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MULTIPLICITY OF HOMOCLINIC SOLUTIONS FOR SECOND-ORDER HAMILTONIAN SYSTEMS

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Abstract. By using a modified function technique and variational methods, we establish the existence of infinitely many homoclinic solutions for a second-order Hamiltonian system $\ddot{u} - L(t)u + F_u(t,u) = 0$, for all $t \in \mathbb{R}$, where no coercive condition for $F(t,u)$ at infinity is imposed.

1. Introduction and statement of main results

This article concerns the existence of homoclinic solutions for the following second-order Hamiltonian system

$$\ddot{u} - L(t)u + F_u(t,u) = 0, \quad \forall t \in \mathbb{R}, \tag{1.1}$$

where $u = (u_1, \dots, u_N) \in \mathbb{R}^N$, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $F \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Here, as usual, we say that a solution u of system (1.1) is a homoclinic solution (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

There have been many papers devoted to the homoclinic solutions of second order Hamiltonian systems via variational methods; see, e.g., [1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 15, 16, 17, 18, 19] and the references therein. If L and F are T-periodic in t, Rabinowitz [10] obtains the existence of one homoclinic solution to system (1.1) as a limit of $2kT$-periodic solutions. The methods and the results are extended by many further works; e.g. see [3] for a significant paper. If L and F are not periodic in t, the problem of existence of homoclinic solutions to system (1.1) is quite different. We now recall some papers. In [4], the author considers the case where $L(t)$ is not periodic and the corresponding linear part is not necessarily positive definite and proves that system (1.1) possesses homoclinic solutions by extending the compact imbedding theorems in [9]. The case is also considered in [16] but $F(t,u)$ is subquadratic satisfying a variant of the Ahmad-Lazer-Paul type condition. By using variant fountain theorem, the authors in [17] also investigate the case when $F(t,u)$ is subquadratic or superquadratic. We should point out that either in the superquadratic or the subquadratic case for $F(t,u)$, which is considered in the above mentioned papers, some kind of coercive conditions at infinity are needed.

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In this paper, by using variational methods, we obtain infinitely many homoclinic solutions of system (1.1) without requiring any coercive condition or even any growth restriction for $F(t,u)$ at infinity when $F(t,u)$ is subquadratic. We introduce the following hypotheses.

- (L1) There exist $a>0$ and $r>0$ such that one of the following two conditions is true,
  - (i) $L \in C^1(\mathbb{R}, \mathbb{R}^N)$ and $|L'(t)| \le a|L(t)|$ for all $|t| \ge r$,
  - (ii) $L \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $L''(t) \le aL(t)$ for all $|t| \ge r$, where $L'(t) = (\mathrm{d}/\mathrm{d}t)L(t)$ and $L''(t) = (\mathrm{d}^2/\mathrm{d}t^2)L(t)$.
- (L2) There exists $\alpha < 1$ such that
$$l(t)|t|^{\alpha-2} \to \infty \text{as } |t| \to \infty,$$
where $l(t)$ is the smallest eigenvalue of $L(t)$; i.e.,
$$l(t) := \inf_{|\xi|=1,\, \xi\in\mathbb{R}^N} \langle L(t)\xi, \xi\rangle.$$
- (F1) $F(t,u) \ge 0$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$ and there exists a constant $1<\mu<2$ such that
$$\langle F_u(t,u), u\rangle \le \mu F(t,u), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N.$$
- (F2) $F(t,0) \equiv 0$ and there exist constants $c_1>0, R_1>0$ and $\frac{1}{2} \le v < 1$ such that
$$|F_u(t,u)| \le c_1|u|^v, \quad \forall t\in\mathbb{R},\ |u| \le R_1.$$
- (F3) There exist constants $L_0>0, L_1>0, d_0>0$, where L_1 is sufficiently large (fixed below), such that
$$F(t,u) \ge d_0|u| > 0, \quad \forall t \in \mathbb{R},\ L_0 \le |u| \le L_1.$$
- (F4) $0 < \underline{b} \equiv \inf_{t\in\mathbb{R},\, |u|=1} F(t,u) \le \sup_{t\in\mathbb{R},\, |u|=1} F(t,u) \equiv \overline{b} < \infty$.

Here and in the sequel, $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the standard inner product and the associated norm in $\mathbb{R}^N$ respectively.

