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EXISTENCE OF INFINITELY MANY PERIODIC SUBHARMONIC SOLUTIONS FOR NONLINEAR NON-AUTONOMOUS NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence of an infinite number of subharmonic periodic solutions to a class of second-order neutral nonlinear functional differential equations. Subdifferentiability of lower semicontinuous convex functions $\varphi(x(t), x(t-\tau))$ and the corresponding conjugate functions are constructed. By combining the critical point theory, Z_2-group index theory and operator equation theory, we obtain the infinite number of subharmonic periodic solutions to such system.

1. Introduction

The existence of periodic solutions of differential equations has attracted the attention of mathematicians during the past few decades. Great progress in this area has been made. In particular, the existence of periodic solutions to ordinary differential equations and partial differential equations without delay variant has been extensively studied. Investigating the existence of periodic solutions to functional differential equations is more challenging due to the structure of such differential equations. Different mathematical methods, such as the averaging method [2], the Massera-Yoshizawa theory [8, 17], the Kaplan-York method of coupled systems [5], the Grafton cone mapping method [3], the Nussbaum method of fixed point theory [11], and Mawhin coincidence degree theory [9] have been used to address the existence of solutions to functional differential equations. Critical point theory has rarely been used in the study of the existence of periodic solutions to functional differential equations. Most results in the literature deal with autonomous functional differential equations. Using critical point theory for nonautonomous functional differential equations appear only in a few publication; see [14, 15, 16].

In this article, we use critical point theory and operator equation theory to study the second-order nonlinear nonautonomous neutral functional differential equation

$$\begin{gathered}[p(t)(x'(t)+x'(t-2\tau))]' + f(t, x(t), x(t-\tau), x(t-2\tau)) = 0,\\ x(0) = 0.\end{gathered} \tag{1.1}$$

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We show that (1.1) has an infinite number of subharmonic periodic solutions. The rest of the paper is organized as follows: In Section 2, we will discuss the variational structure and recall some definitions, lemmas and theorems, which will be referred to throughout this article. The weak solution is considered in section 3. The main result of this article, the existence of multiple subharmonic periodic solutions to (1.1), is given in Section 4.

2. Preliminaries

For system (1.1), if x'' exists and $p(t) \in C^1$, we define operator

$$A = \frac{d}{dt}(p(t)\frac{d}{dt}) = p(t)\frac{d^2}{dt^2} + p'(t)\frac{d}{dt}.$$

Thus,

$$\begin{aligned} A(x(t)+x(t-2\tau)) &= [p(t)(x'(t)+x'(t-2\tau))]' \\ &= p(t)[x''(t)+x''(t-2\tau)] + p'(t)[x'(t)+x'(t-2\tau)]. \end{aligned} \tag{2.1}$$

Substituting (2.1) into (1.1) yields

$$\begin{gathered} A(x(t)+x(t-2\tau)) + f(t, x(t), x(t-\tau), x(t-2\tau)) = 0, \\ x(0) = 0. \end{gathered} \tag{2.2}$$

In this article, we use the following assumptions:

- (A1) $f(t, x_1, x_2, x_3) \in C(\mathbb{R}^4, R)$, and $\frac{\partial f(t,x_1,x_2,x_3)}{\partial t} \neq 0$;
- (A2) there exists a continuously differentiable function $F(t, x_1, x_2) \in C^1(\mathbb{R}^3, \mathbb{R})$ such that
$$F_2'(t, x_1, x_2) + F_1'(t, x_2, x_3) = f(t, x_1, x_2, x_3),$$
where
$$F_1'(t, x_2, x_3) = \frac{\partial F(t, x_2, x_3)}{\partial x_2}, \quad F_2'(t, x_1, x_2) = \frac{\partial F(t, x_1, x_2)}{\partial x_2};$$
- (A3) $F(t+\tau, x_1, x_2) = F(t, x_1, x_2)$ for all $x_1, x_2, \in \mathbb{R}$;
- (A4) $p(t) > 0 \in C^1[0, \tau]$ is an τ-periodic function;
- (A5) $F(t, -x_1, -x_2) = F(t, x_1, x_2)$ and $f(t, -x_1, -x_2, -x_3) = -f(t, x_1, x_2, x_3)$.

Fix an integer $\gamma > 1$ and a real number $\tau > 0$. Then define the set

$$\begin{aligned} H_0[0, 2\gamma\tau] = \Big\{ & x(t) \in L^2[0, 2\gamma\tau] : x' \in L^2[0, 2\gamma\tau],\ x(t) \text{ is } 2\gamma\tau\text{-periodic}, \\ & x(0) = 0,\ x(t) \text{ has compact support in } [0, 2\gamma\tau] \Big\}. \end{aligned}$$

Obviously, $H_0[0, 2\gamma\tau]$ is a Sobolev space, with the inner product and norm:

$$\langle x, y \rangle_{H_0[0,2\gamma\tau]} = \int_0^{2\gamma\tau} x'(t) y'(t) dt,$$
$$\|x\|_{H_0[0,2\gamma\tau]} = \Big(\int_0^{2\gamma\tau} |x'|^2 dt \Big)^{1/2} \quad \forall x, y \in H_0[0, 2\gamma\tau].$$

A function $x(t) \in H_0[0, 2\gamma\tau]$ can be written as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \big(a_k \cos \frac{k\pi}{\gamma\tau} t + b_k \sin \frac{k\pi}{\gamma\tau} t \big).$$

We define the functional

$$I(x)=\int_0^{2\gamma\tau}[p(t)x'(t)x'(t-\tau)-F(t,x(t),x(t-\tau))]dt \tag{2.3}$$

on $H_0[0,2\gamma\tau]$. It follows that for all $x,y\in H_0[0,2\gamma\tau]$ and $\varepsilon>0$, we have

$$\begin{aligned}
I(x+\varepsilon y)=I(x)+\varepsilon\Big(\int_0^{2\gamma\tau}&[p(t)(x'(t)y'(t-\tau)+x'(t-\tau)y'(t))\\
&-(F(t,x(t)+\varepsilon y(t),x(t-\tau)+\varepsilon y(t-\tau))-F(t,x(t),x(t-\tau)))]dt\Big)\\
&+\varepsilon^2\int_0^{2\gamma\tau}[p(t)y'(t)y'(t-\tau))]dt.
\end{aligned}$$

Obviously,

$$\begin{aligned}
\langle I'(x),y\rangle=\int_0^{2\gamma\tau}&[p(t)(x'(t)y'(t-\tau)+x'(t-\tau)y'(t))\\
&-F_1'(t,x(t),x(t-\tau))y(t)-F_2'(t,x(t),x(t-\tau))y(t-\tau)]dt,
\end{aligned} \tag{2.4}$$

where $I'(x)$ denotes the Frechet differential of the function $I(x)$. It follows from the periodicity of $F(t,u_1,u_2)$, $x(t)$ and $y(t)$ that

$$\begin{aligned}
&\int_0^{2\gamma\tau}\frac{d}{dt}(p(t)\frac{dx(t-\tau)}{dt})y(t)dt\\
&=\int_0^{2\gamma\tau}p(t)x''(t-\tau)y(t)dt+\int_0^{2\gamma\tau}p'(t)x'(t-\tau)y(t)dt\\
&=p(t)x'(t-\tau)y(t)\mid_0^{2\gamma\tau}-\int_0^{2\gamma\tau}p(t)x'(t-\tau)y'(t)dt\\
&=-\int_0^{2\gamma\tau}p(t)x'(t-\tau)y'(t)dt,
\end{aligned}$$

and

$$\int_0^{2\gamma\tau}(p(t)x'(t+\tau))'y(t)dt=-\int_0^{2\gamma\tau}p(t)x'(t)y'(t-\tau)dt.$$

In a similar way, we obtain

$$\begin{aligned}
\int_0^{2\gamma\tau}F_2'(t,x(t),x(t-\tau))y(t-\tau)dt&=\int_{-\tau}^{(2\gamma-1)\tau}F_2'(t+\tau,x(t+\tau),x(t))y(t)dt\\
&=\int_0^{2\gamma\tau}F_2'(t,x(t+\tau),x(t))y(t)dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
\langle I'(x),y\rangle=\int_0^{2\gamma\tau}&\Big[-(p(t)(x'(t-\tau)+x'(t+\tau)))'\\
&-F_1'(t,x(t),x(t-\tau))-F_2'(t,x(t+\tau),x(t))\Big]y(t)dt.
\end{aligned}$$

The above equation implies that the corresponding Euler equation of the functional $I(x)$ is

$$(p(t)(x'(t-\tau)+x'(t+\tau))'+[F_1'(t,x(t),x(t-\tau))+F_2'(t,x(t+\tau),x(t))]=0. \tag{2.5}$$

Since (2.5) is equivalent to (1.1) on $H_0[0,2\gamma\tau]$, system (1.1) is the Euler equation of functional $I(x)$. Therefore, it is possible to obtain the $2\gamma\tau$-weakly periodic solutions to system (1.1) by seeking the critical points of functional $I(x)$. Due to the fact that functional $I(x)$ has no supremum or infimum, we can use operator equation theory here and there is no need to seek the critical points of $I(x)$.

Next, we present some preliminaries to be used throughout the whole article. Here we suppose that E is a real Banach space with norm $\|\cdot\|$.

Definition 2.1. A "critical point" of $g \in C^1(E,R)$ is a point $x^* \in E$ such that $g'(x^*) = 0$. A "critical value" of g is a number c such that $g(x^*) = c$ for some critical point x^*. The set $K = \{x \in E : g'(x) = 0\}$ is the "critical set" of "g". We use K_c to denote the set $\{x \in E : g'(x) = 0,\ g(x) = c\}$. The "critical Level" set g_c of g is defined by $g_c = \{x \in E : g(x) \le c\}$.

