to +\infty$, contains a subsequence converging to a critical point of J.

Lemma 2.11 (Dual fountain theorem [21]). *Assume that $(X, \|\cdot\|)$ is a separable Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional satisfying the $(PS)_c^*$ condition. Moreover, for each $k = 1, 2, \dots$, there exist $\rho_k > r_k > 0$ such that*

- (B1) $\inf_{\{u\in Z_k: \|u\| = \rho_k\}} J(u) \ge 0$;
- (B2) $b_k := \max_{\{u\in Y_k: \|u\| = r_k\}} J(u) < 0$;
- (B3) $d_k := \inf_{\{u\in Z_k: \|u\| = \rho_k\}} J(u) \to 0$ *as* $k \to \infty$.

Then J has a sequence of negative critical values which tends to 0.

Proof of Theorem 2.6. According to (F3) and Lemma 2.4, J is an even functional and satisfies the (PS) condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A1) and (A2) hold. Thus, the assertion of conclusion can be obtained from the Fountain theorem.

(A1): From (F0), there exists $c_{16} > 0$ such that

$$|F(x,t)| \le c_{16}(|t| + |t|^q) \quad \text{for all } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Then, using (M1) and Lemma 1.1, for any $u \in Z_k$,

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big) - \int_\Omega |x|^{-p(a+1)+c} F(x,u)\,dx \\ &\ge \frac{m_1}{p\alpha_1}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)^{\alpha_1} - c_{16}\int_\Omega |x|^{-p(a+1)+c}(|u| + |u|^q)\,dx \\ &\ge \frac{m_1}{p\alpha_1}\|u\|_{a,p}^{\alpha_1 p} - c_{17}\beta_k^q\|u\|_{a,p}^q - c_{17}\|u\|_{a,p}, \end{aligned} \tag{2.21}$$

where

$$\beta_k = \sup\big\{\|u\|_{L^q(\Omega, |x|^{-p(a+1)+c})} : \|u\|_{a,p} = 1, u \in Z_k\big\}. \tag{2.22}$$

Now, we deduce from (2.21) that for any $u \in Z_k$, $\|u\|_{a,p} = r_k = \left(\frac{c_{17} q \beta_k^q}{m_1}\right)^{\frac{1}{\alpha_1 p - q}}$,

$$\begin{aligned} J(u) &\geq \frac{m_1}{p\alpha_1}\|u\|_{a,p}^{\alpha_1 p} - c_{17}\beta_k^q \|u\|_{a,p}^q - c_{17}\|u\|_{a,p} \\ &= \frac{m_1}{p\alpha_1}\Big(\frac{c_{17} q \beta_k^q}{m_1}\Big)^{\frac{\alpha_1 p}{\alpha_1 p - q}} - c_{17}\beta_k^q \Big(\frac{c_{17} q \beta_k^q}{m_1}\Big)^{\frac{q}{\alpha_1 p - q}} - c_{17}\Big(\frac{c_{17} q \beta_k^q}{m_1}\Big)^{\frac{1}{\alpha_1 p - q}} \\ &= m_1\Big(\frac{1}{\alpha_1 p} - \frac{1}{q}\Big)\Big(\frac{c_{17} q \beta_k^q}{m_1}\Big)^{\frac{\alpha_1 p}{\alpha_1 p - q}} - c_{17}\Big(\frac{c_{17} q \beta_k^q}{m_1}\Big)^{\frac{1}{\alpha_1 p - q}}, \end{aligned} \tag{2.23}$$

which tends to $+\infty$ as $k \to +\infty$, because $\alpha_1 p < q < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$ and $\beta_k \to 0$ as $k \to \infty$, see Lemma 2.8.

(A2): From (F2), there exists a constant $c_{18} > 0$ such that

$$F(x,t) \geq c_{18}|t|^\mu - c_{18} \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}.$$

Therefore, using (M1), for any $w \in Y_k$ with $\|w\|_{a,p} = 1$ and $1 < t < \rho_k$, we have

$$\begin{aligned} J(tw) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla tw|^p\,dx\Big) - \int_\Omega |x|^{-p(a+1)+c}F(x,tw)\,dx \\ &\leq \frac{m_2}{\alpha_2 p}\Big(\int_\Omega |x|^{-ap}|\nabla tw|^p\,dx\Big)^{\alpha_2} - c_{18}\int_\Omega |x|^{-p(a+1)+c}|tw|^\mu\,dx - c_{19} \\ &= \frac{m_2 t^{\alpha_2 p}}{\alpha_2 p}\|w\|_{a,p}^{\alpha_2 p} - c_{18}t^\mu \int_\Omega |x|^{-p(a+1)+c}|w|^\mu\,dx - c_{19}. \end{aligned} \tag{2.24}$$

Since $\mu > \alpha_2 p$ and $\dim(Y_k) = k$, it is easy to see that $J(u) \to -\infty$ as $\|u\|_{a,p} \to +\infty$ for $u \in Y_k$. $\square$

To prove Theorem 2.7, we need to verify the following lemma.

