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MULTIPLE POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. In this article, we investigate how the coefficient $f(z)$ affects the number of positive solutions of the quasilinear elliptic system

$$\begin{gathered} -\Delta_p u = \lambda g(z)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta} f(z)|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega, \\ -\Delta_p v = \mu h(z)|v|^{q-2}v + \frac{\beta}{\alpha+\beta} f(z)|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $\alpha > 1$, $\beta > 1$ and $1 < p < q < \alpha+\beta = p^* = \frac{Np}{N-p}$ for $N > 2p$.

1. Introduction

Let $\Omega \ni 0$ be a smooth bounded domain in $\mathbb{R}^N$ with $N > 2p$. We are concerned with the quasilinear elliptic problem

$$\begin{gathered} -\Delta_p u = \lambda g(z)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta} f(z)|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega, \\ -\Delta_p v = \mu h(z)|v|^{q-2}v + \frac{\beta}{\alpha+\beta} f(z)|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda, \mu > 0$, $1 < p < q < p^*$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p^*$ and $p^* = \frac{Np}{N-p}$ for $N > 2p$ denotes the critical Sobolev exponent.

In recent years, there have been many papers concerned with the existence and multiplicity of positive solutions for semilinear elliptic problems. Results relating to these problem can be found in Wu [16, 17], Furtado and Paiva [6], Lin et al [11] and the references therein.

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In[1], the authors established the existence and nonexistence of solutions for the system of elliptic equations

$$\begin{gathered}-\Delta u = au + bv + \frac{2\alpha}{\alpha+\beta} u|u|^{\alpha-2}|v|^{\beta} \quad \text{in } \Omega, \\ -\Delta v = bu + cv + \frac{2\beta}{\alpha+\beta}|u|^{\alpha} v|v|^{\beta-2} \quad \text{in } \Omega, \\ u, v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{1.2}$$

As for quasilinear problems, Zhang [19] studied the elliptic equation

$$\begin{gathered}-\Delta_p u + |u|^{p-2}u = f(u) \quad x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N)\end{gathered} \tag{1.3}$$

Using a minimization argument, the author obtained the existence of ground state solutions for (1.3).

In [9], the authors investigated how the shape of the graph of $f(z)$ affects the number of positive solutions of the problem

$$\begin{gathered}-\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u \quad x \in B_1. \\ u|_{\partial\Omega} = 0.\end{gathered} \tag{1.4}$$

By variational methods, Hsu [8] showed the existence of multiple positive solutions for the elliptic system

$$\begin{gathered}-\Delta_p u = \lambda|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega, \\ -\Delta_p v = \mu|v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{1.5}$$

Yin and Yang [18] studied the problem

$$\begin{gathered}-\Delta_p u + |u|^{p-2}u = f_{1\lambda_1}(x)|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta} g_\mu |u|^{\alpha-2}u|v|^{\beta} \quad x \in \Omega, \\ -\Delta_p v + |v|^{p-2}v = f_{2\lambda_2}(x)|v|^{q-2}v + \frac{2\beta}{\alpha+\beta} g_\mu |u|^{\alpha}|v|^{\beta-2}v \quad x \in \Omega, \\ u = v = 0 \quad x \in \partial\Omega.\end{gathered} \tag{1.6}$$

Motivated by the results of the above cited papers, we shall study system (1.2); in particular, the results of the semilinear systems are extended to the quasilinear systems. We can find the related results for $p = 2$ in [11].

In this paper, we assume that f, g and h satisfy the following conditions:

(A1) f, g and h are positive continuous functions in $\overline{\Omega}$.

(A2) There exist k points a^1, a^2,…,a^k in Ω such that

$$f(a^i) = \max_{z\in\Omega} f(z) = 1 \quad \text{for } 1 \le i \le k,$$

and for some $\sigma > N$, $f(z) - f(a^i) = O(|z - a^i|^{\sigma})$ as $z \to a^i$ uniformly in i.

(A3) Choose $\rho_0 > 0$ such that

$$\overline{B_{\rho_0}(a^i)} \cap \overline{B_{\rho_0}(a^j)} = \emptyset \quad \text{for } i \ne j \text{ and } 1 \le i, j \le k,$$

and $\cup_{i=1}^k \overline{B_{\rho_0}(a^i)} \subset \Omega$, where $\overline{B_{\rho_0}(a^i)} = \{z \in \mathbb{R}^N : |z - a^i| \le \rho_0\}$.

Let $E = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ be the Sobolev space with norm

$$\|(u,v)\| = \Big(\int_\Omega (|\nabla u|^p + |\nabla v|^p) dz \Big)^{1/p}.$$

We will show the existence and multiplicity result of nontrivial solutions of (1.1) by looking for critical points of the associated functional

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \frac{1}{p}\|(u,v)\|^p - \frac{1}{p^*}\int_\Omega f(z)|u|^\alpha |v|^\beta dz \\ &\quad - \frac{1}{q}\int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q) dz. \end{aligned}$$

The critical points of the functional $J_{\lambda,\mu}$ are in fact weak solutions of (1.1). By a weak solution (u,v) of (1.1), we mean that $(u,v) \in E$ satisfying

$$\begin{aligned} &\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla \varphi_1 + |\nabla v|^{p-2}\nabla v \nabla \varphi_2) dz - \lambda \int_\Omega g(z)|u|^{q-2}u\varphi_1 dz \\ &- \mu \int_\Omega h(z)|v|^{q-2}v\varphi_2 dz - \frac{\alpha}{p^*}\int_\Omega f(z)|u|^{\alpha-2}u|v|^\beta \varphi_1 dz - \frac{\beta}{p^*}\int_\Omega f(z)|u|^\alpha |v|^{\beta-2}v\varphi_2 dz \\ &= 0, \end{aligned}$$

for any $(\varphi_1, \varphi_2) \in E$

Consider the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in E \backslash \{(0,0)\} : \langle J'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = \|(u,v)\|^p - \int_\Omega f(z)|u|^\alpha |v|^\beta dz - \int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q) dz = 0.$$

Note that the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ contains all nontrivial weak solutions of (1.1).