Definition 2.2. Let $g \in C^1(E,R)$. We say that g satisfies the "Palais-Smale" condition if every sequence $\{x_n\} \subset E$ such that $\{g(x_n)\}$ is bounded and $g'(x_n) \to \theta$ as $n \to \infty$ has a convergent subsequence.

We say that a closed symmetric set $A \subset E$ satisfies property $\Re$ if, for some $n \in Z^+$, there exists an odd continuous function $\varphi : A \to \mathbb{R}^n \setminus \{\theta\}$. Let $N_A \subset Z$ be defined as follows: $n \in \mathbb{N}_A$ if and only if A satisfies property $\Re$ with this n.

Definition 2.3. Let E be a real Banach space, and

$$\Sigma = \big\{A \subset E \setminus \{\theta\} : A \text{ is closed and symmetric}\big\}.$$

Define $\gamma : \Sigma \to Z^+ \cup \{+\infty\}$ as

$$\gamma(A) = \begin{cases} \min N_A & N_A \neq \emptyset \\ 0 & \text{if } A = \emptyset \\ +\infty & \text{if } A \neq \emptyset, \text{ but } N_A = \emptyset. \end{cases}$$

We say that "γ is the genus of Σ". Denote $i_1(g) := \lim_{a\to -0} \gamma(g_a)$ and $i_2(g) := \lim_{a\to -\infty} \gamma(g_a)$.

Lemma 2.4 ([1]). *Let E be a real Banach space, and $I(\cdot) \in C^1(E,R)$ be an even functional that satisfies the Palais-Smale condition. If*

(1) *there exist constants $\rho > 0$, $a > 0$ and a finite dimensional subspace X of E, such that $I(x)|_{X^\perp \cap S_\rho} \ge a$, where $S_\rho = \{x \in E : \|x\|_E = \rho\}$;*
(2) *there exist subspaces $\widehat{X_j}$ of E, $j = \dim(\widehat{X_j})$, and a sequence $\{r_j : r_j > 0\}$, such that $I(x) \le 0$ for $x \in \widehat{X_j} \setminus B_{r_j}$, $(j = 1, 2, \dots)$;*

Then, I possesses an unbounded sequence of critical values.

Next, we consider the subdifferentiability and the conjugate function of the lower semicontinuous convex function $\varphi(x(t), x(t-\tau))$, which has been investigated in [16]. Suppose the Banach space X is a space of all given $n \times \tau$-periodic functions in t, where $n \in \mathbb{N}$ is a positive integer. We use $\overline{R}$ to denote $R \cup \{+\infty\}$, and let $\varphi : X^2 \to \overline{R}$ be a lower semicontinuous convex function. Since φ is not always differentiable in general, we redefine the "derivative" as follows:

Definition 2.5 ([16]). Suppose that $(x_1^*, x_2^*) \in X^* \times X^*$. We say that (x_1^*, x_2^*) is a sub-gradient of φ at point $(x_0(t), (x_0(t-\tau)) \in X$ if

$$\varphi(x_0(t), x_0(t-\tau)) + \langle x_1^*, x(t) - x_0(t)\rangle + \langle x_2^*, x(t-\tau) - x_0(t-\tau)\rangle \le \varphi(x(t), x(t-\tau)).$$

For all $x_0(t) \in X$, the set of all sub-gradients of φ at point $(x_0(t), x_0(t-\tau))$ is called the subdifferential of φ at point $(x_0(t), x_0(t-\tau))$. We use $\partial\varphi(x_0(t), x_0(t-\tau))$ to denote such a set.

Using the definition of subdifferentiability of the function φ, we define the corresponding conjugate function φ^* as

$$\varphi^*(x_1^*, x_2^*) = \sup\{< x_1^*, x(t)\rangle + \langle x_2^*, x(t-\tau)\rangle - \varphi(x(t), x(t-\tau))\},$$

where $\langle\cdot\rangle$ denotes the duality relation of X^* and X. Then there hold the following propositions:

Proposition 2.6 ([16]). *The function φ^* is a lower semicontinuous convex (φ^* may have functional value $+\infty$, but do not have functional value $-\infty$).*

Proposition 2.7 ([16]). *If $\varphi \le \psi$, then $\varphi^* \ge \psi^*$.*

Proposition 2.8 (Yang inequality [16]).

$$\varphi(x(t), x(t-\tau)) + \varphi^*(x_1^*, x_2^*) \ge \langle x_1^*, x(t)\rangle + \langle x_2^*, x(t-\tau)\rangle.$$

Proposition 2.9 ([16]).

$$\varphi(x(t), x(t-\tau)) + \varphi^*(x_1^*, x_2^*) = \langle x_1^*, x(t)\rangle + \langle x_2^*, x(t-\tau)\rangle$$

if and only if $(x_1^*, x_2^*) \in \partial\varphi(x(t), x(t-\tau))$.

Proposition 2.10 ([16]). *The function φ^* is not always equal to $+\infty$.*

Theorem 2.11 ([16]). *If φ is a lower semicontinuous convex function that does not always equal $+\infty$, then $\varphi^{**} = \varphi$.*

Corollary 2.12 ([16]). *Let φ be a lower semicontinuous convex function that does not always equal $+\infty$. Then $(x_1^*, x_2^*) \in \partial\varphi(x(t), x(t-\tau))$ if and only if*

$$(x(t), x(t-\tau)) \in \partial\varphi^*(x_1^*, x_2^*).$$

3. Weak solutions of the operator equation (2.2)

It follows from (2.4) that

$$\begin{aligned}
\langle u(t), A(\omega(t))\rangle &= \int_0^{2\gamma\tau} u(t)(p(t)\omega'(t))'dt \\
&= u(t)(p(t)\omega'(t))|_0^{2\gamma\tau} - \int_0^{2\gamma\tau} p(t)\omega'(t)u'(t)dt \\
&= -\int_0^{2\gamma\tau} p(t)u'(t)d\omega(t) \\
&= -p(t)u'(t)\omega(t)|_0^{2\gamma\tau} + \int_0^{2\gamma\tau} \omega(t)d(p(t)u'(t)) \\
&= \langle Au(t), \omega(t)\rangle,
\end{aligned}$$

$$\langle u(t), A(\omega(t-\tau))\rangle = \langle Au(t+\tau), \omega(t)\rangle,$$
$$\langle u(t-\tau), A(\omega(t))\rangle = \langle A(u(t-\tau)), \omega(t)\rangle.$$

The above discussion leads us to the following definition.

Definition 3.1. For $u \in L^p[0, 2\gamma\tau]$, we say that u is a weak solution of the operator equation (2.2), if

$$\begin{aligned}
&\langle u(t), A(\omega(t-\tau))\rangle + \langle u(t-\tau), A(\omega(t))\rangle \\
&+ \langle \omega(t), F_1'(t, u(t), u(t-\tau))\rangle + \langle \omega(t-\tau), F_2'(t, u(t), u(t-\tau))\rangle = 0,
\end{aligned}$$

for all $\omega(t) \in D(A) \cap L^p[0, 2\gamma\tau]$, where

$$\langle u(t), \upsilon(t)\rangle = \int_0^{2\gamma\tau} u(t)\upsilon(t)dt,$$

$u(t) \in L^p[0, 2\gamma\tau], \upsilon(t) \in L^q[0, 2\gamma\tau]$ for $2 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Here, our aim is to define the conjugate function of $F(t, x(t), x(t-\tau))$ using the definition of subdifferentiability of lower semicontinuous convex functions, and the dual variational structure. First, we assume that $F(t, x(t), x(t-\tau))$ satisfies the following conditions:

(A6) $u = (u_1, u_2) \to F(t, u_1, u_2)$ is a continuously differentiable and strictly convex function, and satisfies

$$F(t, 0, 0) = 0, \quad F_1'(t, 0, 0) = F_2'(t, 0, 0) = 0 \quad \forall t \in [0, 2\gamma\tau];$$

(A7) for $\alpha_2 = 1/p$, there exist constants $M, C > 0$, such that when $|u| = \sqrt{u_1^2 + u_2^2} \geq C$, we have

$$F(t, u_1, u_2) \leq \alpha_2[F_1'(t, u_1, u_2)u_1 + F_2'(t, u_1, u_2)u_2],$$
$$F(t, u_1, u_2) \leq M|u|^{1/\alpha_2};$$

(A8)

$$\lim_{|u|\to 0} \frac{F(t, u_1, u_2)}{|u|^2} = 0.$$

The conjugate function of $F(t, x(t), x(t-\tau))$ is then obtained as

$$\begin{aligned} H(t, \omega(t), \omega(t-2\tau)) = \sup_{x(t)\in L^p[0,2\gamma\tau]} \Big\{ &\langle \omega(t), x(t)\rangle + \langle \omega(t-2\tau), x(t-\tau)\rangle \\ &- F(t, x(t), x(t-\tau))\Big\}, \quad \forall t \in [0, 2\gamma\tau]. \end{aligned}$$

The above discussion indicates that H is a continuously differentiable and strictly convex function. By duality principle (Corollary 2.12), we have

$$(\omega(t), \omega(t-2\tau)) = (F_1'(t, x(t), x(t-\tau)), F_2'(t, x(t), x(t-\tau)))$$

if and only if

$$(H_1'(t, \omega(t), \omega(t-2\tau)), H_2'(t, \omega(t), \omega(t-2\tau))) = (x(t), x(t-\tau)), \tag{3.1}$$

where

$$H_1'(t, \omega(t), \omega(t-2\tau)) = \frac{\partial H(t, \omega(t), \omega(t-2\tau))}{\partial \omega(t)},$$

and

$$H_2'(t, \omega(t), \omega(t-2\tau)) = \frac{\partial H(t, \omega(t), \omega(t-2\tau))}{\partial \omega(t-2\tau)}.$$