Remark 1.1. In fact, if we set
$$M := \tau_\infty\Big(4 + (a_1^4 + 2a_2^4)\Big(L_0 + \frac{8}{d_0(2-\mu)}\Big)^2 + 8c_2\tau_{1+v}^{1+v} + 8c_2\tau_\mu^\mu\Big)^{\frac{1}{2-s}}$$
then the constant L_1 in (F3) can be any constant bigger than M, where $s = \max\{1+v, \mu\}$, τ_{1+v}, τ_μ and τ_∞ are defined in Lemma 2.1, a_1, a_2 are defined in the proof of Theorem 1.2, c_2 is defined in (3.1).

Our main results are the following theorems.

Theorem 1.2. *Suppose that* (L1)–(L2), (F1)–(F4) *are satisfied, and $F(t,u)$ is even in u. Then system* (1.1) *has infinitely many homoclinic solutions.*

Theorem 1.3. *Suppose that $L(t)$ is positive for all t, and satisfies* (L1)–(L2). *Assume that $F(t,u)$ is even in u and*

- (F5) $\lim_{|u|\to 0} \frac{F(t,u)}{|u|^2} = \infty$ *uniformly for $t \in \mathbb{R}$.*

Then system (1.1) *has infinitely many homoclinic solutions which converge to zero.*

Remark 1.4. We point out that there are natural functions $F(t,u)$ satisfying the conditions of Theorem 1.2. For example,

$$F(t,u) = u^{6/5} e^{-\varepsilon u^2}.$$

It is easy to see that, for $\varepsilon > 0$ small, $F(t,u)$ does not satisfy any of the coercive conditions for the problem (1.1) in the above-mentioned papers (c.f. [4, 17, 16]).

2. Variational settings and preliminaries

We first recall the variational settings for system (1.1).

Denote by $\mathcal{A}$ the self-adjoint extension of the operator $-(\mathrm{d}^2/\mathrm{d}t^2)+L(t)$ with the domain $\mathcal{D}(\mathcal{A}) \subset L^2 := L^2(\mathbb{R},\mathbb{R}^N)$. Let $E := \mathcal{D}(|\mathcal{A}|^{1/2})$, the domain of $|\mathcal{A}|^{1/2}$, and define in E the inner product and norm by

$$(u,v)_0 := (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u,v)_2, \quad \|u\|_0 := (u,u)_0^{1/2},$$

where, as usual, $(\cdot,\cdot)_2$ denotes the inner product of L^2. Then E is a Hilbert space. The following lemma is proved in [4].

Lemma 2.1. *If $L(t)$ satisfies condition* (L2)*, then E is compactly embedded in $L^p := L^p(\mathbb{R},\mathbb{R}^N)$ for $1 \le p \le \infty$, which implies that there exists a constant $\tau_p > 0$ such that*

$$|u|_p \le \tau_p \|u\|_0, \ \forall u \in E.$$

By Lemma 2.1, the spectrum $\sigma(\mathcal{A})$ consists of only eigenvalues numbered in $\lambda_1 \le \lambda_2 \le \dots \to \infty$(counted in their multiplicities) and a corresponding system of eigenfunctions $\{e_n\}$, $\mathcal{A}e_n = \lambda_n e_n$, which forms an orthogonal basis of L^2. Assume that $\lambda_1, \dots, \lambda_{n^-} < 0$, $\lambda_{n^-+1} = \dots = \lambda_{\bar{n}} = 0$, and let $E^- := \operatorname{span}\{e_1, \dots, e_{n^-}\}$, $E^0 := \operatorname{span}\{e_{n^-+1}, \dots, e_{\bar{n}}\}$ and $E^+ := \overline{\operatorname{span}\{e_{\bar{n}+1}, \dots\}}$. Then $E = E^- \oplus E^0 \oplus E^+$.

We introduce in E the inner product

$$(u,v) := (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u^0, v^0)_2$$

and the norm

$$\|u\|^2 = (u,u) = \||\mathcal{A}|^{1/2}u\|_2^2 + \|u^0\|_2^2,$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$. Then $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. From now on, the norm $\|\cdot\|$ in E will be used. Hereafter, $(\cdot,\cdot)$ denotes the inner product in E or the pairing between E^* and E.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\oplus_{j\in N} X_j}$ with $\dim X_j < \infty$, for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider the following C^1-functional $\Phi_\lambda : X \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \ \lambda \in [1,2].$$

The following variant of the fountain theorem is established in [19].