Denote

$$S_{\alpha,\beta} = \inf_{u,v \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{\|(u,v)\|^p}{(\int_\Omega |u|^\alpha |v|^\beta dz)^{\frac{p}{\alpha+\beta}}}.$$

Modifying the proof of Alves et al [1, Theorem 5] or from Yin and Yang [18, Lemma 2.2], we can easily obtain that

$$S_{\alpha,\beta} = ((\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}})S, \tag{1.7}$$

where $\alpha + \beta = p^*$ and S is the best Sobolev constant defined by

$$S = \inf_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^p dz}{(\int_\Omega |u|^{p^*} dz)^{p/p^*}} > 0.$$

Recall that S is independent of the domain and is never achieved except when $\Omega = \mathbb{R}^N$. Moreover, S is attained by the function

$$U(z) = [N(N-p)]^{\frac{N-p}{p^2}} / (1 + |z|^{\frac{p}{p-1}})^{\frac{N-p}{p}},$$

so that

$$\|\nabla U\|_{L^p}^p = \|U\|_{L^{p^*}}^{p^*} = S^{N/p}.$$

For $\lambda = \mu = 0$, we consider the quasilinear elliptic system

$$\begin{gathered} -\Delta_p u = \frac{\alpha}{\alpha+\beta} f(z)|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega, \\ -\Delta_p v = \frac{\beta}{\alpha+\beta} f(z)|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \Omega, \\ (u,v) \in E, \end{gathered} \tag{1.8}$$

Related to this system, we define the energy functional

$$J_{0,0}(u,v) = \frac{1}{p}\|(u,v)\|^p - \frac{1}{\alpha+\beta}\int_\Omega f(z)|u|^\alpha|v|^\beta dz,$$

and

$$\theta_{0,0} = \inf_{(u,v)\in\mathcal{N}_{0,0}} J_{0,0}(u,v),$$

where

$$\mathcal{N}_{0,0} = \{(u,v) \in E\backslash\{(0,0)\} | \langle J'_{0,0}(u,v),(u,v)\rangle = 0\}.$$

Moreover, if $f \equiv \max_{z\in\Omega} f(z) = 1$, we define

$$J_{\max}(u,v) = \frac{1}{p}\|(u,v)\|^p - \frac{1}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta dz,$$

and

$$\theta_{\max} = \inf_{(u,v)\in\mathcal{N}_{\max}} J_{\max}(u,v),$$

where

$$\mathcal{N}_{\max} = \{(u,v) \in E\backslash\{(0,0)\} : \langle J'_{\max}(u,v),(u,v)\rangle = 0\}.$$

The paper is organized as follows. Firstly, we study the argument of the Nehari manifold $\mathcal{N}_{\lambda,\mu}$. Next, we show the existence of a positive solution $(u_0,v_0) \in \mathcal{N}_{\lambda,\mu}$ of (1.1). Finally, in Section 4, we show that the condition (A2) affects the number of positive solutions of (1.1), that is, there are at least k critical points $(u_i,v_i) \in \mathcal{N}_{\lambda,\mu}$ of $J_{\lambda,\mu}$ such that $J_{\lambda,\mu}(u_i,v_i) = \gamma^i_{\lambda,\mu}$((PS)-value) for $1 \le i \le k$.

Inspired by [11, 18], we establish the following theorem.

Theorem 1.1. *System* (1.1) *admits at least one positive solution* $(u_0,v_0) \in \mathcal{N}_{\lambda,\mu}$.

Theorem 1.2. *Assume* (A1)–(A3) *hold, then there exists a positive number* Λ^* *such that* (1.1) *admits at least k positive solutions for any* $0 < \lambda + \mu < \Lambda^*$.

2. Preliminaries

Lemma 2.1 ([7, Lemma 2.1])**.** *Let $D \subset \mathbb{R}^N$ (possibly unbounded) be a smooth domain. If $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $W^{1,p}_0(D)$, and $u_n \to u$, $v_n \to v$ almost everywhere in D, then*

$$\lim_{n\to\infty}\int_D |u_n-u|^\alpha|v_n-v|^\beta dz = \lim_{n\to\infty}\int_D |u_n|^\alpha|v_n|^\beta dz - \int_\Omega |u|^\alpha|v|^\beta dz.$$

Note that $J_{\lambda,\mu}$ is not bounded from below in E. But from the following lemma, we have that $J_{\lambda,\mu}$ is bounded from below on the Nehari manifold $\mathcal{N}_{\lambda,\mu}$.

Lemma 2.2. *The energy functional $J_{\lambda,\mu}$ is bounded from below on the Nehari manifold $\mathcal{N}_{\lambda,\mu}$.*

Proof. For any $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$J_{\lambda,\mu}(u,v) = (\frac{1}{p} - \frac{1}{q})\|(u,v)\|_E^p + (\frac{1}{q} - \frac{1}{p^*}) \int_\Omega f(z)|u|^\alpha |v|^\beta dz > 0.$$

Thus, $J_{\lambda,\mu}$ is bounded from below on $\mathcal{N}_{\lambda,\mu}$. □

Then, we define

$$\theta_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

Lemma 2.3. (i) *There exist positive number ζ and d_0 such that $J_{\lambda,\mu}(u,v) \geq d_0$ for $\|(u,v)\|_E = \zeta$;*
(ii) *There exists $(\overline{u},\overline{v}) \in E\backslash\{(0,0)\}$ such that $\|(u,v)\|_E > \zeta$ and $J_{\lambda,\mu}(\overline{u},\overline{v}) < 0$.*

Proof. (i) Combining (1.7), the Hölder's inequality ($q_1 = \frac{p^*}{p^*-q}$, $q_2 = \frac{p^*}{q}$, $\frac{1}{q_1} + \frac{1}{q_2} = 1$) with the Sobolev embedding theorem, we have

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \frac{1}{p}\|(u,v)\|_E^p - \frac{1}{p^*}\int_\Omega f(z)|u|^\alpha|v|^\beta dz - \frac{1}{q}\int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q)dz \\ &\geq \frac{1}{p}\|(u,v)\|_E^p - \frac{1}{p^*} S_{\alpha,\beta}^{-\frac{p^*}{p}} \|(u,v)\|_E^{p^*} \\ &\quad - \frac{1}{q}\max\{\|g\|_\infty, \|h\|_\infty\}|\Omega|^{\frac{p^*-q}{p^*}} S^{-\frac{q}{p}}(\lambda+\mu)\|(u,v)\|^q . \end{aligned}$$

Thus, there exist positive numbers ζ, d_0 such that $J_{\lambda,\mu}(u,v) \geq d_0$ for $\|(u,v)\|_E = \zeta$.