Example 3.2. *Let* $F(x(t), x(t-\tau)) = \frac{1}{p}\big(\sqrt{x^2(t) + x^2(t-\tau)}\big)^p$, *then*

$$H(\omega(t), \omega(t-2\tau)) = \frac{1}{q}\Big(\sqrt{\omega^2(t) + \omega^2(t-2\tau)}\Big)^q$$

Proof.

$$\begin{aligned} H(\omega(t),\omega(t-2\tau)) &= \sup_{x(t)\in L^p[0,2\gamma\tau]}\Big\{\langle\omega(t),x(t)\rangle \\ &\quad + \langle\omega(t-2\tau),x(t-\tau)\rangle - \frac{1}{p}\Big(\sqrt{x^2(t)+x^2(t-\tau)}\Big)^p\Big\} \\ &= \sup_{\lambda>0}\Big\{\sqrt{\omega^2(t)+\omega^2(t-2\tau)}\ \ \lambda - \frac{1}{p}\lambda^p\Big\} \\ &= \frac{1}{q}\Big(\sqrt{\omega^2(t)+\omega^2(t-2\tau)}\Big)^q. \end{aligned}$$

□

We use $R(A)$ to denote the value field of operator A. Then, it is obvious that $R(A)$ is a closed set. Suppose P is the orthogonal projection operator of $R(A)$ and $\widehat{K} = A^{-1}P$. Then it is easy to see that $\widehat{K}$ maps a continuous operator into a compact operator of $L^q[0,2\gamma\tau] \to L^q[0,2\gamma\tau]$. Denote

$$\begin{aligned} E = \Big\{&(\upsilon(t),\upsilon(t-2\tau)) \in L^q[0,2\gamma\tau]\times L^q[0,2\gamma\tau] : \upsilon(0)=0, \langle\phi(t-2\tau),\upsilon(t)\rangle \\ &= \langle\phi(t-2\tau),\upsilon(t-2\tau)\rangle = \langle\phi(t-\tau),\upsilon(t)\rangle \\ &= \langle\phi(t-2\tau),\upsilon(t-2\tau)\rangle = 0,\ \forall\phi(t)\in\Re(A)\cap L^p[0,2\gamma\tau], \phi(0)=0\Big\}, \end{aligned}$$

where $\Re(A) = \{u \in D(A) : A(u(t)+u(t-2\tau)) = 0\}$.

Remark 3.3. Actually, for all $x(0) = 0$, $x(t) \in L^p[0,2\gamma\tau]$ or $x(t) \in L^q[0,2\gamma\tau]$, $x(t)$ can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{\gamma\tau}t + b_k\sin\frac{k\pi}{\gamma\tau}t).$$

Hence, $\langle\upsilon(t),\phi(t)\rangle = 0$ if and only if

$$\langle\phi(t-2\tau),\upsilon(t)\rangle = \langle\phi(t-2\tau),\upsilon(t-2\tau)\rangle = \langle\phi(t-\tau),\upsilon(t)\rangle = \langle\phi(t-2\tau),\upsilon(t-2\tau)\rangle = 0.$$

Thus E can also be written as

$$\begin{aligned} E = \Big\{&(\upsilon(t),\upsilon(t-2\tau)) \in L^q[0,2\gamma\tau]\times L^q[0,2\gamma\tau] : \upsilon(0)=0, \langle\phi(t),\upsilon(t)\rangle = 0, \\ &\forall\phi(t)\in\Re(A)\cap L^p[0,2\gamma\tau], \phi(0)=0\Big\}. \end{aligned}$$

From

$$\begin{aligned} &A(x(t)+x(t-2\tau)) \\ &= -\sum_{k=1}^{\infty}p(t)(\frac{k\pi}{\gamma\tau})^2[a_k(\cos\frac{k\pi}{\gamma\tau}t+\cos\frac{k\pi}{\gamma\tau}(t-2\tau)) + b_k(\sin\frac{k\pi}{\gamma\tau}t+\sin\frac{k\pi}{\gamma\tau}(t-2\tau))] \\ &\quad + \sum_{k=1}^{\infty}p'(t)\frac{k\pi}{\gamma\tau}[-a_k(\sin\frac{k\pi}{\gamma\tau}t+\sin\frac{k\pi}{\gamma\tau}(t-2\tau)) + b_k(\cos\frac{k\pi}{\gamma\tau}t+\cos\frac{k\pi}{\gamma\tau}(t-2\tau))] \\ &= -\sum_{k=1}^{\infty}2p(t)(\frac{k\pi}{\gamma\tau})^2\cos\frac{k\pi}{\gamma}[(a_k\cos\frac{k\pi}{\gamma}+b_k\sin\frac{k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}t \\ &\quad + (a_k\sin\frac{k\pi}{\gamma}-b_k\cos\frac{k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}t] \end{aligned}$$

$$+\sum_{k=1}^{\infty} 2p'(t)\frac{k\pi}{\gamma\tau}\cos\frac{k\pi}{\gamma}[-a_k\sin\frac{k\pi}{\gamma\tau}(t-\tau)+b_k\cos\frac{k\pi}{\gamma\tau}(t-\tau)]$$

it follows that

$$A(x(t)+x(t-2\tau))=0 \Longleftrightarrow \cos\frac{k\pi}{\gamma}=0.$$

From the above discussion, it is easy to see that

$$\Re(A)=\big\{x(t)\in L^p[0,2\gamma\tau]: x(t)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{(\frac{1}{2}+n)\pi}{\tau}t+b_n\sin\frac{(\frac{1}{2}+n)\pi}{\tau}t)\big\}.$$

Here, we need that $\{(\upsilon(t),\upsilon(t-2\tau)),(\chi(t-\tau),\chi(t-2\tau))\}$ satisfies

$$\begin{gathered}\chi(t-\tau)=\widehat{K}(\upsilon(t-2\tau))+H_1'(t,\upsilon(t),\upsilon(t-2\tau)),\\ \chi(t-2\tau)=\widehat{K}(\upsilon(t))+H_2'(t,\upsilon(t),\upsilon(t-2\tau)),\end{gathered}\tag{3.2}$$

where $(\upsilon(t),\upsilon(t-2\tau))\in E$, $\chi(t)\in\Re(A)\cap L^p[0,2\gamma\tau]$; that is, $(\chi(t-\tau),\chi(t-2\tau))\in E^{\perp}$.

Suppose that $\{(\upsilon(t),\upsilon(t-2\tau)),(\chi(t-\tau),\chi(t-2\tau))\}$ is a solution of (3.2). We let $u(t)=H_1'(t,\upsilon(t),\upsilon(t-2\tau)), u(t-\tau)=H_2'(t,\upsilon(t),\upsilon(t-2\tau))$. It then follows from duality principle and (3.2) that

$$\begin{aligned}&\langle u(t-\tau),A(z(t))\rangle+\langle u(t),A(z(t-\tau))\rangle\\ &+\langle z(t),F_{u_1}'(t,u(t),u(t-\tau))\rangle+\langle z(t-\tau),F_{u_2}'(t,u(t),u(t-\tau))\rangle\\ &=\langle H_2'(t,\upsilon(t),\upsilon(t-2\tau)),A(z(t))\rangle+\langle H_1'(t,\upsilon(t),\upsilon(t-2\tau)),A(z(t-\tau))>\\ &\quad+\langle z(t),\upsilon(t)\rangle+\langle z(t-\tau),\upsilon(t-2\tau)\rangle\\ &=\langle\chi(t-2\tau)-\widehat{K}(\upsilon(t)),A(z(t))\rangle+\langle\chi(t-\tau)-\widehat{K}(\upsilon(t-2\tau)),A(z(t-\tau))\rangle\\ &\quad+\langle z(t),\upsilon(t)\rangle+\langle z(t),\upsilon(t-\tau)\rangle\\ &\quad-\langle\upsilon(t),z(t)\rangle-\langle\upsilon(t-\tau),z(t)\rangle+\langle z(t),\upsilon(t)\rangle+\langle z(t),\upsilon(t-\tau)\rangle\\ &=0,\quad\forall z(t)\in D(A)\cap L^p[0,2\gamma\tau].\end{aligned}$$

The above equation indicates that $u(t)$ is a weak solution of (1.1).

4. Main results

In this section, we seek the solutions of operator equation (3.2) by using critical point theory. Let $\upsilon=(\upsilon(t),\upsilon(t-2\tau))$, and

$$K\begin{pmatrix}\upsilon(t)\\ \upsilon(t-2\tau)\end{pmatrix}=\begin{pmatrix}0&\widehat{K}\\ \widehat{K}&0\end{pmatrix}\begin{pmatrix}\upsilon(t)\\ \upsilon(t-2\tau)\end{pmatrix}=\begin{pmatrix}\widehat{K}\upsilon(t-2\tau)\\ \widehat{K}\upsilon(t)\end{pmatrix}.$$

It is easy to see that $\langle K(\upsilon),\psi\rangle=\langle\upsilon,K(\psi)\rangle=\langle\overline{K}\upsilon(t-2\tau),\psi(t)\rangle+\langle\overline{K}\upsilon(t),\psi(t-2\tau)\rangle$, where $\psi=(\psi(t),\psi(t-2\tau))$. The previous equation implies that the operator K is a symmetric operator.