Proposition 2.2. *Assume that the functional Φ_λ defined above satisfies the following conditions.*

- (T1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times X$.*
- (T2) *$B(u) \ge 0$ for all $u \in X$; $B(u) \to \infty$ as $\|u\| \to \infty$ in any finite dimensional subspace of X.*

(T3) *There exist* $\rho_k > r_k > 0$ *such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0 > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u), \quad \forall \lambda \in [1,2],$$

and

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0, \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exist $\lambda_n \to 1$, $u_{\lambda_n} \in Y_n$ *such that*

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \to \eta_k \in [\xi_k(2), \beta_k(1)], \quad \text{as } n \to \infty$$

Particularly, if $\{u_{\lambda_n}\}$ *has a convergent subsequence for every* k, *then* Φ_1 *has infinitely many nontrivial critical points* $\{u_k\} \subset X \backslash \{\theta\}$ *satisfying* $\Phi_1(u_k) \to 0^-$ *as* $k \to \infty$.

We shall use a result from [7]. For this purpose, we first recall the definition of genus.

Definition 2.3. Let X be a real Banach space and A a subset of X. The set A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{\theta\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

Remark 2.4 ([8, 11])**.** 1. For any bounded symmetric neighborhood Ω of the origin in $\mathbb{R}^m$ it holds that $\gamma(\partial\Omega) = m$.

2. Let A, B be closed symmetric subsets of X which do not contain the origin. If there is an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.

The following proposition is established in [7].

Proposition 2.5. *Let* X *be an infinite dimensional Banach space and let* $I \in C^1(X, \mathbb{R})$ *satisfy the following two conditions:*

(A1) $I(u)$ *is even, bounded from below,* $I(\theta) = 0$ *and* $I(u)$ *satisfies the Palais-Smale condition* (PS)

(A2) *For each* $k \in \mathbb{N}$, *there exists an* $A_k \in \Gamma_k$ *such that* $\sup_{u \in A_k} I(u) < 0$.

Then $I(u)$ *admits a sequence of critical points* u_k *such that* $I(u_k) \leq 0$, $u_k \neq \theta$ *and* $\lim_{k \to \infty} u_k = \theta$.

3. Proofs of the main results

3.1. **Proof of Theorem 1.2.** By (F1), (F2) and (F4), we obtain

$$|F(t,u)| \leq c_2(|u|^{1+v} + |u|^{\mu}), \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.1}$$

for some $c_2 > 0$. By (F3), there exists a constant $\delta_0 > 0$ such that

$$F(t,u) \geq \frac{d_0}{2}|u| > 0, \quad \forall t \in \mathbb{R},\ L_0 \leq |u| \leq L_1 + \delta_0. \tag{3.2}$$

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(y) \equiv 1$, if $y \leq L_1$, $\chi(y) \equiv 0$, if $y \geq L_1 + \delta_0$ and $\chi'(y) < 0$, if $y \in (L_1, L_1 + \delta_0)$. Set

$$G(t,u) := \chi(|u|)F(t,u) + \frac{d_0}{2}(1 - \chi(|u|))|u|.$$

Then $G \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $G(t,u)\ge 0$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^N$. It is easily seen that

$$\langle G_u(t,u),u\rangle = \chi(|u|)\langle F_u(t,u),u\rangle + \chi'(|u|)|u|(F(t,u)-\frac{d_0}{2}|u|)+\frac{d_0}{2}(1-\chi(|u|))|u|.$$

Hence,by (F1), (3.2) and the definition of χ, we have

$$\langle G_u(t,u),u\rangle \le \mu G(t,u),\quad \forall (t,u)\in\mathbb{R}\times\mathbb{R}^N. \tag{3.3}$$

Without loss of generality, we assume that $d_0\le 1$. Combining (3.1) and (3.2), we obtain

$$G(t,u)\le 2c_2(|u|^{1+v}+|u|^{\mu}),\ \forall (t,u)\in\mathbb{R}\times\mathbb{R}^N, \tag{3.4}$$

and

$$G(t,u)\ge \frac{d_0}{2}|u|>0,\quad \forall t\in\mathbb{R},\ |u|\ge L_0. \tag{3.5}$$

Let

$$\begin{aligned}\varphi(u)&=\frac12\int_{\mathbb{R}}(|\dot u|^2+\langle L(t)u,u\rangle)\mathrm{d}t-\int_{\mathbb{R}}G(t,u)\,\mathrm{d}t\\&=\frac12\|u^+\|-\frac12\|u^-\|^2-\int_{\mathbb{R}}G(t,u)\,\mathrm{d}t\\&=\varphi_1(u)+\varphi_2(u)\end{aligned}$$

where $\varphi_1(u)=\frac12\|u^+\|-\frac12\|u^-\|^2$, $\varphi_2(u)=\int_{\mathbb{R}}G(t,u)\mathrm{d}t$ for $u=u^-+u^0+u^+\in E$. By [4], we have the following lemma.