(ii) Note that

$$\begin{aligned} J_{\lambda,\mu}(su,sv) &= \frac{s^p}{p}\|(u,v)\|_E^p - \frac{s^{p^*}}{p^*}\int_\Omega f(z)|u|^\alpha|v|^\beta dz \\ &\quad - \frac{s^q}{q}\int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q)dz, \end{aligned}$$

for any $(u,v) \in E\backslash\{(0,0)\}$, then we have $\lim_{s\to\infty} J_{\lambda,\mu}(su,sv) = -\infty$. Thus, for fixed $(u,v) \in E\backslash\{(0,0)\}$, there exists $\overline{s} > 0$ such that $\|(\overline{s}u,\overline{s}v)\|_E > \zeta$ and $J_{\lambda,\mu}(\overline{s}u,\overline{s}v) < 0$. Let $(\overline{u},\overline{v}) = (\overline{s}u,\overline{s}v)$, then we finish the proof. □

Define $\Phi_{\lambda,\mu} = \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle$, then for $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned} &\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle \\ &= p\|(u,v)\|_E^p - p^*\int_\Omega f(z)|u|^\alpha|v|^\beta dz - q\int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q)dz \\ &= (p-p^*)\|(u,v)\|_E^p + (p^*-q)\int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q)dz \\ &= (p-q)\|(u,v)\|_E^p + (q-p^*)\int_\Omega f(z)|u|^\alpha|v|^\beta dz < 0. \end{aligned}$$

Lemma 2.4. *If $(u_0,v_0) \in \mathcal{N}_{\lambda,\mu}$ satisfies $J_{\lambda,\mu}(u_0,v_0) = \min_{(u,v)\in\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \theta_{\lambda,\mu}$, then (u_0,v_0) is a nontrivial solution of* (1.1).

Proof. Since $\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle < 0$ for each $(u,v) \in \mathcal{N}_{\lambda,\mu}$ and $J_{\lambda,\mu}(u_0,v_0) = \min_{(u,v)\in\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$, by the Lagrange multiplier theorem, there is $\kappa \in \mathbb{R}$ such that $J'_{\lambda,\mu}(u_0,v_0) = \kappa\Phi'_{\lambda,\mu}(u_0,v_0)$ in E^{-1}, where E^{-1} is the dual space of E. Then we have

$$0 = \langle J'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle = \kappa\langle \Phi'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle.$$

Thus $\kappa = 0$ and $J'_{\lambda,\mu}(u_0, v_0) = 0$ in E^{-1}. Therefore, (u_0, v_0) is a nontrivial solution of (1.1) and $J_{\lambda,\mu}(u_0, v_0) = \theta_{\lambda,\mu}$. □

Lemma 2.5. *For each $(u, v) \in E\backslash\{(0, 0)\}$, there is a positive number $s_{u,v}$ such that $(s_{u,v}u, s_{u,v}v) \in \mathcal{N}_{\lambda,\mu}$ and $J_{\lambda,\mu}(s_{u,v}u, s_{u,v}v) = \sup_{s\geq 0} J_{\lambda,\mu}(su, sv)$.*

Proof. Let $\varphi(s) = J_{\lambda,\mu}(su, sv)$ for fixed $(u, v) \in E\backslash\{(0, 0)\}$, then we have

$$\begin{aligned}\varphi(s) = J_{\lambda,\mu}(su, sv) &= \frac{s^p}{p}\|(u, v)\|^p_E - \frac{s^{p^*}}{p^*}\int_\Omega f(z)|u|^\alpha|v|^\beta dz \\ &\quad - \frac{s^q}{q}\int_\Omega(\lambda g(z)|u|^q + \mu h(z)|v|^q)dz.\end{aligned}$$

It is easy to see that $\varphi(0) = 0$ and $\lim_{s\to\infty} \varphi(s) = -\infty$, then by Lemma 2.3 (i), we obtain that $\sup_{s\geq 0} \varphi(s)$ is achieved at some $s_{u,v} > 0$. Thus, we have $\varphi'(s_{u,v}) = 0$; that is, $(s_{u,v}u, s_{u,v}v) \in \mathcal{N}_{\lambda,\mu}$ and we competed the proof. □

Lemma 2.6. *$\theta_{\lambda,\mu} \geq d_0 > 0$ for some constant d_0.*

Combining Lemma 2.3 (i) with Lemma 2.5, we can easily obtain the result of the above lemma.

3. (PS)-condition in E for $J_{\lambda,\mu}$

First, we give the definition of the Palais-Smale sequence and (PS)-condition in E for the energy functional J.

Definition 3.1. Let $c \in \mathbb{R}$, E be a Banach space and $J \in C^1(E, \mathbb{R})$,

(i) $\{(u_n, v_n)\}$ is a $(PS)_c$-sequence in E for J if $J(u_n, v_n) = c + o_n(1)$ and $J'(u_n, v_n) = o_n(1)$ strongly in E^{-1} as $n \to \infty$, where E^{-1} is the dual space of E.

(ii) We say that J satisfies the $(PS)_c$-condition in E if any $(PS)_c$-sequence in E for J has a convergent subsequence.

Applying Ekeland's variational principle and using the similar argument as in Cao and Zhou [4] or Tarantello [14], we have the following lemma.

Lemma 3.2. *There exist a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\}$ in $\mathcal{N}_{\lambda,\mu}$ for $J_{\lambda,\mu}$.*

Next, we show that $J_{\lambda,\mu}$ satisfies the $(PS)_c$-condition for $c \in (0, \frac{1}{N}(S_{\alpha,\beta})^{N/p})$ in E.

Lemma 3.3. *$J_{\lambda,\mu}$ satisfies the $(PS)_c$-condition in E for $c \in (0, \frac{1}{N}(S_{\alpha,\beta})^{N/p})$.*

The proof of the above lemma is similar to the proof in [11, Lemma 3.3]; thus it is omitted here.

4. Existence of k solutions

Recall that the best Sobolev constant S is defined as

$$S = \inf_{u\in W^{1,p}(\mathbb{R}^N)\backslash\{0\}} \frac{\|\nabla u\|^p_{L^p}}{\|u\|^p_{L^{p^*}}}.$$

Moreover, $U(z) = \frac{[N(N-p)]^{\frac{N-p}{p^2}}}{(1+|z|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}$ is a minimizer of S, and we can easily get that

$$\|\nabla U\|^p_{L^p} = \|U\|^{p^*}_{L^{p^*}} = S^{N/p}.$$

Let $\psi_i(z) \in C_0^\infty(\Omega)$ be a cut function such that

$$\psi_i(z) = \begin{cases} 1, & |z-a^i| < \rho_0/2, \\ 0 \le \psi_i(z) \le 1, & \rho_0/2 \le |z-a^i| \le \rho_0, \\ 0, & |z-a^i| > \rho_0, \end{cases}$$

where $1 \le i \le k$, and $|\nabla \psi_i(z)| \le C$. Then, we define

$$u_\epsilon^i(z) = \epsilon^{(p-N)/p}\psi_i(z)U(\frac{z-a^i}{\epsilon}) = C_1\epsilon^{\frac{N-p}{p(p-1)}}\psi_i(z)(\epsilon^{\frac{p}{p-1}} + |z-a^i|^{\frac{p}{p-1}})^{(p-N)/p},$$

where $C_1 = [N(N-p)]^{\frac{N-p}{p^2}}$.