Remark 4.1. Let $x(t)=\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{\gamma\tau}t+b_k\sin\frac{k\pi}{\gamma\tau}t)$. Then

$$\begin{aligned}&Ax(t)\\ &=-p(t)\sum_{k=1}^{\infty}(\frac{k\pi}{\gamma\tau})^2[a_k\cos\frac{k\pi}{\gamma\tau}t+b_k\sin\frac{k\pi}{\gamma\tau}t]+p'(t)\sum_{k=1}^{\infty}\frac{k\pi}{\gamma\tau}[-a_k\sin\frac{k\pi}{\gamma\tau}t+b_k\cos\frac{k\pi}{\gamma\tau}t]\end{aligned}$$

$$= -\sum_{k=1}^{\infty}[(\frac{k\pi}{\gamma\tau})^2 p(t)a_k - \frac{k\pi}{\gamma\tau}p'(t)b_k]\cos\frac{k\pi}{\gamma\tau}t + [(\frac{k\pi}{\gamma\tau})^2 p(t)b_k + \frac{k\pi}{\gamma\tau}p'(t)a_k]\sin\frac{k\pi}{\gamma\tau},$$

for all $t \in [0, 2\gamma\tau]$. For example, if $x(t) = \cos\frac{k\pi}{\gamma\tau}t$, we have

$$A(\cos\frac{k\pi}{\gamma\tau}t) = -(\frac{k\pi}{\gamma\tau})^2 p(t)\cos\frac{k\pi}{\gamma\tau}t - \frac{k\pi}{\gamma\tau}p'(t)\sin\frac{k\pi}{\gamma\tau}t.$$

Letting $y(t) = -(\frac{k\pi}{\gamma\tau})^2 p(t)\cos\frac{k\pi}{\gamma\tau}t - \frac{k\pi}{\gamma\tau}p'(t)\sin\frac{k\pi}{\gamma\tau}t$, from $\widehat{K}y(t) = \widehat{K}A(\cos\frac{k\pi}{\gamma\tau}t) = \cos\frac{k\pi}{\gamma\tau}t$ and the linearity of the operator $\widehat{K}$ in x, it follows that

$$-(\frac{k\pi}{\gamma\tau})^2 p(t)\widehat{K}(\cos\frac{k\pi}{\gamma\tau}t) - \frac{k\pi}{\gamma\tau}p'(t)\widehat{K}(\sin\frac{k\pi}{\gamma\tau}t) = \cos\frac{k\pi}{\gamma\tau}t. \tag{4.1}$$

Similarly, we have

$$A(\sin\frac{k\pi}{\gamma\tau}t) = -(\frac{k\pi}{\gamma\tau})^2 p(t)\sin\frac{k\pi}{\gamma\tau}t + \frac{k\pi}{\gamma\tau}p'(t)\cos\frac{k\pi}{\gamma\tau}t$$

and

$$-(\frac{k\pi}{\gamma\tau})^2 p(t)\widehat{K}(\sin\frac{k\pi}{\gamma\tau}t) + \frac{k\pi}{\gamma\tau}p'(t)\widehat{K}(\cos\frac{k\pi}{\gamma\tau}t) = \sin\frac{k\pi}{\gamma\tau}t. \tag{4.2}$$

By (4.1) and (4.2), we obtain

$$\begin{aligned}
&\widehat{K}(\cos\frac{k\pi}{\gamma\tau}t)\\
&= \frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[p'(t)\sin\frac{k\pi}{\gamma\tau}t - \frac{k\pi}{\gamma\tau}p(t)\cos\frac{k\pi}{\gamma\tau}t]\\
&= \frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[(p'(t)\cos\frac{2k\pi}{\gamma} + \frac{k\pi}{\gamma\tau}p(t)\sin\frac{2k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}(t-2\tau)\\
&\quad + (p'(t)\sin\frac{2k\pi}{\gamma} - \frac{k\pi}{\gamma\tau}p(t)\cos\frac{2k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}(t-2\tau)],
\end{aligned}$$

$$\begin{aligned}
&\widehat{K}(\sin\frac{k\pi}{\gamma\tau}t)\\
&= -\frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[p'(t)\cos\frac{k\pi}{\gamma\tau}t + \frac{k\pi}{\gamma\tau}p(t)\sin\frac{k\pi}{\gamma\tau}t]\\
&= -\frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[(p'(t)\cos\frac{2k\pi}{\gamma} + \frac{k\pi}{\gamma\tau}p(t)\sin\frac{2k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}(t-2\tau)\\
&\quad + (-p'(t)\sin\frac{2k\pi}{\gamma} + \frac{k\pi}{\gamma\tau}p(t)\cos\frac{2k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}(t-2\tau)],
\end{aligned}$$

$$\begin{aligned}
&\widehat{K}(\cos\frac{k\pi}{\gamma\tau}(t-2\tau))\\
&= \frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[p'(t)\sin\frac{k\pi}{\gamma\tau}(t-2\tau) - \frac{k\pi}{\gamma\tau}p(t)\cos\frac{k\pi}{\gamma\tau}(t-2\tau)]\\
&= \frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[(p'(t)\cos\frac{2k\pi}{\gamma} - \frac{k\pi}{\gamma\tau}p(t)\sin\frac{2k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}t\\
&\quad + (-p'(t)\sin\frac{2k\pi}{\gamma} - \frac{k\pi}{\gamma\tau}p(t)\cos\frac{2k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}t],
\end{aligned}$$

and

$$
\begin{aligned}
&\widehat{K}(\sin\frac{k\pi}{\gamma\tau}(t-2\tau)) \\
&= -\frac{1}{\frac{k\pi}{\gamma\tau}(p'^2+(\frac{k\pi}{\gamma\tau})^3p^2(t)}[p'(t)\cos\frac{k\pi}{\gamma\tau}(t-2\tau)+\frac{k\pi}{\gamma\tau}p(t)\sin\frac{k\pi}{\gamma\tau}(t-2\tau)] \\
&= -\frac{1}{\frac{k\pi}{\gamma\tau}(p'^2+(\frac{k\pi}{\gamma\tau})^3p^2(t)}[(p'(t)\cos\frac{2k\pi}{\gamma}-\frac{k\pi}{\gamma\tau}p(t)\sin\frac{2k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}t \\
&\quad +(p'(t)\sin\frac{2k\pi}{\gamma}+\frac{k\pi}{\gamma\tau}p(t)\cos\frac{2k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}t].
\end{aligned}
$$

Theorem 4.2. *Under the assumptions* (A1)–(A8)*, problem* (1.1) *has an infinite number of nontrivial* $2\gamma\tau$*-periodic solutions.*

The solutions of (3.2) can be obtained through seeking critical points of the functional $J(\upsilon)$, which is defined by

$$
\begin{aligned}
J(\upsilon) &= \frac{1}{2}\langle(\upsilon),\upsilon\rangle+\int_0^{2\gamma\tau}H(t,\upsilon)dt \\
&= \frac{1}{2}\langle\widehat{K}\upsilon(t-2\tau),\upsilon(t)\rangle+\frac{1}{2}\langle\widehat{K}\upsilon(t),\upsilon(t-2\tau)\rangle \\
&\quad +\int_0^{2\gamma\tau}H(t,\upsilon(t),\upsilon(t-2\tau))dt \quad \forall(\upsilon(t),\upsilon(t-2\tau))\in E.
\end{aligned}
\tag{4.3}
$$

Here, J can be considered as the restriction to E of function $\widehat{J}$ defined on $L^q[0,2\gamma\tau]\times L^q[0,2\gamma\tau]$, since both functions share same components on E. Also, we have

$$\widehat{J}'(\upsilon)=K(\upsilon)+H'(\upsilon).$$

We note that

$$\langle\widehat{J}'(\upsilon)-J'(\upsilon),z\rangle=0,\quad \forall\upsilon\in E,\ z=(z(t),z(t-2\tau))\in E.$$

There exist $\chi(t)\in\Re(A)$ and $\chi_\upsilon=(\chi_\upsilon(t-2\tau),\chi_\upsilon(t-\tau))\in E^\perp$ such that

$$\widehat{J}'(\upsilon)-J'(\upsilon)=\chi_\upsilon.$$

Thus, if υ^* is a critical point of $J'(\upsilon^*)=0$ on E, then there exists $\chi^*_{\upsilon^*}=(\chi^*_{\upsilon^*}(t-\tau),\chi^*_{\upsilon^*}(t-2\tau))\in E^\perp$, such that

$$K(\upsilon^*)+H'(\upsilon^*)=\chi^*_{\upsilon^*}.$$

The above discussion indicates that $\{\upsilon^*,\chi^*_{\upsilon^*}\}$ is a solution of (3.2). That is to say, $\{(\upsilon^*(t),\upsilon^*(t-2\tau)),(\chi^*_{\upsilon^*}(t-\tau),\chi^*_{\upsilon^*}(t-2\tau))\}$ is a solution of equation (3.2).

Lemma 4.3. *The following two conditions are equivalent:*

(1) $F(t,u_1,u_2)\le\alpha_2[F'_1(t,u_1,u_2)u_1+F'_2(t,u_1,u_2)u_2]$ *for all* $t\in[0,2\gamma\tau]$*, when* $|u|=\sqrt{u_1^2+u_2^2}\ge C$.

(2) $F(t,\beta u_1,\beta u_2)\ge\beta^{1/\alpha_2}F(t,u_1,u_2)>0$ *for all* $\beta\ge1$, $t\in[0,2\gamma\tau],|u|\ge C$.

Proof. For all $u=(u_1,u_2)$ with $|u|\ge C$, let $\Phi(\beta)=F(t,\beta u_1,\beta u_2)$, and $\Psi(\beta)=\beta^{1/\alpha_2}F(t,u_1,u_2)$.

