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EXISTENCE OF THREE POSITIVE SOLUTIONS FOR AN m-POINT BOUNDARY-VALUE PROBLEM ON TIME SCALES

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ABSTRACT. We study an m-point boundary-value problem on time scales. By using a fixed point theorem, we prove the existence of at least three positive solutions, under suitable growth conditions imposed on the nonlinear term. An example is given to illustrate our results.

1. Introduction

The theory of dynamic equation on time scales (or measure chains) was initiated by Stefan Hilger in his Ph. D. thesis in 1988 [12] (supervised by Bernd Aulbach) as a means of unifying structure for the study of differential equations in the continuous case and study of finite difference equations in the discrete case. In recent years, it has found a considerable amount of interest and attracted the attention of many researchers; see for example [1, 3, 4, 8, 9, 20, 23, 24, 26]. It is still a new area, and research in this area is rapidly growing. The study of time scales has led to several important applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models [6, 13, 21, 25].

Throughout the remainder of this article, let $\mathbb{T}$ be a closed nonempty subset of R, and let $\mathbb{T}$ have the subspace topology on R. In some of the current literature, $\mathbb{T}$ is called a time scale. For convenience, we make the blanket assumption that $0, T$ are points in $\mathbb{T}$.

Sang and Xi [19] considered the following p-Laplacian dynamic equation on time scales

$$(\phi_p(u^\Delta(t)))^\nabla + a(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}},$$
$$\phi_p(u^\Delta(0)) = \sum_{i=1}^{m-2} a_i\phi_p(u^\Delta(\xi_i)), \quad u(T) = \sum_{i=1}^{m-2} b_iu(\xi_i),$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$.

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He [11] studied the existence of at least two positive solutions by way of a new double fixed-point theorem for the equation

$$
\begin{gathered}
[\varphi_p(u^\Delta(t))]^\nabla + a(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \\
u(0) - B_0(u^\Delta(\eta)) = 0 \quad u^\Delta(T) = 0, \text{or} \\
u^\Delta(0) = 0, \quad u(T) + B_1(u^\Delta(\eta)) = 0,
\end{gathered}
$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta \in (0, \rho(t))_{\mathbb{T}}$.

Anderson et al [4] showed the existence of at least one solution for the corresponding boundary-value problem

$$
\begin{gathered}
[g(u^\Delta(t))]^\nabla + c(t)f(u(t)) = 0, \quad t \in (a,b), \\
u(a) - B_0(u^\Delta(\upsilon)) = 0, \quad u^\Delta(b) = 0,
\end{gathered}
$$

where $g(z) = |z|^{p-2}z$, $p > 1$, and $\upsilon \in (a,b) \subset \mathbb{T}$.

In recent years, much attention has been paid to the existence of positive solutions of boundary value problems (BVPs) on time scales for $p(t) \equiv 1$ and $\varphi(u) = |u|^{p-2}u$, $p > 1$; see [2, 4, 11, 14, 15, 22, 23] and the references therein. The key condition used in the above papers is the oddness of a p-Laplacian operator. Nevertheless, we define a new operator which improves and generalizes a p-Laplacian operator for some $p > 1$, and φ is not necessary odd. In addition, there are not many results concerning increasing homeomorphism and positive homomorphism on time scales; see [16, 17].

Motivated by works mentioned above, in this paper, we study the existence of at least three positive solutions to the following p-Laplacian multipoint BVP on time scales

$$
[\varphi(p(t)u^\Delta(t))]^\nabla + a(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}^K \cap \mathbb{T}_K}, \tag{1.1}
$$

$$
u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u^\Delta(T) = 0, \tag{1.2}
$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$, $p \in C([0,T]_{\mathbb{T}}, (0,+\infty))$ and $\xi_i \in [0,T]_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < T$, $0 < \sum_{i=1}^{m-2} a_i < 1$ $a : \mathbb{T} \to [0,+\infty)$ is ld-continuous and not identically zero on any closed subinterval of $[0,T]_{\mathbb{T}}$. The usual notation and terminology for time scales as can be found in [5, 6], will be used here.