Next, we show that

$$\sup_{s\ge 0} J_{\lambda,\mu}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) < \frac{1}{N}(S_{\alpha,\beta})^{N/p} \quad \text{uniformly in } i.$$

Lemma 4.1. *If there exists $0 < \epsilon_0 < \min\{1, \rho_0/2\}$ such that for $0 < \epsilon < \epsilon_0$, then we have*

$$\sup_{s\ge 0} J_{\lambda,\mu}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) < \frac{1}{N}(S_{\alpha,\beta})^{N/p} \quad \textit{uniformly in } i.$$

Moreover,

$$0 < \theta_{\lambda,\mu} < \frac{1}{N}(S_{\alpha,\beta})^{N/p}.$$

Proof. From Hsu [10, Lemma 4.3] and after a detailed calculation, we have the following estimates

$$\begin{aligned} \|u_\epsilon^i\|_{L^{p^*}}^p &= \|U\|_{L^{p^*}(\mathbb{R}^N)}^p + O(\epsilon^{N-p}), \\ \|\nabla u_\epsilon^i\|_{L^p}^p &= \|\nabla U\|_{L^p(\mathbb{R}^N)}^p + O(\epsilon^{N-p}). \end{aligned} \tag{4.1}$$

For $0 < \epsilon < \rho_0/2$ and $N > 2p$, we have

$$\|u_\epsilon^i\|_{L^p}^p = \int_{B_{\frac{\rho_0}{2}}(a^i)} [\epsilon^{(p-N)/p}U(\frac{z-a^i}{\epsilon})]^p dz + O(\epsilon^{N-p}) \ge C_2\epsilon^\theta + O(\epsilon^{N-p}), \tag{4.2}$$

where $\theta = N - \frac{(N-p)q}{p} > 0$.

When $\lambda = \mu = 0$, we consider the functional $J_{0,0} : E \to R$ given by

$$J_{0,0}(u,v) = \frac{1}{p}\|(u,v)\|_E^p - \frac{1}{p^*}\int_\Omega f(z)|u|^\alpha|v|^\beta dz.$$

First, we claim that

$$\sup_{s\ge 0} J_{0,0}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) \le \frac{1}{N}(S_{\alpha,\beta})^{N/p} + O(\epsilon^{N-p}).$$

From assumption (A2), when $\sigma > N$, we have

$$\Big(\int_\Omega f(z)(u_\epsilon^i)^{p^*}dz\Big)^{p/p^*} = \|u_\epsilon^i\|_{L^{p^*}}^p + O(\epsilon^{N-p}) = \|U\|_{L^{p^*}(\mathbb{R}^N)}^p + O(\epsilon^{N-p}). \tag{4.3}$$

The equalities in (4.1) combined with (4.3) lead to

$$\frac{\|\nabla u_\epsilon^i\|_{L^p}^p}{(\int_\Omega f(z)(u_\epsilon^i)^{p^*}dz)^{p/p^*}} = \frac{\|\nabla U\|_{L^p(\mathbb{R}^N)}^p + O(\epsilon^{N-p})}{\|U\|_{L^{p^*}(\mathbb{R}^N)}^p + O(\epsilon^{N-p})} = S + O(\epsilon^{N-p}). \tag{4.4}$$

Using the fact that

$$\sup_{s\geq 0}(\frac{s^p}{p}A - \frac{s^{p^*}}{p^*}B) = \frac{1}{N}A(\frac{A}{B})^{\frac{N-p}{p}} = \frac{1}{N}(\frac{A}{B^{p/p^*}})^{N/p},$$

for any $A > 0$ and $B > 0$. By (4.4), we obtain that

$$\begin{aligned}
&\sup_{s\geq 0} J_{0,0}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) \\
&= \sup\{\frac{1}{p}\int_\Omega (s\sqrt[p]{\alpha}\nabla u_\epsilon^i)^p + (s\sqrt[p]{\beta}\nabla u_\epsilon^i)^p dz - \frac{1}{p^*}\int_\Omega f(z)|s\sqrt[p]{\alpha}u_\epsilon^i|^\alpha |s\sqrt[p]{\beta}u_\epsilon^i|^\beta dz\} \\
&= \sup\{\frac{s^p}{p}\int_\Omega (\alpha+\beta)|\nabla u_\epsilon^i|^p dz - \frac{s^{p^*}}{p^*}\int_\Omega f(z)\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}}|u_\epsilon^i|^{p^*} dz\} \\
&= \frac{1}{N}(\frac{\int_\Omega (\alpha+\beta)|\nabla u_\epsilon^i|^p dz}{(\int_\Omega f(z)\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}}|u_\epsilon^i|^{p^*} dz)^{p/p^*}})^{N/p} \\
&= \frac{1}{N}\Big\{[(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}}]\Big(\frac{\int_\Omega |\nabla u_\epsilon^i|^p dz}{(\int_\Omega f(z)|u_\epsilon^i|^{p^*} dz)^{p/p^*}}\Big)\Big\}^{N/p} \\
&= \frac{1}{N}(S_{\alpha,\beta})^{N/p} + O(\epsilon^{N-p}).
\end{aligned}$$