(2) $\Rightarrow$ (1): By $\Phi(\beta)\ge\Psi(\beta),\forall\beta\ge1$ and $\Phi(1)=\Psi(1)$, it is easy to see that $\Phi'(1)\ge\Psi'(1)$. In other words,

$$F'_1(t,u_1,u_2)u_1+F'_2(t,u_1,u_2)u_2\ge\frac{1}{\alpha_2}F(t,u_1,u_2).$$

(1) $\Rightarrow$ (2): Since

$$\begin{aligned}\Phi'(\beta) &= F_1'(t,\beta u_1,\beta u_2)u_1 + F_2'(t,\beta u_1,\beta u_2)u_2 \\ &= \frac{1}{\beta}[F_1'(t,\beta u_1,\beta u_2)\beta u_1 + F_2'(t,\beta u_1,\beta u_2)\beta u_2] \geq \frac{1}{\alpha_2\beta}\Phi(\beta),\end{aligned}$$

we have

$$F(t,\beta u_1,\beta u_2) \geq \beta^{1/\alpha_2}F(t,u_1,u_2) > 0, \quad \forall \beta \geq 1,\ t \in [0,2\gamma\tau].$$

□

Lemma 4.4. *Suppose that $F(t,u_1,u_2)$ satisfies* (A6)–(A7). *Then there exist constants $m > 0$ and $M > 0$, such that*

$$\begin{aligned}&F(t,u_1,u_2) \geq m(\sqrt{u_1^2+u_2^2})^{1/\alpha_2}, \quad \forall t \in [0,2\gamma\tau],\ \text{when } |u| \geq C, \\ &|F'(t,u_1,u_2)| \leq (2^{1/\alpha_2}M - m)(\sqrt{u_1^2+u_2^2})^{\frac{1}{\alpha_2}-1},\ \text{when } |u| \geq C,\end{aligned}$$

where $|F'(t,u_1,u_2)| = \sqrt{|F_1'(t,u_1,u_2)|^2 + |F_2'(t,u_1,u_2)|^2}$.

Proof. Let

$$m = \min_{(u_1,u_2)\in\partial B_C} \frac{F(t,u_1,u_2)}{C^{1/\alpha_2}},$$

where B_C is a ball of radius $C > 0$ centered at the point $\theta = (0,0)$. It follows from (A6) that $m > 0$. Lemma 4.3 and (A7) yield

$$\begin{aligned}F(t,u_1,u_2) &\geq F\big(t, \frac{Cu_1}{\sqrt{u_1^2+u_2^2}}, \frac{Cu_2}{\sqrt{u_1^2+u_2^2}}\big)\Big(\frac{\sqrt{u_1^2+u_2^2}}{C}\Big)^{1/\alpha_2} \\ &\geq m\Big(\sqrt{u_1^2+u_2^2}\Big)^{1/\alpha_2}.\end{aligned}$$

The convexity of function F indicates that

$$F(t,u_1,u_2) + F_1'(t,u_1,u_2)(z_1-u_1) + F_2'(t,u_1,u_2)(z_2-u_2) \leq F(t,z_1,z_2).$$

Let $z = (z_1,z_2)$ run all over the ball $B_{|u|}(z)$ of radius $|u|$ centered at the point z, and choose the maximum of $F_1'(t,u_1,u_2)(z_1-u_1) + F_2'(t,u_1,u_2)(z_2-u_2)$. Then we obtain the inequality

$$|F'(t,u_1,u_2)|\sqrt{u_1^2+u_2^2} \leq M\Big(\sqrt{z_1^2+z_2^2}\Big)^{\frac{1}{\alpha_2}} - m\Big(\sqrt{u_1^2+u_2^2}\Big)^{1/\alpha_2}.$$

Since $z \leq 2|u|$, we know that

$$|F'(t,u_1,u_2)| \leq (2^{1/\alpha_2}M - m)\Big(\sqrt{u_1^2+u_2^2}\Big)^{\frac{1}{\alpha_2}-1}.$$

□

Lemma 4.5. *$H \in C^1(\mathbb{R}^3, R)$ is a strictly convex function and satisfies*

$$H_1'(t,0,0) = H_2'(t,0,0) = 0, \quad H(t,0,0) = 0, \quad \forall t \in [0,2\gamma\tau];$$

$$\frac{C_{\alpha_2}}{M}|\omega|^{\frac{1}{1-\alpha_2}} - C_1 \leq H(t,\omega(t),\omega(t-2\tau)) \leq \frac{C_{\alpha_2}}{m}|\omega|^{\frac{1}{1-\alpha_2}} + C_2; \tag{4.4}$$

$$C'_{\alpha_2}|\omega|^{\frac{\alpha_2}{1-\alpha_2}} - C_4 \leq |H'(t,\omega(t),\omega(t-2\tau))| \leq C_{\alpha_2}\Big(\frac{2^{\frac{1}{1-\alpha_2}}}{m} - \frac{1}{M}\Big)|\omega|^{\frac{\alpha_2}{1-\alpha_2}} + C_3, \tag{4.5}$$

where $C_1, \dots, C_4$ are constants, $C_{\alpha_2}, C'_{\alpha_2}$ are constants depending on α_2, and

$$|\omega| = \sqrt{\omega^2(t) + \omega^2(t-2\tau)},$$

$$|H'(t,\omega(t),\omega(t-2\tau))| = \sqrt{|H'_1(t,\omega(t),\omega(t-2\tau))|^2 + |H'_2(t,\omega(t),\omega(t-2\tau))|^2}.$$

In addition, the function H satisfies

$$\lim_{|\omega|\to 0} \frac{H(t,\omega(t),\omega(t-2\tau))}{|\omega|^2} = \infty. \tag{4.6}$$

Proof. From Corollary 2.12 and $F'_1(t,0,0) = F'_2(t,0,0) = 0$ we have $H'_1(t,0,0) = H'_2(t,0,0) = 0$, for all $t \in [0, 2\gamma\tau]$. Using the definition of H, we know that $H(t,0,0) = 0$.

We first consider (4.4). It follows from (A7) that $F(t,u_1,u_2) \le M|u|^{1/\alpha_2} + C_1$ for all $u = (u_1,u_2) \in \mathbb{R}^2$. Hence, from Proposition 2.7 and Example 3.2, it is not difficult to see that

$$H(t,\omega(t),\omega(t-2\tau)) \ge \frac{C_{\alpha_2}}{M}|\omega|^{\frac{1}{1-\alpha_2}} - C_1,$$

where

$$C_{\alpha_2} = 2^{\frac{1}{1-\alpha_2}}/M^{\frac{\alpha_2}{1-\alpha_2}-1}(\alpha_2^{\frac{\alpha_2}{1-\alpha_2}} - \alpha_2^{\frac{1}{1-\alpha_2}}).$$

A similar argument as the one in the proof of Lemma 4.4 indicates that there exists a constant C_2 such that

$$F(t,u_1,u_2) \ge m|u|^{1/\alpha_2} - C_2.$$

Therefore,

$$H(t,\omega(t),\omega(t-2\tau)) \le \frac{C_{\alpha_2}}{m}|\omega|^{\frac{1}{1-\alpha_2}} + C_2.$$

Next we consider (4.5). Using a similar argument as used in the proof of Lemma 4.4, we obtain the estimate

$$|H'(t,\omega(t),\omega(t-2\tau))| \le C_{\alpha_2}(\frac{2^{\frac{1}{1-\alpha_2}}}{m} - \frac{1}{M})|\omega|^{\frac{\alpha_2}{1-\alpha_2}} + C_3,$$

where $C_3 = \max\{C_1 + C_2, \sup_{|\omega|<1}|H'(t,\omega(t),\omega(t-2\tau))|\}$. Lemma 4.4 and the duality principle yield

$$(u_1,u_2) = (H'_1(t,\omega(t),\omega(t-2\tau)), H'_2(t,\omega(t),\omega(t-2\tau)))$$

if and only if

$$(\omega(t),\omega(t-2\tau)) = (F'_1(t,u_1,u_2), F'_2(t,u_1,u_2)).$$

When $|H'(t,\omega(t),\omega(t-2\tau))| \ge C$, we have

$$|\omega| \le (2^{1/\alpha_2}M - m)|H'|^{\frac{1}{\alpha_2}-1}.$$

Thn there exists a constant M_C such that when $|u| = \sqrt{u_1^2 + u_2^2} = |H'(t,\omega(t),\omega(t-2\tau)| \le C$, we have

$$|\omega| = |F'(t,u_1,u_2)| \le M_C.$$

Let

$$C'_{\alpha_2} = (2^{1/\alpha_2}M - m)^{\frac{\alpha_2}{\alpha_2-1}}, \quad C_4 = C'_{\alpha_2}M_C^{\frac{\alpha_2}{1-\alpha_2}}.$$

We then obtain that

$$|H'(t,\omega(t),\omega(t-2\tau))| \ge C'_{\alpha_2}|\omega|^{\frac{\alpha_2}{1-\alpha_2}} - C_4.$$

Now, we consider (4.6). It follows from (A8) that for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $|u| = \sqrt{u_1^2 + u_2^2} < \delta$, we have

$$F(t, u_1, u_2) \le \varepsilon\sqrt{u_1^2 + u_2^2}.$$

Therefore, for all $K > 0$, if we choose $\varepsilon = \frac{1}{4K}$ and $\eta = 2\varepsilon\delta(\varepsilon)$, then when $\sqrt{\omega^2(t) + \omega^2(t-2\tau)} < \eta$, we obtain

$$H(t, \omega(t), \omega(t-2\tau)) \ge \frac{1}{4\varepsilon}(\omega^2(t) + \omega^2(t-2\tau)) = K|\omega|^2;$$

i.e.,

$$\lim_{|\omega|\to 0} \frac{H(t, \omega(t), \omega(t-2\tau))}{|\omega|^2} = \infty.$$

□

Lemma 4.6. *There exist constants C_δ and C'_δ depending on δ, such that*

$$H(t, \omega(t), \omega(t-2\tau)) \ge \begin{cases} C_\delta|\omega|^2, & \text{when } |\omega| \le \delta, \\ C'_\delta|\omega|^q, & \text{when } |\omega| \ge \delta. \end{cases}$$

In addition, when $\delta \to +0$, $C_\delta \to +\infty$.