A projection $\varphi : R \to R$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:

(i) if $x \le y$, then $\varphi(x) \le \phi(y), \forall x, y \in R$;
(ii) φ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\varphi(xy) = \varphi(x)\varphi(y), \forall x, y \in R$.

If the above conditions hold, then it implies that φ is homogeneous and generates a p-Laplacian operator. It is well known that the p-Laplacian operator is odd. Nevertheless, the operator which we defined above is not necessarily odd.

Throughout this article we assume that the following conditions are satisfied:

(H1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $0 < \sum_{i=1}^{m-2} a_i < 1$;
(H2) $p \in C([0,T]_{\mathbb{T}}, [0,\infty))$and nondecreasing on $[0,T]_{\mathbb{T}}$;
(H3) $a : \mathbb{T} \to [0,\infty)$ is ld-continuous and not identical zero on any closed subinterval of $[0,T]_{\mathbb{T}}$.

The rest of article is arranged as follows. In Section 2, we state some definitions, notation, lemmas and prove several preliminary results. The main theorem on the existence of at least three positive solutions and its proof are presented in Section 3. In last section 4, we give an example to demonstrate our results.

2. Preliminaries

In this section, we provide some background materials from theory of cones in Banach spaces. The following definitions can be found in the book by Deimling [7] and in the book by Guo and Lakshmikantham [10].

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geq 0$ imply $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$ imply $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. We say the map α is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.3. Given a nonnegative continuous functional γ on a cone P of E, for each $d > 0$ we define the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

Let the Banach space $E = C_{ld}([0, T]_{\mathbb{T}}, \mathrm{R})$ with norm $\|u\| = \sup_{t \in [0,T]_{\mathbb{T}}} |u(t)|$ and define the cone $P \subset E$ by

$$P = \{u \in E | u(t) \text{ is a concave and nonnegative nondecreasing function on } [0, T]_{\mathbb{T}^{K} \cap \mathbb{T}_{K}}\}.$$

Lemma 2.4. *If $\sum_{i=1}^{m-2} \alpha_i \neq 1$ then for $h \in C_{ld}[0, T]_{\mathbb{T}}$,*

$$[\varphi(p(t)u^{\Delta}(t))]^{\nabla} + h(t) = 0, \quad t \in [0, T]_{\mathbb{T}^{K} \cap \mathbb{T}_{K}}, \tag{2.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u^{\Delta}(T) = 0 \tag{2.2}$$

has the unique solution

$$\begin{aligned} u(t) = &\int_0^t \frac{1}{p(s)} \varphi^{-1}\Big(\int_s^T h(\tau)\nabla\tau\Big)\Delta s \\ &+ \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \frac{1}{p(s)} \varphi^{-1}\Big(\int_s^T h(\tau)\nabla\tau\Big)\Delta s. \end{aligned} \tag{2.3}$$

Proof. Let u be as in (2.3), taking the delta derivative of (2.3), we have

$$u^{\Delta}(t) = \frac{1}{p(t)} \varphi^{-1}\Big(\int_t^T h(\tau)\nabla\tau\Big),$$

moreover, we get

$$\varphi(p(t)u^{\Delta}(t)) = \int_t^T h(\tau)\nabla\tau,$$

taking the nabla derivative of this expression yields $[\varphi(p(t)u^{\Delta}(t))]^{\nabla}=-h(t)$. Routine calculations verify that u satisfies the boundary value conditions in (2.2), so that u given in (2.3) is a solution of (2.1) and (2.2). It is easy to see that the BVP

$$[\varphi(p(t)u^{\Delta}(t))]^{\nabla}=0, \quad u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i), \quad u^{\Delta}(T)=0$$

has only the trivial solution. Thus u in (2.3) is the unique solution of (2.1) and (2.2). $\square$