Since $J_{\lambda,\mu}$ is continuous in E, $J_{\lambda,\mu}(0,0) = 0$, and from (4.1), we see that the set $\{(\sqrt[p]{\alpha}u_\epsilon^i, \sqrt[p]{\beta}u_\epsilon^i)\}$ is uniformly bounded in E for any $0 < \epsilon < \min\{1, \rho_0/2\}$, then there exists $s_0 > 0$ such that

$$\sup_{0\leq s\leq s_0} J_{\lambda,\mu}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) < \frac{1}{N}(S_{\alpha,\beta})^{N/p} \quad \text{uniformly in } i,$$

for any $0 < \epsilon < \min\{1, \frac{\rho_0}{2}\}$

Let $g_{\rm inf} = \inf_{z\in\overline{\Omega}} g(z) > 0$ and $h_{\rm inf} = \inf_{z\in\overline{\Omega}} h(z) > 0$, then we have

$$\begin{aligned}
&\sup_{s\geq s_0} J_{\lambda,\mu}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) \\
&\leq \sup_{s\geq s_0} J_{0,0}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) - \frac{s_0^q}{q}\int_\Omega (\lambda g(z)|\sqrt[p]{\alpha}u_\epsilon^i|^q + \mu h(z)|\sqrt[p]{\beta}u_\epsilon^i|^q) dz \\
&\leq \frac{1}{N}(S_{\alpha,\beta})^{N/p} + O(\epsilon^{N-p}) - \frac{s_0^q}{q}(\lambda+\mu)m\int_{B_{\frac{\rho_0}{2}}(a^i)} (u_\epsilon^i)^q dz \\
&\leq \frac{1}{N}(S_{\alpha,\beta})^{N/p} + O(\epsilon^{N-p}) - \frac{s_0^q}{q}C_2 m(\lambda+\mu)\epsilon^\theta,
\end{aligned}$$

where $m = \min\{\alpha^{\frac{q}{p}}g_{\rm inf}, \beta^{\frac{q}{p}}h_{\rm inf}\}$ and $\theta = N - \frac{(N-p)q}{p} > 0$.

Since $p < q < p^*$, it follows that $0 < \theta = N - \frac{(N-p)q}{p} < p < N-p$ for $N > 2p$. Thus, we can choose $\epsilon_0 > 0$ such that $\epsilon_0 < \min\{1, \frac{\rho_0}{2}\}$ and $O(\epsilon^{N-p}) - \frac{s_0^q}{q}C_2 m(\lambda + \mu)\epsilon^\theta < 0$ for any $0 < \epsilon < \epsilon_0$. Therefore, we have for any $0 < \epsilon < \epsilon_0$,

$$\sup_{s\geq 0} J_{\lambda,\mu}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) < \frac{1}{N}(S_{\alpha,\beta})^{N/p} \quad \text{uniformly in } i.$$

Combining Lemma 2.5 with Lemma 2.6, we obtain

$$0 < \theta_{\lambda,\mu} \leq J_{\lambda,\mu}(s_\epsilon^i\sqrt[p]{\alpha}u_\epsilon^i, s_\epsilon^i\sqrt[p]{\beta}u_\epsilon^i)$$

$$= \sup_{s\geq 0} J_{\lambda,\mu}(s\sqrt[p]{\alpha}u_\epsilon^i, s\sqrt[p]{\beta}u_\epsilon^i) < \frac{1}{N}(S_{\alpha,\beta})^{N/p}.$$

Hence, the proof is complete

Proof of Theorem 1.1. From Lemma 3.2, we have that there is a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ for $J_{\lambda,\mu}$ satisfying $J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o_n(1)$ and $J'_{\lambda,\mu}(u_n, v_n) = o_n(1)$ in E^{-1}. Combining Lemma 4.1 with Lemma 3.3, we obtain $0 < \theta_{\lambda,\mu} < \frac{1}{N}(S_{\alpha,\beta})^{N/p}$ and then there exist a subsequence (still denoted by $\{(u_n, v_n)\}$) and $(u_0, v_0) \in E$ such that $(u_n, v_n) \to (u_0, v_0)$ strongly in E. By direct computation, we can easily prove that (u_0, v_0) is a nontrivial solution of (1.1) and $J_{\lambda,\mu}(u_0, v_0) = \theta_{\lambda,\mu}$. Using the fact that $J_{\lambda,\mu}(u_0, v_0) = J_{\lambda,\mu}(|u_0|, |v_0|)$ and $(|u_0|, |v_0|) \in \mathcal{N}_{\lambda,\mu}$ and by Lemma 2.4, we may assume that $u_0 \geq 0$, $v_0 \geq 0$. Thus, by the maximum principle, we can get that $u_0 > 0$ and $v_0 > 0$ in Ω. That is, (1.1) admits a positive solution $(u_0, v_0) \in \mathcal{N}_{\lambda,\mu}$. □

Now we study the effect of the coefficient $f(z)$. Then, we want to construct the k compact (PS)-sequences.

From the assumptions (A2) and (A3), choose $\rho_0 > 0$ such that

$$\overline{B_{\rho_0}(a^i)} \cap \overline{B_{\rho_0}(a^j)} = \emptyset \quad \text{for } i \neq j \text{ and } 1 \leq i, j \leq k,$$

and $\cup_{i=1}^k \overline{B_{\rho_0}(a^i)} \subset \Omega$ and $f(a^i) = \max_{z\in\Omega} f(z) = 1$.

Then we define $M = \{a^i | 1 \leq i \leq k\}$ and $M_{\rho_0/2} = \cup_{i=1}^k \overline{B_{\rho_0/2}(a^i)}$. Suppose $\cup_{i=1}^k \overline{B_{\rho_0}(a^i)} \subset B_{r_0}(0)$ for some $r_0 > 0$.

Let $Q : E\backslash\{(0,0)\} \to \mathbb{R}^N$ be given by

$$Q(u, v) = \frac{\int_\Omega \chi(z)|u|^\alpha |v|^\beta dz}{\int_\Omega |u|^\alpha |v|^\beta dz},$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$ satisfying

$$\chi(z) = \begin{cases} z, & |z| \leq r_0, \\ r_0 z/|z|, & |z| > r_0. \end{cases}$$

For each $1 \leq i \leq k$, we define

$$\begin{gathered} D^i_{\lambda,\mu} = \{(u, v) \in \mathcal{N}_{\lambda,\mu} \| Q(u, v) - a^i| < \rho_0\}, \\ \partial D^i_{\lambda,\mu} = \{(u, v) \in \mathcal{N}_{\lambda,\mu} \| Q(u, v) - a^i| = \rho_0\}, \\ \gamma^i_{\lambda,\mu} = \inf_{(u,v)\in D^i_{\lambda,\mu}} J_{\lambda,\mu}(u, v), \\ \widetilde{\gamma^i_{\lambda,\mu}} = \inf_{(u,v)\in \partial D^i_{\lambda,\mu}} J_{\lambda,\mu}(u, v). \end{gathered}$$

From Lemma 2.5, there exists $s^i_\epsilon > 0$ such that $(s^i_\epsilon \sqrt[p]{\alpha}u^i_\epsilon, s^i_\epsilon \sqrt[p]{\beta}u^i_\epsilon) \in \mathcal{N}_{\lambda,\mu}$ for each $1 \leq i \leq k$. Then we have the following lemma.