Lemma 2.12. *Assume that* (M1), (M2), (F0), (F2) *are satisfied. Then the functional J satisfies the $(PS)_c^*$ condition.*

Proof. Let $\{u_{n_j}\} \subset X$ be such that $u_{n_j} \in Y_{n_j}$ and $J(u_{n_j}) \to 0$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$ as $n_j \to \infty$. Similar to the process of verifying the (PS) condition in the proof of Lemma 2.4, we can get the boundedness of $\{\|u_{n_j}\|_{a,p}\}$. Going, if necessary, to a subsequence, we can assume that $\{u_{n_j}\}$ converges weakly to u in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence,

$$\begin{aligned} \lim_{n_j\to\infty} J'(u_{n_j})(u_{n_j} - u) &= \lim_{n_j\to\infty} J'(u_{n_j})(u_{n_j} - v_{n_j}) + \lim_{n_j\to\infty} J'(u_{n_j})(v_{n_j} - u) \\ &= \lim_{n_j\to\infty} (J|_{Y_{n_j}})'(u_{n_j})(u_{n_j} - v_{n_j}) = 0. \end{aligned} \tag{2.25}$$

From the proof of Lemma 2.4, J' is of (S_+) type, so we can conclude that $u_{n_j} \to u$ as $n_j \to \infty$, furthermore we have $J'(u_{n_j}) \to J'(u)$.

Let us prove $J'(u) = 0$, i.e., u is a critical point of J. Indeed, taking arbitrarily $w_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned} J'(u)(w_k) &= (J'(u) - J'(u_{n_j}))(w_k) + J'(u_{n_j})(w_k) \\ &= (J'(u) - J'(u_{n_j}))(w_k) + (J|_{Y_{n_j}})'(u_{n_j})(w_k). \end{aligned} \tag{2.26}$$

Going to limit in the right hand-side of (2.26) reaches $J'(u)(w_k) = 0$ for all $w_k \in Y_k$. Thus, $J'(u) = 0$ and the functional J satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$. $\square$

Proof of Theorem 2.7. From (F0), (F2), (F3) and Lemma 2.12, we know that J is an even functional and satisfies the $(\mathrm{PS})_c^*$ condition, the assertion of conclusion can be obtained from Dual fountain theorem.

(B1): For any $v \in Z_k$, $\|v\|_{a,p} = 1$ and $0 < t < 1$, using (M1) and (2.18), we have

$$\begin{aligned} &J(tv)\\ &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla tv|^p\,dx\Big) - \int_\Omega |x|^{-p(a+1)+c}F(x,tv)\,dx\\ &\ge \frac{m_1}{\alpha_1 p}t^{\alpha_1 p}\|v\|_{a,p}^{\alpha_1 p} - \epsilon t^{\alpha_1 p}\int_\Omega |x|^{-p(a+1)+c}|v|^{\alpha_1 p}\,dx - c_\epsilon t^q\int_\Omega |x|^{-p(a+1)+c}|v|^q\,dx\\ &\ge \Big(\frac{m_1}{\alpha_1 p} - \epsilon c_{20}\Big)t^{\alpha_1 p} - c_{21}\beta_k^q t^q. \end{aligned} \tag{2.27}$$

Let $0 < \epsilon < \frac{M_1}{\alpha_1 p c_{20}}$. Since $q > \alpha_1 p$, taking $\rho_k = t$ small enough and sufficiently large k, for $v \in Z_k$ with $\|v\|_{a,p} = 1$, we have $J(tv) \ge 0$. So for sufficiently large k,

$$\inf_{\{u\in Z_k:\|u\|_{a,p}=\rho_k\}} J(u) \ge 0;$$

i.e., (B1) is satisifed.