Lemma 2.5. *If $u\in P$, then*

$$u(t)\geq \frac{t}{T}\|u\|, \quad t\in[0,T]_{\mathbb{T}},$$

where

$$\|u\|=\sup_{t\in[0,T]_{\mathbb{T}}}|u(t)|.$$

Proof. Since $u^{\Delta\nabla}(t)\leq 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0<t<T$,

$$u(t)-u(0)=\int_0^t u^{\Delta}(s)\Delta s\geq tu^{\Delta}(t),$$
$$u(T)-u(t)=\int_t^T u^{\Delta}(s)\Delta s\leq (T-t)u^{\Delta}(t)$$

from which we have

$$u(t)\geq\frac{tu(T)+(T-t)u(0)}{T}\geq\frac{t}{T}u(T)=\frac{t}{T}\|u\|.$$

The proof is complete. $\square$

Let us define the mapping A from P to E by the formula

$$\begin{aligned}(Au)(t)&=\int_0^t\frac{1}{p(s)}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau)\nabla\tau\Big)\Delta s\\ &\quad+\frac{\sum_{i=1}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i}\int_0^{\xi_i}\frac{1}{p(s)}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau)\nabla\tau\Big)\Delta s, \quad u\in P.\end{aligned}\tag{2.4}$$

Lemma 2.6. *The mapping $A:P\to P$ is completely continuous.*

Proof. For each $u\in P$, we have $(Au)(t)\geq 0$, for all $t\in[0,T]_{\mathbb{T}}$. Taking the delta derivative of (2.4), we have

$$(Au)^{\Delta}(t)=\frac{1}{p(t)}\varphi^{-1}\Big(\int_t^T a(\tau)f(u(\tau))\nabla\tau\Big).$$

Clearly, $(Au)^{\Delta}(t)$ is a continuous function and $(Au)^{\Delta}(t)\geq 0$, that is $(Au)(t)$ is decreasing on $[0,T]_{\mathbb{T}}$.

(i) If $t\in[0,T]_{\mathbb{T}^{\kappa}\cap\mathbb{T}_{\kappa}}$ is left scattered, we have

$$(Au)^{\Delta\nabla}(t)=\frac{(Au)^{\Delta}(\rho(t))-(Au)^{\Delta}(t)}{\rho(t)-t}\leq 0.$$

(ii) If $t \in [0,T]_{\mathbb{T}^K \cap \mathbb{T}_K}$ is a left dense, we have

$$(Au)^{\Delta\nabla}(t) = \lim_{s\to t} \frac{(Au)^{\Delta}(t) - (Au)^{\Delta}(s)}{t-s} \le 0.$$

By (i) and (ii), we have $(Au)^{\Delta\nabla}(t) \le 0$, $t \in [0,T]_{\mathbb{T}^K \cap \mathbb{T}_K}$; i.e., (Au) is concave on $[0,T]_{\mathbb{T}}$. This implies that $Au \in P$ and $A : P \to P$. With standard argument one may show that $A : P \to P$ is completely continuous. $\square$

The following fixed point theorem is fundamental for the proofs of our main results.

Theorem 2.7 ([18])**.** *Let P be a cone in a Banach space E. Let α, β and γ be three increasing, nonnegative and continuous functionals on P, satisfying for some $c > 0$ and $M > 0$ such that*

$$\gamma(x) \le \beta(x) \le \alpha(x), \|u\| \le M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exists a completely continuous operator $T : \overline{P(\gamma,c)} \to P$ and $0 < a < b < c$ such that

(S1) *$\gamma(Tx) < c$, for all $x \in \partial P(\gamma, c)$;*
(S2) *$\beta(Tx) > b$, for all $x \in \partial P(\beta, b)$;*
(S3) *$P(\alpha, a) \neq \emptyset$, and $\alpha(Tx) < a$, for all $x \in \partial P(\alpha, a)$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$0 \le \alpha(x_1) < a < \alpha(x_2), \beta(x_2) < b < \beta(x_3), \gamma(x_3) < c.$$

3. Main results

We define the increasing, nonnegative, continuous functionals:

$$\gamma(u) = \max_{t\in[0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1),$$
$$\beta(u) = \min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_1),$$
$$\alpha(u) = \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}).$$

Clearly for every $u \in P$

$$\gamma(u) \le \beta(u) \le \alpha(u).$$

Moreover, for each $u \in P$, Lemma 2.5 implies $\gamma(u) = u(\xi_1) \ge \frac{\xi_1}{T}\|u\|$. That is, $\|u\| \le \frac{T}{\xi_1}\gamma(u)$ for all $u \in P$.