Lemma 4.2. *There exists $\epsilon_1 \in (0, \epsilon_0)$ such that if $0 < \epsilon < \epsilon_1$, then*

$$Q(s^i_\epsilon \sqrt[p]{\alpha}u^i_\epsilon, s^i_\epsilon \sqrt[p]{\beta}u^i_\epsilon) \in M_{\rho_0/2}$$

for each $1 \leq i \leq k$.

The proof of the above lemma follows from the same argument as in [11, Lemma 4.2], and is omitted here.

Before we show that $\gamma^i_{\lambda,\mu} < \widetilde{\gamma^i_{\lambda,\mu}}$ for sufficiently small λ, μ, we give the following lemma.

Lemma 4.3. $\theta_{\max} = \frac{1}{N}(S_{\alpha,\beta})^{N/p}$.

The proof of the above lemma follows from the same argument as in [11, Lemma 4.3], and it is omitted here.

Lemma 4.4. $\theta_{0,0} = \theta_{\max}$.

Proof. Using the fact that $f(z) \le \max_{z\in\Omega} f(z) = 1$, we obtain $\theta_{\max} \le \theta_{0,0}$. From the proof of Lemma 4.1,

$$\begin{aligned}\sup_{s\ge 0} J_{0,0}(s\sqrt[p]{\alpha}u^i_\epsilon, s\sqrt[p]{\beta}u^i_\epsilon) &\le \frac{1}{N}(S_{\alpha,\beta})^{N/p} + O(\epsilon^{N-p})\\ &= \theta_{\max} + O(\epsilon^{N-p}),\end{aligned}$$

uniformly in i. Similarly to Lemma 2.5, we can get that there is a sequence $\{t^i_\epsilon\} \subset \mathbb{R}^+$ such that $(t^i_\epsilon\sqrt[p]{\alpha}u^i_\epsilon, t^i_\epsilon\sqrt[p]{\beta}u^i_\epsilon) \in \mathcal{N}_{0,0}$ and

$$\begin{aligned}\theta_{0,0} &\le J_{0,0}(t^i_\epsilon\sqrt[p]{\alpha}u^i_\epsilon, t^i_\epsilon\sqrt[p]{\beta}u^i_\epsilon)\\ &= \sup_{s\ge 0} J_{0,0}(s\sqrt[p]{\alpha}u^i_\epsilon, s\sqrt[p]{\beta}u^i_\epsilon)\\ &\le \frac{1}{N}(S_{\alpha,\beta})^{N/p} + O(\epsilon^{N-p})\\ &= \theta_{\max} + O(\epsilon^{N-p}).\end{aligned}$$

Let $\epsilon \to 0^+$, we obtain that $\theta_{0,0} \le \theta_{\max}$. Therefore, we have $\theta_{0,0} = \theta_{\max}$ and the proof is complete. $\square$

Using the ideas in [11], we give the following Lemmas.

Lemma 4.5. *There exists a positive number η_0 such that if $(u,v) \in \mathcal{N}_{0,0}$ and $J_{0,0}(u,v) \le \theta_{0,0}(= \theta_{\max} = \frac{1}{N}(S_{\alpha,\beta})^{N/p}) + \eta_0$, then $Q(u,v) \in M_{\rho_0/2}$.*

Proof. Suppose by contradiction that there exists a sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{0,0}$ such that $J_{0,0}(u_n,v_n) = \theta_{0,0} + o_n(1)$ as $n \to \infty$ and $Q(u_n,v_n) \notin M_{\rho_0/2}$ for all $n \in \mathbf{N}$. A similar argument as in Lemma 2.5, we obtain that there is a sequence $\{t^n_{\max}\} \subset \mathbb{R}^+$ such that $(t^n_{\max}u_n, t^n_{\max}v_n) \in \mathcal{N}_{\max}$ and

$$\begin{aligned}0 < \theta_{\max} &\le J_{\max}(t^n_{\max}u_n, t^n_{\max}v_n) \le J_{0,0}(t^n_{\max}u_n, t^n_{\max}v_n) \le J_{0,0}(u_n,v_n)\\ &= \theta_{0,0}(= \theta_{\max} = \frac{1}{N}(S_{\alpha,\beta})^{N/p}) + o_n(1), \quad \text{as } n \to \infty.\end{aligned}$$

From Ekeland's variational principle, there exists a $(PS)_{\theta_{\max}}$-sequence $\{(U_n,V_n)\}$ for $J_{\max}$ and $\|(U_n - t^n_{\max}u_n, V_n - t^n_{\max}v_n)\|_E = o_n(1)$. Now, we will show that

$$\int_\Omega |U_n|^\alpha |V_n|^\beta dz \not\to 0 \quad \text{as } n \to \infty.$$

Assuming the contrary and using that $\|(U_n,V_n)\|^p_E = \int_\Omega |U_n|^\alpha|V_n|^\beta dz + o_n(1)$ as $n \to \infty$, we obtain

$$\theta_{\max} + o_n(1) = J_{\max}(U_n,V_n)$$

$$\begin{aligned}&= \frac{1}{p}\|(U_n,V_n)\|_E^p - \frac{1}{p^*}\int_\Omega |U_n|^\alpha |V_n|^\beta dz + o_n(1)\\ &= (\frac{1}{p}-\frac{1}{p^*})\int_\Omega |U_n|^\alpha|V_n|^\beta dz + o_n(1) = o_n(1),\end{aligned}$$

which is a contradiction. Thus, we obtain that

$$\int_\Omega |u_n|^\alpha |v_n|^\beta dz \not\to 0 \quad \text{as } n\to\infty.$$

Therefore, from (Lions [12] or Willem [15]), there exist sequences $\{\delta_n\}\subset\mathbb{R}^+$ and $\{y_n\}\subset\Omega$ such that

$$\int_{B_{\delta_n}(y_n)} |U_n|^\alpha|V_n|^\beta dz \ge C_0 \tag{4.5}$$

for some positive constant C_0. Let

$$(\widetilde{U_n},\widetilde{V_n}) = (\delta_n^{\frac{N-p}{p}} U_n(\delta_n z + y_n), \delta_n^{\frac{N-p}{p}} V_n(\delta_n z + y_n)),$$

then we can easily get $\frac{1}{\delta_n}\operatorname{dist}(y_n,\partial\Omega)\to\infty$ as $n\to\infty$, and there exist a subsequence (still denoted by $\{(\widetilde{U_n},\widetilde{V_n})\}$) and $(\widetilde{U},\widetilde{V})\in W^{1,p}(\mathbb{R}^N)\times W^{1,p}(\mathbb{R}^N)$ such that $\widetilde{U_n}\to\widetilde{U}$ and $\widetilde{V_n}\to\widetilde{V}$ strongly in $W^{1,p}(\mathbb{R}^N)$.