(B2): For $v \in Y_k$, $\|v\|_{a,p} = 1$ and $0 < t < \rho_k < 1$, we have

$$\begin{aligned} J(tv) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla tv|^p\,dx\Big) - \int_\Omega |x|^{-p(a+1)+c}F(x,tv)\,dx\\ &\le \frac{m_2}{\alpha_2 p}\Big(\int_\Omega |x|^{-ap}|\nabla tv|^p\,dx\Big)^{\alpha_2} - C_2\int_\Omega |x|^{-p(a+1)+c}|tv|^r\,dx\\ &= \frac{m_2}{\alpha_2 p}t^{\alpha_2 p}\|v\|_{a,p}^{\alpha_2 p} - C_2 t^r\int_\Omega |x|^{-p(a+1)+c}|v|^r\,dx. \end{aligned} \tag{2.28}$$

Condition $\alpha_2 p < r < \min\{p^*, \frac{p(N-(a+1)p+c)}{N-(a+1)p}\}$ implies that there exists a constant $r_k \in (0,\rho_k)$ such that $J(tv) < 0$ when $t = r_k$. Hence, we obtain from (2.28) that

$$b_k := \max_{\{u\in Y_k:\|u\|_{a,p}=r_k\}} J(u) < 0,$$

so (B2) is satisfied.

(B3): Because $Y_k \cap Z_k \ne \emptyset$ and $r_k < \rho_k$ we have

$$d_k := \inf_{\{u\in Z_k:\|u\|_{a,p}\le\rho_k\}} J(u) \le b_k := \max_{\{u\in Y_k:\|u\|_{a,p}=r_k\}} J(u) < 0. \tag{2.29}$$

From (2.27), for $v \in Z_k$, $\|v\|_{a,p} = 1$, $0 \le t \le \rho_k$ and $u = tv$ we have

$$\begin{aligned} J(u) &= J(tv)\\ &\ge \Big(\frac{m_1}{\alpha_1 p} - \epsilon c_{20}\Big)t^{\alpha_1 p} - c_{21}\beta_k^q t^q\\ &\ge -c_{21}\beta_k^q t^q. \end{aligned} \tag{2.30}$$

From (2.29) and (2.30), $d_k \to 0$ as $k \to \infty$; i.e., (B3) is satisfied. $\square$

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3. Corrigendum posted on August 21, 2014

A reader pointed out that no function $M(t)$ can satisfy both hypotheses (M1) and (M2). In response, we present a proof of our results with a modified assumption (F2), and without assumption (M2).

Modified assumptions. We delete the assumption (M2), and re-state the following:

(M1) There exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}, \quad \forall t \in \mathbb{R}^+$$

(The original (M1) implies $\alpha_1 = \alpha_2$, so we rename constant α.);

(F2) There exists a positive constant $\mu > \frac{m_2}{m_1}\alpha p$ such that

$$0 < \mu F(x,t) = \mu \int_0^t f(x,s)\,ds \leq f(x,t)t$$

for all $x \in \Omega$ and $|t| \geq T > 0$ (The constant μ has been redefined);

New Lemma 2.4. *Assume that* (M1), (F0), (F2) *are satisfied. Then the functional* J *satisfies the Palais-Smale condition in the space* X.

Proof. Let $\{u_m\} \subset X$ be a sequence such that

$$J(u_m) \to \bar{c} < \infty, \quad J'(u_m) \to 0 \quad \text{in } X^* \text{ as } m \to \infty, \tag{3.1}$$

where X^* is the dual space of X.

We shall show that the sequence $\{u_m\}$ is bounded in X. Indeed, from (3.1), (M1) and (F2), for all m large enough, we have

$$\begin{aligned}
&1 + \bar{c} + \|u_m\|_{a,p} \\
&\geq J(u_m) - \frac{1}{\mu} J'(u_m)(u_m) \\
&= \frac{1}{p}\widehat{M}\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big) - \frac{1}{\mu} M\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx \\
&\quad - \int_\Omega |x|^{-p(a+1)+c} F(x,u_m)\,dx + \frac{1}{\mu}\int_\Omega |x|^{-p(a+1)+c} f(x,u_m)u_m\,dx \\
&\geq \frac{m_1}{\alpha p}\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)^\alpha - \frac{m_2}{\mu}\Big(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\Big)^\alpha \\
&\quad - \int_\Omega |x|^{-p(a+1)+c}\big(\frac{1}{\mu} f(x,u_m)u_m - F(x,u_m)\big)\,dx \\
&\geq \big(\frac{m_1}{\alpha p} - \frac{m_2}{\mu}\big)\|u_m\|_{a,p}^{\alpha p} - c_4.
\end{aligned} \tag{3.2}$$

Since $\alpha p > 1$ and $\mu > \frac{m_2}{m_1}\alpha p$, from (3.2) it follows that $\{u_m\}$ is bounded. Then with similar arguments as in the proof of the original Lemma 2.4 we can show that J satisfies the Palais-Smale condition. □

Theorem 2.2 remains unchanged. However, Theorems 2.3, 2.6, 2.7 and Lemma 2.12 need to be stated without assumption (M2). Their proofs are similar to the original proofs, but using the new Lemma 2.4, and replacing α_1 and α_2 by α.

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