For simplicity, we use the following symbols:

$$\lambda_1 = \frac{1}{p(0)}\Big(\xi_1 + \frac{\sum_{i=1}^{m-2} a_i T}{1 - \sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_0^T a(\tau)\nabla\tau\Big),$$
$$\lambda_2 = \frac{1}{p(T)}\Big(\xi_1 + \frac{\sum_{i=1}^{m-2} a_i T}{1 - \sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_{\xi_{m-2}}^T a(\tau)\nabla\tau\Big),$$
$$\lambda_3 = \frac{1}{p(0)}\Big(\xi_{m-2} + \frac{\sum_{i=1}^{m-2} a_i T}{1 - \sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_0^T a(\tau)\nabla\tau\Big).$$

Theorem 3.1. *Suppose that conditions* (H1), (H2), (H3) *are satisfied. Let $0 < a < \frac{\xi_1}{T}b < b < \frac{\lambda_2}{\lambda_1}c$, and suppose that f satisfies the following conditions:*

(i) *$f(u) < \varphi(c/\lambda_1)$ for all $u \in [0, Tc/\xi_1]$;*

(ii) $f(u) > \varphi(b/\lambda_2)$ *for all* $u \in [b, Tb/\xi_1]$;
(iii) $f(u) < \varphi(a/\lambda_3)$ *for all* $u \in [0, Ta/\xi_1]$.

Then there exist at least three positive solutions u_1, u_2, u_3 *of* (1.1) *and*(1.2) *such that*

$$0 \le \alpha(u_1) < a < \alpha(u_2), \quad \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c.$$

Proof. Define the completely continuous operator A by (2.4). Let $u \in \partial P(\gamma, c)$, then $(Au)(t) \ge 0$ for $t \in [0,T]_{\mathbb{T}}$. By Lemma 2.6 we know that $A : \overline{P(\gamma, c)} \to P$.

Now, we show that all the conditions of Theorem 2.7 are satisfied. To verify (S1) of Theorem 2.7 holds, we choose $u \in \partial P(\gamma, c)$. Then $\gamma(u) = \max_{t\in[0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1) = c$. If we recall that $\|u\| \le \frac{T}{\xi_1}\gamma(u) = \frac{T}{\xi_1}c$. Therefore

$$0 \le u(t) \le \frac{T}{\xi_1}c, \quad \text{for all } t \in [0,T]_{\mathbb{T}}.$$

As a consequence of (i),

$$f(u(s)) < \varphi(c/\lambda_1), \quad \text{for } s \in [0,T]_{\mathbb{T}}.$$

Since $Au \in P$, we have

$$\begin{aligned}
\gamma(Au) &= (Au)(\xi_1)\\
&= \int_0^{\xi_1} \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau))\nabla\tau\Big)\Delta s\\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau))\nabla\tau\Big)\Delta s\\
&\le \frac{1}{p(0)}\int_0^{\xi_1} \varphi^{-1}\Big(\int_0^T a(\tau)f(u(\tau))\nabla\tau\Big)\Delta s\\
&\quad + \frac{\frac{1}{p(0)}\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \varphi^{-1}\Big(\int_0^T a(\tau)f(u(\tau))\nabla\tau\Big)\Delta s\\
&< \frac{1}{p(0)}\Big(\xi_1 + \frac{\sum_{i=1}^{m-2} a_iT}{1-\sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_0^T a(\tau)\nabla\tau\Big)\frac{c}{\lambda_1} = c.
\end{aligned}$$

Thus, (S1) of Theorem 2.7 is satisfied.