Proof. Equality (4.6) implies that

$$\lim_{|\omega|\to 0} \frac{H(t, \omega(t), \omega(t-2\tau))}{|\omega|^2} = \infty.$$

Thus, when $\delta \to +0$, we have $C_\delta := \inf\{H(t, \omega(t), \omega(t-2\tau))/|\omega|^2 : |\omega| \le \delta\} \to +\infty$. That is to say,

$$H(t, \omega(t), \omega(t-2\tau)) \ge C_\delta|\omega|^2 \tag{4.7}$$

when $|\omega| \le \delta$.

Next, we consider the second part of the inequality. For all $\omega_0 = (\omega_0(t), \omega_0(t-2\tau))$, $|\omega_0| = 1$, let $\phi_{\omega_0}(\beta) = H(t, \beta\omega_0(t), \beta\omega_0(t-2\tau))$. Thus,

$$\phi'_{\omega_0}(\beta) = H'_1(t, \beta\omega_0(t), \beta\omega_0(t-2\tau))\omega_0(t) + H'_2(t, \beta\omega_0(t), \beta\omega_0(t-2\tau))\omega_0(t-2\tau).$$

The convexity of function ϕ_{ω_0} implies that for all $\beta > 0$,

$$H'_1(t, \beta\omega_0(t), \beta\omega_0(t-2\tau))\omega_0(t) + H'_2(t, \beta\omega_0(t), \beta\omega_0(t-2\tau))\omega_0(t-2\tau) \ge \frac{1}{\beta}\phi_{\omega_0}(\beta).$$

It follows from (4.7) that

$$H'_1(t, \delta\omega_0(t), \delta\omega_0(t-2\tau))\omega_0(t) + H'_2(t, \delta\omega_0(t), \delta\omega_0(t-2\tau))\omega_0(t-2\tau) \ge C_\delta \cdot \delta.$$

We note that H is a convex function. Thus,

$$\begin{aligned} &H(t, s\omega_0(t), s\omega_0(t-2\tau)) \\ &\ge H'_1(t, \delta\omega_0(t), \delta\omega_0(t-2\tau))(s-\delta)\omega_0(t) \\ &\quad + H'_2(t, \delta\omega_0(t), \delta\omega_0(t-2\tau))(s-\delta)\omega_0(t-2\tau) + H(t, \delta\omega_0(t), \delta\omega_0(t-2\tau)) \\ &\ge C_\delta \cdot \delta(s-\delta) + C_\delta\delta^2 = C_\delta\delta s, \quad \forall s > 0, \end{aligned}$$

which indicates that

$$H(t, \omega(t), \omega(t-2\tau)) \ge C_\delta \cdot \delta|\omega|. \tag{4.8}$$

From Lemma 4.5, we know that there exists $T > 0$ satisfying

$$H(t, \omega(t), \omega(t-2\tau)) \geq \frac{C_{\alpha_2}}{2M} |\omega|^q. \tag{4.9}$$

Let $C'_\delta = \min\{\frac{C_{\alpha_2}}{2M}, T^{1-q}\delta C_\delta\}$. It follows from (4.7), (4.8) and (4.9) that

$$H(t, \omega(t), \omega(t-2\tau)) \geq \begin{cases} C_\delta |\omega|^2, & |\omega| \leq \delta, \\ C'_\delta |\omega|^q, & |\omega| \geq \delta. \end{cases}$$

□

Lemma 4.7. *Suppose that* $\upsilon_m = (\upsilon_m(t), \upsilon_m(t-2\tau)) \rightharpoonup \upsilon = (\upsilon(t), \upsilon(t-2\tau))$ *(weakly convergent sequence on* $L^q([0, 2\gamma\tau]))^2$*) and satisfies*

$$\int_0^{2\gamma\tau} H(t, \upsilon_m) dt \to \int_0^{2\gamma\tau} H(t, \upsilon) dt.$$

Then we have

$$\int_0^{2\gamma\tau} H(t, \upsilon_m - \upsilon) dt \to 0.$$

Proof. We accomplish the proof by carrying out the following two steps.
Step 1. Show that the terms in $\{H(t, \upsilon_m)\}$ have equicontinuous integrals, that is to say, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all m,

$$\int_\Omega H(t, \upsilon_m) dt < \varepsilon \quad \text{when} \quad \mu(\Omega) < \delta.$$

Since H is a convex function, one has

$$H'_1(t, \upsilon)(\upsilon(t) - \upsilon_m(t)) + H'_2(t, \upsilon)(\upsilon(t-2\tau) - \upsilon_m(t-2\tau)) \leq H(t, \upsilon_m) - H(t, \upsilon).$$

It follows from the above equality and $\upsilon_m \rightharpoonup \upsilon$ that

$$\int_0^{2\gamma\tau} H(t, \upsilon) dt \leq \liminf_{m\to\infty} \int_0^{2\gamma\tau} H(t, \upsilon_m) dt. \tag{4.10}$$

Since $H \geq 0$, the assumption $\int_0^{2\gamma\tau} H(t, \upsilon) dt \to \int_0^{2\gamma\tau} H(t, \upsilon_m) dt$ implies that

$$\lim_{m\to\infty} \int_\Omega H(t, \upsilon_m) dt = \int_\Omega H(t, \upsilon) dt \quad \text{for all measurable sets } \Omega. \tag{4.11}$$

Now we prove that the terms in $\{H(t, \upsilon_m)\}$ have equicontinuous integrals. Suppose, to the contrary, that there exist $\varepsilon_0 > 0$, functions $\upsilon_{m_k} = (\upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau))$ and measurable sets Ω_k, such that

$$\int_\Omega H(t, \pm\upsilon) dt < \varepsilon_0, \quad \text{for all measurable sets } \Omega \text{ and } \mu(\Omega) < \delta. \tag{4.12}$$

We note that

$$\int_{\Omega_k} H(t, \upsilon_{m_k}) dt \geq \varepsilon_0, \quad \mu(\Omega_k) < \frac{\delta}{2^k}.$$

If we choose $\Omega_0 = \bigcup_{k=1}^\infty \Omega_k$, then it is not difficult to see that $\mu(\Omega_0) < \delta$ and

$$\int_{\Omega_0} H(t, \upsilon_{m_k}) dt \geq \int_{\Omega_k} H(t, \upsilon_{m_k}) dt \geq \varepsilon_0,$$

which contradicts to (4.10) and (4.11).
Step 2. For all $b > 0$, we divide $[0, 2\gamma\tau]$ into the following three subsets:

$$Q_1 = \{t \in [0, 2\gamma\tau] : |\upsilon| = \sqrt{\upsilon^2(t) + \upsilon^2(t-2\tau)} > b\},$$

$$Q_2^m = \{t \in [0, 2\gamma\tau] : |\upsilon| \le b, |\upsilon_m - \upsilon| \ge \delta\},$$
$$Q_3^m = \{t \in [0, 2\gamma\tau] : |\upsilon| \le b, |\upsilon_m - \upsilon| < \delta\},$$

where $|\upsilon_m - \upsilon| = \sqrt{(\upsilon_m(t) - \upsilon(t))^2 + (\upsilon_m(t-2\tau) - \upsilon(t-2\tau))^2}$. By inequality (4.4), there exist constants K and L satisfying

$$H(t, 2z(t), 2z(t-2\tau)) \le KH(t, z(t), z(t-2\tau)) + L, \forall (z(t), z(t-2\tau)) \in \mathbb{R}^2.$$

Notice that H is convex on Q_1. Thus,

$$H(t, \upsilon_m - \upsilon) \le \frac{1}{2}[H(t, 2\upsilon_m) + H(t, -2\upsilon)] \le \frac{K}{2}(H(t, \upsilon_m) + H(t, -\upsilon)) + L.$$

By step 1, we may choose a constant b large enough and fix it such that $\mu(Q_1)$ is small enough such that

$$\int_{Q_1} H(t, \upsilon_m - \upsilon)dt \le \frac{K}{2}\int_{Q_1}(H(t, \upsilon_m) + H(t, -\upsilon))dt + L\mu(Q_1) < \frac{\varepsilon}{3}. \tag{4.13}$$

For the fixed b, choose a value of δ small enough satisfying

$$\int_{Q_3^m} H(t, \upsilon_m - \upsilon)dt < \frac{\varepsilon}{3} \tag{4.14}$$

and fix δ. For the fixed b and δ, let

$$\begin{aligned}\kappa = \inf_{|\omega - z| \ge \delta, |z| \le b} \Big[&H(t, \omega) - H(t, z) - H_1'(t, \omega)(\omega(t) - z(t)) \\ &- H_2'(t, \omega)(\omega(t-2\tau) - z(t-2\tau))\Big].\end{aligned}$$

It is easy to see that $\kappa > 0$, which indicates that

$$\begin{aligned}\kappa\mu(Q_2^m) &\le \int_{Q_2^m}\Big[H(t, \upsilon_m) - H(t, \upsilon) - H_1'(t, \upsilon)(\upsilon_m(t) - \upsilon(t)) \\ &\quad - H_2'(t, \upsilon)(\upsilon_m(t-2\tau) - \upsilon(t-2\tau))\Big]dt \\ &\le \int_0^{2\gamma\tau}\Big[H(t, \upsilon_m) - H(t, \upsilon) - H_1'(t, \upsilon)(\upsilon_m(t) - \upsilon(t)) \\ &\quad - H_2'(t, \upsilon)(\upsilon_m(t-2\tau) - \upsilon(t-2\tau))\Big]dt.\end{aligned}$$

Therefore, $\mu(Q_2^m) \to 0$ when $m \to \infty$. Thus, $\int_{Q_2^m} H(t, \upsilon_m) \to 0$.