Lemma 3.1. *Suppose that* (L1)–(L2), (F1)–(F4) *are satisfied. Then* $\varphi_2\in C^1(E,\mathbb{R})$ *and* $\varphi_2': E\to E^*$ *is compact. Moreover,*

$$(\varphi_2'(u),v)=\int_{\mathbb{R}}\langle G_u(t,u),v\rangle\,\mathrm{d}t,$$
$$(\varphi'(u),v)=(u^+,v^+)-(u^-,v^-)-\int_{\mathbb{R}}\langle G_u(t,u),v\rangle\,\mathrm{d}t$$

for all $u,v\in E=E^-\oplus E^0\oplus E^+$ *with* $u=u^-+u^0+u^+$ *and* $v=v^-+v^0+v^+$. *Correspondingly, the nontrivial critical points of* φ *in* E *are the homoclinic solutions of the system*

$$\ddot u-L(t)u+G_u(t,u)=0,\quad \forall t\in\mathbb{R}. \tag{3.6}$$

To prove Theorem 1.2 using Proposition 2.2, we define the functionals

$$A(u)=\frac12\|u^+\|^2,\quad B(u)=\frac12\|u^-\|^2+\int_{\mathbb{R}}G(t,u)\ \mathrm{d}t, \tag{3.7}$$

$$\Phi_\lambda(u)=A(u)-\lambda B(u)=\frac12\|u^+\|^2-\lambda\Big(\frac12\|u^-\|^2+\int_{\mathbb{R}}G(t,u)\ \mathrm{d}t\Big) \tag{3.8}$$

for all $u=u^-+u^0+u^+\in E=E^-\oplus E^0\oplus E^+$ and $\lambda\in[1,2]$.

By the similar arguments as in [17], we obtain the following two Lemmas. For the completeness of this paper we will give their proofs.

Lemma 3.2. *Suppose that* (F1)–(F3) *are satisfied. Then* $B(u)\ge 0$ *for all* $u\in E$ *and* $B(u)\to\infty$ *as* $\|u\|\to\infty$ *in any finite-dimensional subspace of* E.

Proof. By $G(t,u) \geq 0$ and (3.7), we have $B(u) \geq 0$. For any finite-dimensional subspace $E_0 \subset E$, there exists a constant $\varepsilon > 0$ such that

$$m(\{t \in \mathbb{R} : |u(t)| \geq \varepsilon\|u\|\}) \geq \varepsilon, \quad \forall u \in E_0 \setminus \{\theta\}, \tag{3.9}$$

where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}$. The proof of the claim is standard(e.g. see [17, 15]). Let

$$\Lambda_u = \{t \in \mathbb{R} : |u(t)| \geq \varepsilon\|u\|\}, \quad \forall u \in E_0 \setminus \{\theta\},$$

where ε is given in (3.9). Then

$$m(\Lambda_u) \geq \varepsilon, \ \forall u \in E_0 \setminus \{\theta\}. \tag{3.10}$$

Combining with (3.5) and (3.10), for any $u \in E_0$ with $\|u\| \geq L_0/\varepsilon$, we have

$$\begin{aligned} B(u) &= \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} G(t,u)\, \mathrm{d}t \\ &\geq \int_{\Lambda_u} G(t,u)\, \mathrm{d}t \\ &\geq \int_{\Lambda_u} \frac{d_0}{2}|u|\, \mathrm{d}t \\ &\geq d_0\varepsilon\|u\| \cdot m(\Lambda_u)/2 \\ &\geq d_0\varepsilon^2\|u\|/2. \end{aligned}$$

This implies that $B(u) \to \infty$ as $\|u\| \to \infty$ in any finite-dimensional subspace of $E_0 \subset E$. The proof is completed. □

Lemma 3.3. *Suppose that* (L2), (F1)-(F4) *are satisfied. Then there exist a positive integer k_1 and two sequences $0 < r_k < \rho_k \to 0$ as $k \to \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1,$$

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2],$$

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N},$$

where $Y_k = \bigoplus_{j=1}^k X_j = \operatorname{span}\{e_1, \dots, e_k\}$ and $Z_k = \overline{\bigoplus_{j=k}^\infty X_j} = \overline{\operatorname{span}\{e_k, \dots\}}$ for all $k \in \mathbb{N}$.