Secondly, we prove that (S2) of Theorem 2.7 is fulfilled. For this, we choose $u \in \partial P(\beta, b)$. Then $\beta(u) = \min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_1) = b$. This means $u(t) \ge b, t \in [\xi_i, T]_{\mathbb{T}}$ and since $u \in P$, we have $b \le u(t) \le \|u\| = u(T)$ for $t \in [\xi_1, T]_{\mathbb{T}}$. Note that $\|u\| \le \frac{T}{\xi_1}\gamma(u) = \frac{T}{\xi_1}\beta(u) = \frac{T}{\xi_1}b$ for all $u \in P$. Therefore,

$$b \le u(t) \le \frac{T}{\xi_1}b, \quad \text{for all } t \in [\xi_1, T]_{\mathbb{T}}.$$

From (ii), we have

$$f(u(s)) > \varphi\left(\frac{b}{\lambda_2}\right), \quad \text{for } s \in [\xi_1, T]_{\mathbb{T}},$$

and so

$$\begin{aligned}
\beta(Au) &= (Au)(\xi_1)\\
&= \int_0^{\xi_1} \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau))\nabla\tau\Big)\Delta s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \frac{1}{p(s)} \varphi^{-1}\Big(\int_s^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&> \frac{1}{p(T)} \int_0^{\xi_1} \varphi^{-1}\Big(\int_{\xi_{m-2}}^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&\quad + \frac{\frac{1}{p(T)} \sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \varphi^{-1}\Big(\int_{\xi_{m-2}}^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&> \frac{1}{p(T)}\Big(\xi_1 + \frac{\sum_{i=1}^{m-2} a_i T}{1-\sum_{i=1}^{m-2} a_i}\Big) \varphi^{-1}\Big(\int_{\xi_{m-2}}^T a(\tau) \nabla\tau\Big) \frac{b}{\lambda_2} = b.
\end{aligned}
$$

Thus, (S2) of Theorem 2.7 is satisfied.

Finally we prove that (S3) of Theorem 2.7 is also satisfied. We note that $u(t) = a/2$, $t \in [0,T]_{\mathbb{T}}$ is a member of $P(\alpha, a)$ and $\alpha(u) = \frac{a}{2} < a$. Therefore $P(\alpha, a) \neq \emptyset$. Now let $u \in P(\alpha, a)$. Then $\alpha(u) = \max_{t \in [0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}) = a$. This implies that $0 \le u(t) \le a$ for $t \in [0, \xi_{m-2}]_{\mathbb{T}}$. Recalling that $\|u\| \le \frac{T}{\xi_{m-2}} \gamma(u) \le \frac{T}{\xi_{m-2}} \alpha(u) \le \frac{T}{\xi_1} a$ for all $u \in P$, we have

$$
0 \le u(t) \le \frac{T}{\xi_1} a, \quad \text{for all } t \in [0,T]_{\mathbb{T}}.
$$

From assumption (iii), we obtain

$$
f(u(s)) < \varphi\Big(\frac{a}{\lambda_3}\Big), \quad \text{for } s \in [0,T]_{\mathbb{T}},
$$

and so

$$
\begin{aligned}
\alpha(Au) &= (Au)(\xi_{m-2}) \\
&= \int_0^{\xi_{m-2}} \frac{1}{p(s)} \varphi^{-1}\Big(\int_s^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \frac{1}{p(s)} \varphi^{-1}\Big(\int_s^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&\le \frac{1}{p(0)} \int_0^{\xi_{m-2}} \varphi^{-1}\Big(\int_0^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&\quad + \frac{\frac{1}{p(0)} \sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^T \varphi^{-1}\Big(\int_0^T a(\tau) f(u(\tau)) \nabla\tau\Big) \Delta s \\
&< \frac{1}{p(0)}\Big(\xi_{m-2} + \frac{\sum_{i=1}^{m-2} a_i T}{1-\sum_{i=1}^{m-2} a_i}\Big) \varphi^{-1}\Big(\int_0^T a(\tau) \nabla\tau\Big) \frac{a}{\lambda_3} = a.
\end{aligned}
$$

Then condition (S3) of Theorem 2.7 is satisfied. So Theorem 2.7 implies that A has a least three fixed points which are positive solutions u_1, u_2, u_3 belonging to $\overline{P(\gamma, c)}$ of (1.1) and (1.2) such that

$$
0 \le \alpha(u_1) < a < \alpha(u_2), \quad \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c.
$$