A similar argument on Q_1 suggests that there exists an n_0 such that when $m > n_0$, we have

$$\int_{Q_2^m} H(t, \upsilon_m - \upsilon)dt < \frac{\varepsilon}{3}. \tag{4.15}$$

Thus, it follows from (4.13), (4.14) and (4.15) that

$$\lim_{m\to\infty}\int_0^{2\gamma\tau} H(t, \upsilon_m - \upsilon)dt = 0.$$

□

Corollary 4.8. $\upsilon_m = (\upsilon_m(t), \upsilon_m(t-2\tau)) \to \upsilon = (\upsilon(t), \upsilon(t-2\tau))$ *(* $L^q([0, 2\gamma\tau]) \times L^q([0, 2\gamma\tau])$*) if and only if*

$$\int_0^{2\gamma\tau} H(t, \upsilon_m - \upsilon)dt = 0.$$

Proof. (⇒). Since $\upsilon_m \to \upsilon$ implies $\upsilon_m \rightharpoonup \upsilon$ (weakly), by inequality (4.4) and the continuity of the composition operator, we know that $H(t,\upsilon_m) \to H(t,\upsilon)$ $(L^1([0,2\gamma\tau]))$. That is to say, $\int_0^{2\gamma\tau} H(t,\upsilon_m)dt \to \int_0^{2\gamma\tau} H(t,\upsilon)dt$. Thus, the conclusion can be obtained from Lemma 4.7.

(⇐). By Lemma 4.6, there exist constants B_1 and $B_2 > 0$ such that

$$\begin{aligned}\int_0^{2\gamma\tau} H(t,\upsilon)dt &\geq B_1\int_{|\upsilon|\geq\delta}|\upsilon|^q dt + B_2\int_{|\upsilon|<\delta}|\upsilon|^2 dt\\ &\geq B_1\int_{|\upsilon|\geq\delta}|\upsilon|^q dt + B_2(2\gamma\tau)^{\frac{2}{q-2}}\Big(\int_{|\upsilon|<\delta}|\upsilon|^q dt\Big)^{2/q}\\ &\geq C_\delta \min\Big\{\int_0^{2\gamma\tau}|\upsilon|^q dt, \Big(\int_0^{2\gamma\tau}|\upsilon|^q dt\Big)^{2/q}\Big\}.\end{aligned}$$

For δ small enough, $C_\delta > 0$ is a constant, implying that the conclusion holds. □

Proof of Theorem 4.2. We use Lemma 2.4, and do the following three steps:

Step 1. Show that J satisfies the (PS) condition in E. Let $\{\upsilon_n = (\upsilon_n(t),\upsilon_n(t-2\tau))\} \subset E$ and choose constants C_1, C_2 such that

$$C_1 \leq J(\upsilon_n) \leq C_2, \tag{4.16}$$

$$J'(\upsilon_n) \to \theta. \tag{4.17}$$

Next, we need to show that $\{\upsilon_n\}$ has a convergent subsequence in E. By

$$z_m = K\upsilon_m + H'(t,\upsilon_m) - \chi_m \to \theta,$$

and

$$C_1 \leq \frac{1}{2}\langle K\upsilon_m, \upsilon_m\rangle + \int_0^{2\gamma\tau} H(t,\upsilon_m)dt \leq C_2,$$

where $z_m = (z_m(t), z_m(t-\tau)), \upsilon_m = (\upsilon_m(t),\upsilon_m(t-2\tau))$ and $\chi_m = (\chi_m(t-\tau),\chi_m(t-2\tau))$, one can obtain that there exists $n(\varepsilon) > 0$ for all $\varepsilon > 0$ such that when $m \geq m(\varepsilon)$, the following inequality holds

$$\begin{aligned}&\int_0^{2\gamma\tau} H(t,\upsilon_m)dt - \frac{1}{2}[H_1'(t,\upsilon_m)\upsilon_m(t) + H_2'(t,\upsilon_m)\upsilon_m(t-2\tau)]\\ &\leq C_2 + \frac{\varepsilon}{2}(\|\upsilon_m(t)\|_{L^q} + \|\upsilon_m(t-2\tau)\|_{L^q}) = C_2 + \varepsilon\|\upsilon_m(t)\|_{L^q}.\end{aligned} \tag{4.18}$$

Moreover, by Lemmas 4.3 and 4.4, there exist constants α_2, C_3, C_4 and C_5 such that

$$\begin{aligned}&H(t,\omega) - \frac{1}{2}H_1'(t,\omega)\omega(t) - \frac{1}{2}H_2'(t,\omega)\omega(t-2\tau)\\ &= \frac{1}{2}z(t)F_1'(t,z(t),z(t-\tau)) + \frac{1}{2}z(t-\tau)F_2'(t,z(t),z(t-\tau)) - F(t,z(t),z(t-\tau))\\ &\geq (\frac{1}{2\alpha_2} - 1)F(t,z(t),z(t-\tau)) - C_3\\ &\geq m|z|^{1/\alpha_2}(\frac{1}{2\alpha_2} - 1) - C_4\\ &\geq |\omega|^q - C_5,\end{aligned} \tag{4.19}$$

where

$$\omega(t) = F_1'(t,z(t),z(t-\tau)), \quad \omega(t-2\tau) = F_2'(t,z(t),z(t-\tau)),$$

$$z(t) = H_1'(t, \omega(t), \omega(t-2\tau)), \quad z(t-\tau) = H_2'(t, \omega(t), \omega(t-2\tau)),$$
$$|\omega| = \sqrt{\omega^2(t) + \omega^2(t-2\tau)}, \quad |z| = \sqrt{z^2(t) + z^2(t-\tau)}.$$

It then follows from (4.18) and (4.19) that

$$\|\upsilon_m(t)\|_{L^q[0,2\gamma\tau]} = \|\upsilon_m(t-2\tau)\|_{L^q[0,2\gamma\tau]} \le C_6 \quad \text{(constant)},$$

that is to say, $\{\upsilon_n\}$ is bounded.

Now, we show that $\{\upsilon_n\}$ has a convergent subsequence. The fact that $L^q[0, 2\gamma\tau]$ is a reflexive Banach space implies that there exists a subsequence of $\{\upsilon_n\}$ which is weakly convergent in $L^q[0, 2\gamma\tau]$. We use $\{\upsilon_{m_k}\}$ to denote it. Therefore, $\upsilon_{m_k}(t) \rightharpoonup \upsilon^*(t)$, $\upsilon_{m_k}(t-2\tau) \rightharpoonup \upsilon^*(t-2\tau)$. Since H is a convex function, one can obtain the inequality

$$\begin{aligned}
&H(t, \upsilon^*(t), \upsilon^*(t-2\tau)) + H_1'(t, \upsilon^*(t), \upsilon^*(t-2\tau))(\upsilon_{m_k}(t) - \upsilon^*(t)) \\
&+ H_2'(t, \upsilon^*(t), \upsilon^*(t-2\tau))(\upsilon_{m_k}(t-2\tau) - \upsilon^*(t-2\tau)) \\
&\le H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau)).
\end{aligned}$$

The above discussion indicates that

$$\int_0^{2\gamma\tau} H(t, \upsilon^*(t), \upsilon^*(t-2\tau))dt \le \varliminf_{k\to\infty} \int_0^{2\gamma\tau} H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau))dt. \quad (4.20)$$

On the other hand, the convexity of function H yields

$$\begin{aligned}
&H(t, \upsilon^*(t), \upsilon^*(t-2\tau)) \\
&\ge H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau)) + H_1'(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau))(\upsilon^*(t) - \upsilon_{m_k}(t)) \\
&\quad + H_2'(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau))(\upsilon^*(t-2\tau) - \upsilon_{m_k}(t-2\tau)) \\
&= H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau)) + (-K\upsilon_{m_k} + z_{m_k} + \chi_{m_k}) \cdot (\upsilon^* - \upsilon_{m_k}).
\end{aligned}$$

Since the operators A and K are compact and $(z_{m_k}(t), z_{m_k}(t-\tau)) \to \theta$, one can obtain that

$$\limsup_{k\to\infty} \int_0^{2\gamma\tau} H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau))dt \le \int_0^{2\gamma\tau} H(t, \upsilon^*(t), \upsilon^*(t-2\tau))dt. \quad (4.21)$$

Then, using Lemma 4.7, Corollary 4.8, (4.20) and (4.21), one can obtain that

$$(\upsilon_{m_k}(t), \upsilon_{m_k}(t-2\tau)) \to (\upsilon^*(t), \upsilon^*(t-2\tau)).$$

Step 2. Now we prove that there exist constants $\rho, r > 0$ such that

$$J|_{\partial\Omega_r} \ge \rho > 0, \quad (4.22)$$

where

$$\begin{aligned}
\partial\Omega_r = \Big\{&(\upsilon(t), \upsilon(t-2\tau)) \in L^q[0, 2\gamma\tau] \times L^q[0, 2\gamma\tau] : \\
&\|\upsilon(t)\|_{L^q[0,2\gamma\tau]} = \|\upsilon(t-2\tau)\|_{L^q[0,2\gamma\tau]} = r\Big\}.
\end{aligned}$$