From (4.5), we deduce that $\widetilde{U}\neq 0$ and $\widetilde{V}\neq 0$. Using that Ω is a bounded domain and $\{y_n\}\subset\Omega$, there exists a subsequence $\{\delta_n\}$ such that $\delta_n\to 0$ and we can suppose the subsequence $y_n\to y_0\in\overline{\Omega}$ as $n\to\infty$.

Next, we will show that $y_0\in M$. In fact, since $J_{0,0}(t^n_{\max}u_n, t^n_{\max}v_n) = \theta_{\max} + o_n(1)$ and $\|(U_n - t^n_{\max}u_n, V_n - t^n_{\max}v_n)\|_E = o_n(1)$ as $n\to\infty$. Combining the Lebesgue dominated convergence theorem with the fact that $\frac{1}{\delta_n}\operatorname{dist}(y_n,\partial\Omega)\to\infty$ as $n\to\infty$, we obtain

$$\begin{aligned}(S_{\alpha,\beta})^{N/p} &= \frac{\theta_{\max}}{\frac{1}{p}-\frac{1}{p^*}} = \int_\Omega f(z)|U_n|^\alpha|V_n|^\beta dz + o_n(1)\\ &= (\frac{1}{\delta_n})^N\int_\Omega f(z)|\widetilde{U_n}(\frac{z-y_n}{\delta_n})|^\alpha|\widetilde{V_n}(\frac{z-y_n}{\delta_n})|^\beta + o_n(1)\\ &= f(y_0)(S_{\alpha,\beta})^{N/p}.\end{aligned}$$

Then, $f(y_0)=1$; that is, $y_0\in M$.

On the other hand, since $\|(U_n - t^n_{\max}u_n, V_n - t^n_{\max}v_n)\|_E = o_n(1)$ and $\widetilde{U_n}\to\widetilde{U}$ and $\widetilde{V_n}\to\widetilde{V}$ strongly in $W^{1,p}(\mathbb{R}^N)$, we have

$$\begin{aligned}Q(u_n,v_n) &= \frac{\int_\Omega \chi(z)|t^n_{\max}u_n|^\alpha|t^n_{\max}v_n|^\beta dz}{\int_\Omega |t^n_{\max}u_n|^\alpha|t^n_{\max}v_n|^\beta dz}\\ &= \frac{(\frac{1}{\delta_n})^N\int_\Omega \chi(z)|\widetilde{U_n}(\frac{z-y_n}{\delta_n})|^\alpha|\widetilde{V_n}(\frac{z-y_n}{\delta_n})|^\beta dz}{(\frac{1}{\delta_n})^N\int_\Omega |\widetilde{U_n}(\frac{z-y_n}{\delta_n})|^\alpha|\widetilde{V_n}(\frac{z-y_n}{\delta_n})|^\beta dz} + o_n(1)\\ &= y_0 + o_n(1) \quad \text{as } n\to\infty,\end{aligned}$$

which leads to a contradiction. Thus, there exists $\eta_0>0$ such that if $(u,v)\in\mathcal{N}_{0,0}$ and $J_{0,0}(u,v)\le\theta_{0,0}(=\theta_{\max}=\frac{1}{N}(S_{\alpha,\beta})^{N/p})+\eta_0$, then $Q(u,v)\in M_{\rho_0/2}$. □

Lemma 4.6. *If $(u,v)\in\mathcal{N}_{\lambda,\mu}$ and $J_{\lambda,\mu}(u,v)\le\theta_{0,0}(=\theta_{\max}=\frac{1}{N}(S_{\alpha,\beta})^{N/p})+\frac{\eta_0}{2}$, then there exists a positive number Λ^* such that $Q(u,v)\in M_{\rho_0/2}$ for $0<\lambda+\mu<\Lambda^*$.*

Proof. We use the similar computation in Lemma 2.5 to obtain that there is a unique positive number

$$t = t(u,v) = \Big(\frac{\|(u,v)\|_E^p}{\int_\Omega f(z)|u|^\alpha |v|^\beta dz}\Big)^{\frac{N-p}{p^2}}$$

such that $(tu,tv) \in \mathcal{N}_{0,0}$. We want to show that there exists $\Lambda > 0$ such that if $0 < \lambda + \mu < \Lambda$, then $t < \xi$ for some constant $\xi > 0$ (independent of u and v).

Indeed, for $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned}
\theta_{\max} + \frac{\eta_0}{2} &\geq J_{\lambda,\mu}(u,v) \\
&= (\frac{1}{p} - \frac{1}{q})\|(u,v)\|_E^p + (\frac{1}{q} - \frac{1}{p^*})\int_\Omega f(z)|u|^\alpha |v|^\beta dz \\
&\geq \frac{q-p}{pq}\|(u,v)\|_E^p.
\end{aligned}$$

Then

$$\|(u,v)\|_E^p \leq \xi_1 = \frac{pq}{q-p}(\theta_{\max} + \frac{\eta_0}{2}). \tag{4.6}$$

Moreover,

$$\begin{aligned}
0 < d_0 \leq \theta_{\lambda,\mu} &\leq J_{\lambda,\mu}(u,v) \\
&= (\frac{1}{p} - \frac{1}{p^*})\|(u,v)\|_E^p - (\frac{1}{q} - \frac{1}{p^*})\int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q)dz \\
&\leq \frac{1}{N}\|(u,v)\|_E^p.
\end{aligned}$$

Then,

$$\|(u,v)\|_E^p \geq \xi_2 = N d_0. \tag{4.7}$$

Furthermore,

$$\begin{aligned}
\int_\Omega f(z)|u|^\alpha |v|^\beta dz &= \|(u,v)\|_E^p - \int_\Omega (\lambda g(z)|u|^q + \mu h(z)|v|^q)dz \\
&\geq \xi_2 - \max\{\|g\|_\infty, \|h\|_\infty\}|\Omega|^{\frac{p^*-q}{p^*}} S^{-\frac{q}{p}}(\lambda+\mu)\xi_1^{\frac{q}{p}}.
\end{aligned}$$