The proof is complete. $\square$

4. Examples

In this section, we give an example to illustrate our results. Let $\mathbb{T} = \{(\frac{2}{3})^{\mathbb{N}_0}\} \cup \{1 - (\frac{2}{3})^{\mathbb{N}_0}\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. If we choose $a_1 = a_2 = 1/4$, $\xi_1 = 1/3$, $\xi_2 = 2/3$, $T = 1$, $a(t) \equiv 1$, and $p(t) \equiv 1$. Consider the following BVP on the time scale $\mathbb{T}$:

$$[\varphi(u^{\Delta}(t))]^{\nabla} + f(u(t)) = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{4.1}$$

$$u(0) = \frac{1}{4}u\Big(\frac{1}{3}\Big) + \frac{1}{4}u\Big(\frac{2}{3}\Big), \quad u^{\Delta}(1) = 0, \tag{4.2}$$

where

$$\varphi(u) = \begin{cases} \frac{u^5}{1+u^2}, & u \le 0,\\ u^2, & u > 0, \end{cases}$$

and

$$f(u) = \begin{cases} 0.1, & 0 \le u \le 3,\\ 0.1 + \frac{90(u-3)}{4\sqrt{3}-3}, & 3 \le u \le 4\sqrt{3},\\ 90.1, & 4\sqrt{3} \le u. \end{cases}$$

We take $a = 1$, $b = 4\sqrt{3}$, $c = 75$. By simple calculations, we have

$$\lambda_1 = \frac{1}{p(0)}\Big(\xi_1 + \frac{\sum_{i=1}^{m-2} a_i T}{1 - \sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_0^T a(\tau)\nabla\tau\Big) = \frac{4}{3},$$

$$\lambda_2 = \frac{1}{p(T)}\Big(\xi_1 + \frac{\sum_{i=1}^{m-2} a_i T}{1 - \sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_{\xi_{m-2}}^T a(\tau)\nabla\tau\Big) = \frac{4\sqrt{3}}{9},$$

$$\lambda_3 = \frac{1}{p(0)}\Big(\xi_{m-2} + \frac{\sum_{i=1}^{m-2} a_i T}{1 - \sum_{i=1}^{m-2} a_i}\Big)\varphi^{-1}\Big(\int_0^T a(\tau)\nabla\tau\Big) = \frac{5}{3}.$$

It is easy to see that

$$0 < a < \frac{\xi_1}{T}b < b < \frac{\lambda_2}{\lambda_1}c,$$

and that f satisfies

$$f(u) < \varphi\Big(\frac{c}{\lambda_1}\Big) = \Big(\frac{75}{\frac{4}{3}}\Big)^2 \approx 3164.0625, \quad u \in [0, 225];$$

$$f(u) > \varphi\Big(\frac{b}{\lambda_2}\Big) = \Big(\frac{4\sqrt{3}}{\frac{4\sqrt{3}}{9}}\Big)^2 = 81, \quad u \in [4\sqrt{3}, 12\sqrt{3}];$$

$$f(u) < \varphi\Big(\frac{a}{\lambda_3}\Big) = \Big(\frac{1}{\frac{5}{3}}\Big)^2 = \frac{9}{25}, \quad u \in [0, 3].$$

By Theorem 3.1, we see that BVP (4.1) and (4.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{t \in [0,\frac{2}{3}]_{\mathbb{T}}} \{u_i(t)\} \le 75, \quad \text{for } i = 1,2,3;$$

$$0 \le \max_{t \in [0,2/3]_{\mathbb{T}}} \{u_1(t)\} < 1 < \max_{t \in [0,\frac{2}{3}]_{\mathbb{T}}} \{u_2(t)\};$$

$$\min_{t \in [\frac{1}{3},\frac{2}{3}]_{\mathbb{T}}} \{u_2(t)\} < 4\sqrt{3} < \min_{t \in [\frac{1}{3},\frac{2}{3}]_{\mathbb{T}}} \{u_3(t)\}, \quad \max_{t \in [0,\frac{1}{3}]_{\mathbb{T}}} \{u_3(t)\} \le 75.$$

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