Let $\beta = \|\widehat{K}\|_{\mathcal{L}(L^p, L^q)}$. Choose $\delta > 0$ so that constant C_δ is large enough. Take r small enough such that, by Lemma 4.6, when $\|\upsilon(t)\|_{L^q} = r$, there exists a constant $C_7 > 0$ satisfying

$$C_\delta \int_{|\upsilon|<\delta} |\upsilon(t)|^2 dt - 4\beta\Big(\int_{|\upsilon|<\delta} |\upsilon(t)|^q dt\Big)^{2/q} \ge C_7\Big(\int_{|\upsilon|<\delta} |\upsilon(t)|^q dt\Big)^{2/q}, \quad (4.23)$$

$$C'_\delta \int_{|\upsilon|\ge\delta} |\upsilon(t)|^q dt - 4\beta\Big(\int_{|\upsilon|\ge\delta} |\upsilon(t)|^q dt\Big)^{2/q} \ge C_7\Big(\int_{|\upsilon|\ge\delta} |\upsilon(t)|^q dt\Big)^{2/q}, \tag{4.24}$$

where $|\upsilon| = \sqrt{\upsilon^2(t) + \upsilon^2(t-2\tau)}$. Since

$$a^c + b^c \le (a+b)^c \le 2^c(a^c + b^c),$$

where $a, b > 0$ and $c > 1$, by (4.23) and (4.24), we have

$$\begin{aligned}
J(\upsilon) &= \frac{1}{2}\langle \widehat{K}(\upsilon(t)), \upsilon(t-2\tau)\rangle + \frac{1}{2}\langle \widehat{K}(\upsilon(t-2\tau)), \upsilon(t)\rangle \\
&\quad + \int_0^{2\gamma\tau} H(t, \upsilon(t), \upsilon(t-2\tau)) \\
&\ge -\frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} - \frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} + C_\delta \int_{|\upsilon|<\delta} |\upsilon(t)|^2 dt \\
&\quad + C_\delta \int_{|\upsilon|<\delta} |\upsilon(t-2\tau)|^2 dt + C'_\delta \int_{|\upsilon|\ge\delta} (\sqrt{\upsilon^2(t) + \upsilon^2(t-2\tau)})^q dt \\
&\ge -\frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} - \frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} + C_\delta \int_{|\upsilon|<\delta} |\upsilon(t)|^2 dt \\
&\quad + C_\delta \int_{|\upsilon|<\delta} |\upsilon(t-2\tau)|^2 dt + C'_\delta \int_{|\upsilon|\ge\delta} |\upsilon(t)|^q dt \\
&\ge C_7\Big[\Big(\int_{|\upsilon|<\delta} |\upsilon(t)|^q dt\Big)^{2/q} + \Big(\int_{|\upsilon|\ge\delta} |\upsilon(t)|^q dt\Big)^{2/q}\Big] + C_\delta \int_{|\upsilon|<\delta} |\upsilon(t-2\tau)|^2 dt \\
&\ge \frac{C_7}{2^{2/q}}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} = \frac{C_7}{2^{2/q}} r^2.
\end{aligned}$$

Then, substituting $\rho = \frac{C_7}{2^{2/q}} r^2$ in the above inequality yields (4.22).

Step 3. Finally, since $J(\theta) = 0$ and $J(\upsilon)$ is an even function in υ, if we let $\upsilon_j(t) = \sin \frac{j\pi}{\gamma\tau} t, j = 1, 2, \dots$, then the linear spaces can be defined as

$$E_j = \operatorname{span}\{\upsilon_1, \upsilon_2, \dots, \upsilon_j\}.$$

From Remark 4.1 we have

$$\begin{aligned}
&\widehat{K}(\sin \frac{k\pi}{\gamma\tau} t) \\
&= -\frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[(p'(t)\cos\frac{2k\pi}{\gamma} + \frac{k\pi}{\gamma\tau}p(t)\sin\frac{2k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}(t-2\tau) \\
&\quad + (-p'(t)\sin\frac{2k\pi}{\gamma} + \frac{k\pi}{\gamma\tau}p(t)\cos\frac{2k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}(t-2\tau)]
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{K}(\sin \frac{k\pi}{\gamma\tau}(t-2\tau)) \\
&= -\frac{1}{\frac{k\pi}{\gamma\tau}(p'^2 + (\frac{k\pi}{\gamma\tau})^3 p^2(t)}[(p'(t)\cos\frac{2k\pi}{\gamma} - \frac{k\pi}{\gamma\tau}p(t)\sin\frac{2k\pi}{\gamma})\cos\frac{k\pi}{\gamma\tau}t \\
&\quad + (p'(t)\sin\frac{2k\pi}{\gamma} + \frac{k\pi}{\gamma\tau}p(t)\cos\frac{2k\pi}{\gamma})\sin\frac{k\pi}{\gamma\tau}t].
\end{aligned}$$

By assumption (A4), for any $p(t) \in C^1$, we have

$$p(t) = a_0 + \sum_{k=0}^{+\infty}(a_k \cos\frac{2k\pi}{\tau}t + b_k \sin\frac{2k\pi}{\tau}t) > 0,$$

$$p'(t) = \sum_{k=0}^{+\infty}(-a_k\frac{2k\pi}{\tau}\sin\frac{2k\pi}{\tau}t + b_k\frac{2k\pi}{\tau}\cos\frac{2k\pi}{\tau}t)\,.$$

Therefore,

$$\begin{aligned} J(\upsilon_j) &= \frac{1}{2}\langle\widehat{K}\upsilon_j(t-2\tau), \upsilon_j(t)\rangle + \frac{1}{2}\langle\widehat{K}\upsilon_j(t), \upsilon_j(t-2\tau)\rangle + \int_0^{2\gamma\tau} H(t,\upsilon_j)dt \\ &= -\frac{1}{2}\int_0^{2\gamma\tau} \frac{\frac{j\pi}{\gamma\tau}p(t)\cos\frac{2j\pi}{\gamma}}{\frac{j\pi}{\gamma\tau}(p'^2 + (\frac{j\pi}{\gamma\tau})^3p^2(t)}|\upsilon_j(t)|^2dt \\ &\quad - \frac{1}{2}\int_0^{2\gamma\tau} \frac{\frac{j\pi}{\gamma\tau}p(t)\cos\frac{2j\pi}{\gamma}}{\frac{j\pi}{\gamma\tau}(p'^2 + (\frac{j\pi}{\gamma\tau})^3p^2(t)}|\upsilon_j(t-2\tau)|^2dt] + \int_0^{2\gamma\tau} H(t,\upsilon_j)dt, \end{aligned} \tag{4.25}$$

which implies that there exists $t_0 \in [0,\tau]$ such that

$$\begin{aligned} J(\upsilon_j) = -\frac{1}{2}\frac{\frac{j\pi}{\gamma\tau}p(t_0)\cos\frac{2j\pi}{\gamma}}{\frac{j\pi}{\gamma\tau}(p'(t_0))^2 + (\frac{j\pi}{\gamma\tau})^3p^2(t_0)}\Big[\int_0^{2\gamma\tau}|\upsilon_j(t)|^2dt \\ + \int_0^{2\gamma\tau}|\upsilon_j(t-2\tau)|^2dt\Big] + \int_0^{2\gamma\tau} H(t,\upsilon_j)dt. \end{aligned}$$

Without loss of generality, we may choose $j_1 < j_2 < \dots < j_k < \dots$, such that

$$\cos\frac{2j_1\pi}{\gamma} > 0,\ \cos\frac{2j_2\pi}{\gamma} > 0,\ \dots,\ \cos\frac{2j_k\pi}{\gamma} > 0.$$

Thus,

$$\frac{\frac{j_k\pi}{\gamma\tau}p(t)\cos\frac{2j_k\pi}{\gamma}}{\frac{j_k\pi}{\gamma\tau}(p'^2 + (\frac{j_k\pi}{\gamma\tau})^3p^2(t)} > 0 \quad \forall t \in [0, 2\gamma\tau],\ j_k \in \mathbb{N}.$$

Choosing $\upsilon_{j_k}(t) = \sin\frac{j_k\pi}{\gamma\tau}t, (k = 1, 2, \dots,)$, we can define

$$E_{j_k} = \operatorname{span}\{\upsilon_{j_1}, \upsilon_{j_2}, \dots, \upsilon_{j_k}\},$$

and

$$\lambda_M^{-j_k} = \max_{t\in[0,2\gamma\tau, j_k\in\mathbb{N}]}\Big\{-\frac{\frac{j_k\pi}{\gamma\tau}p(t)\cos\frac{2j_k\pi}{\gamma}}{\frac{j_k\pi}{\gamma\tau}(p'^2 + (\frac{j_k\pi}{\gamma\tau})^3p^2(t)}\Big\} < 0.$$

It is obvious that $\dim E_{j_k} = k$. $(k = 1, 2, \dots,)$. Hence, when $\phi \in E_{j_k}$, it follows from (4.25) that

$$\begin{aligned} J(\phi) &\le \frac{\lambda_M^{-j_k}}{2}[\int_0^{2\gamma\tau}|\phi(t)|^2dt + \int_0^{2\gamma\tau}|\phi(t-2\tau)|^2dt] \\ &\quad + \frac{C_{\alpha_2}}{m}\int_0^{2\gamma\tau}(\sqrt{\phi^2(t) + \phi^2(t-2\tau)})^q dt + 2\gamma\tau C_2 \\ &= \lambda_M^{-j_k}\|\phi\|_{L^2}^2 + \frac{C_{\alpha_2}}{m}\int_0^{2\gamma\tau}|\phi|^q dt + 2\gamma\tau C_2. \end{aligned}$$

We notice that $q < 2, \lambda_M^{-j_k} < 0$. Therefore, when $\phi \in E_{j_k} \setminus B_{R_{j_k}}$, there exists $R_{j_k} > 0$ such that $J(\phi) \leq 0$. Thus, from steps 1, 2, 3 and Lemma 2.4, we obtain Theorem 4.2. □

To conclude this section, we present the following remark.

Remark 4.9. Theorem 4.2 introduces a method based on the variational structure and the operator theory to study the periodic solutions to the second-order nonlinear functional differential systems. This method is different from those in the literature such as the fixed point theory, the coincidence degree theory, or the Fourier analysis method. The method has applications in the study of periodic solutions of second-order nonlinear functional differential systems.

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