Thus, there exists $\Lambda > 0$ such that for $0 < \lambda + \mu < \Lambda$,

$$\int_\Omega f(z)|u|^\alpha |v|^\beta dz \geq \xi_2 - \max\{\|g\|_\infty, \|h\|_\infty\}|\Omega|^{\frac{p^*-q}{p^*}} S^{-\frac{q}{p}}\Lambda\xi_1^{\frac{q}{p}} > 0. \tag{4.8}$$

Therefore, combining (4.6), (4.7) with (4.8), we have that $t < \xi$ for some constant $\xi > 0$ (independent of u and v) for some $0 < \lambda + \mu < \Lambda$. Then

$$\begin{aligned}
\theta_{\max} + \frac{\eta_0}{2} &\geq J_{\lambda,\mu}(u,v) = \sup_{s\geq 0} J_{\lambda,\mu}(su,sv) \geq J_{\lambda,\mu}(tu,tv) \\
&= \frac{1}{p}\|(tu,tv)\|_E^p - \frac{1}{p^*}\int_\Omega f(z)|tu|^\alpha |tv|^\beta dz \\
&\quad - \frac{1}{q}\int_\Omega (\lambda g(z)|tu|^q + \mu h(z)|tv|^q)dz \\
&\geq J_{0,0}(tu,tv) - \frac{1}{q}\int_\Omega (\lambda g(z)|tu|^q + \mu h(z)|tv|^q)dz,
\end{aligned}$$

which leads to

$$\begin{aligned}
J_{0,0}(tu,tv) &\le \theta_{\max} + \frac{\eta_0}{2} + \frac{1}{q}\int_\Omega (\lambda g(z)|tu|^q + \mu h(z)|tv|^q)dz \\
&\le \theta_{\max} + \frac{\eta_0}{2} + \frac{1}{q}\max\{\|g\|_\infty, \|h\|_\infty\}|\Omega|^{\frac{p^*-q}{p^*}} S^{-\frac{q}{p}}(\lambda+\mu)\|(tu,tv)\|_E^q \\
&< \theta_{\max} + \frac{\eta_0}{2} + \frac{1}{q}\max\{\|g\|_\infty, \|h\|_\infty\}|\Omega|^{\frac{p^*-q}{p^*}} S^{-\frac{q}{p}}\xi^q\xi_1^{\frac{q}{p}}(\lambda+\mu).
\end{aligned}$$

Therefore, there exists $\Lambda^* \in (0,\Lambda)$ such that for $0 < \lambda+\mu < \Lambda^*$,

$$J_{0,0}(tu,tv) \le \theta_{\max} + \eta_0,$$

where $(tu,tv) \in \mathcal{N}_{0,0}$. By Lemma 4.5, we obtain

$$Q(tu,tv) = \frac{\int_{\mathbb{R}^N} \chi(z)|tu|^\alpha |tv|^\beta dz}{\int_{\mathbb{R}^N} |tu|^\alpha |tv|^\beta dz} \in M_{\rho_0/2},$$

or

$$Q(u,v) \in M_{\rho_0/2} \quad \text{for } 0 < \lambda+\mu < \Lambda^*.$$

The proof is complete. □

Next, we show that $\gamma^i_{\lambda,\mu} < \widetilde{\gamma^i_{\lambda,\mu}}$ for any $0 < \lambda+\mu < \Lambda^*$. In fact, from Lemmas 4.1 and 4.2, we obtain that there exists $0 < \epsilon_1 \le \epsilon_0$ such that

$$\gamma^i_{\lambda,\mu} \le J_{\lambda,\mu}(s^i_\epsilon \sqrt[p]{\alpha}u^i_\epsilon, s^i_\epsilon \sqrt[p]{\beta}u^i_\epsilon) < \frac{1}{N}(S_{\alpha,\beta})^{N/p}, \tag{4.9}$$

for any $0 < \epsilon < \epsilon_1$.

By Lemma 4.6, we obtain

$$\widetilde{\gamma^i_{\lambda,\mu}} \ge \theta_{\max} \quad (= \frac{1}{N}(S_{\alpha,\beta})^{N/p}) + \frac{\eta_0}{2}, \tag{4.10}$$

for any $0 < \lambda+\mu < \Lambda^*$.

Thus, for each $1 \le i \le k$, from (4.9) and (4.10), we have that $\gamma^i_{\lambda,\mu} < \widetilde{\gamma^i_{\lambda,\mu}}$ for any $0 < \lambda+\mu < \Lambda^*$. Therefore, $\gamma^i_{\lambda,\mu} = \inf_{u \in D^i_{\lambda,\mu} \cup \partial D^i_{\lambda,\mu}} J_{\lambda,\mu}(u)$ for any $0 < \lambda+\mu < \Lambda^*$.

Ekeland's variational principle combined with the standard computation leads to the following lemma.

Lemma 4.7. *For each $1 \le i \le k$, there is a $(PS)_{\gamma^i_{\lambda,\mu}}$-sequence $\{(u_n,v_n)\} \subset D^i_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$.*

Proof of Theorem 1.2. From Lemma 4.7, for each $1 \le i \le k$, there is a $(PS)_{\gamma^i_{\lambda,\mu}}$-sequence $\{(u_n,v_n)\} \subset D^i_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$. And from (4.9), we have

$$\gamma^i_{\lambda,\mu} < \frac{1}{N}(S_{\alpha,\beta})^{N/p}.$$

Lemma 3.3 implies $J_{\lambda,\mu}$ satisfies the $(PS)_c$-condition for $c \in (-\infty, \frac{1}{N}(S_{\alpha,\beta})^{N/p})$ in E. Thus, we obtain that $J_{\lambda,\mu}$ has at least k critical points in $\mathcal{N}_{\lambda,\mu}$ for any $0 < \lambda+\mu < \Lambda^*$.

Set $u_+ = \max\{u,0\}$ and $v_+ = \max\{v,0\}$, then replace $\int_\Omega f(z)|u|^\alpha|v|^\beta dz$ and $\int_\Omega(\lambda g(z)|u|^q + \mu h(z)|v|^q)dz$ of the functional $J_{\lambda,\mu}$ by the terms $\int_\Omega f(z)u_+^\alpha v_+^\beta dz$ and $\int_\Omega(\lambda g(z)u_+^q + \mu h(z)v_+^q)dz$ respectively. Thereby, we have that (1.1) has k nonnegative solutions. By the maximum principle, we obtain that (1.1) admits k positive solutions. Thus, the proof is complete. □

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