Proof. Let $l_k = \sup_{u \in Z_k, \|u\|=1} |u|_{1+v}^{1+v}, \forall k \in \mathbb{N}$. Then $l_k \to 0$ as $k \to \infty$ (cf.[14, Lemma 3.8]). Choose k large enough such that $Z_k \subset E^+$. Noticing (F2) and $F(t,u) = G(t,u)$ as $|u| \leq R_1$, we have $G(t,u) \leq c_1|u|^{1+v}$ for $|u| \leq R_1$. Therefore, for any $u \in Z_k$ with $\|u\| \leq R_1/\tau_\infty$, we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} G(t,u)\, \mathrm{d}t \geq \frac{1}{2}\|u\|^2 - 2c_1 l_k \|u\|^{v+1}.$$

Set $\rho_k = (8c_1 l_k)^{\frac{1}{1-v}}$. There exists a positive $k_1 > \bar{n} + 1$ such that $\rho_k < R_1/\tau_\infty$ for all $k \geq k_1$. Thus, for any $k \geq k_1$, we have

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq \rho_k^2/4 > 0.$$

Noticing that $\Phi_\lambda(\theta) = 0$, we have

$$0 \geq \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \geq -2c_1 l_k \rho_k^{v+1}, \quad \forall k \geq k_1.$$

Thus,

$$\xi_k(\lambda) := \inf_{u\in Z_k, \|u\|\le\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k\to\infty \text{ uniformly for } \lambda\in[1,2].$$

Since $\dim Y_k < \infty$, there exists a constant $C_k > 0$ such that $|u|_\mu \ge C_k\|u\|$, $\forall u \in Y_k$. By (F1) and (F4), for any $k\in\mathbb{N}$ and $|u|\le 1$, we have $G(t,u)\ge \underline{b}|u|^\mu$. For any $k\in\mathbb{N}$ and for all $u\in Y_k$ with $\|u\| < \tau_\infty^{-1}$, we have

$$\begin{aligned}\Phi_\lambda(u) &\le \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}} G(t,u)\,\mathrm{d}t\\ &\le \frac{1}{2}\|u\|^2 - \underline{b}|u|_\mu^\mu\\ &\le \frac{1}{2}\|u\|^2 - \underline{b}C_k^\mu\|u\|^\mu, \quad \forall\lambda\in[1,2].\end{aligned}$$

Hence, for $0<r_k<\min\{\rho_k, \tau_\infty^{-1}, (2\underline{b}C_k^\mu)^{\frac{1}{2-\mu}}\}$, we have

$$\beta_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k\in\mathbb{N}.$$

The proof is complete. □

Proof of Theorem 1.2. By $F(t,u) = F(t,-u)$ and the definition of $G(t,u)$, we obtain that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda,u)\in[1,2]\times E$. By Lemma 2.1 and (3.4), we know that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda\in[1,2]$. Combining with Lemmas 3.2-3.3 and Proposition 2.2, for each $k\ge k_1$ there exist $\lambda_n\to 1$, $u_{\lambda_n}^k \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}^k) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}^k)\to\eta_k\in[\xi_k(2),\beta_k(1)], \quad \text{as } n\to\infty. \tag{3.11}$$

Next we will prove that $\{u_{\lambda_n}^k\}$ is bounded and possesses a strong convergent subsequence in E. By Proposition 2.2, we will get infinitely many nontrivial critical points of $\varphi := \Phi_1$. That is, we will get infinitely many homoclinic solutions of system (3.6). By noting that $F(t,u) = G(t,u)$ for $|u|\le L_1$, our proof will be finished if we can find an upper bound $M(\neq\infty)$ of $|u|_\infty$ independent of L_1. For the notational simplicity, we set $u_n = u_{\lambda_n}^k$ for all $n\in\mathbb{N}$, $k\ge k_1$.

Now we prove that $\{u_n\}$ is bounded in E. By Lemma 3.3, there exists $k_2>0$ such that $|\xi_k(\lambda)|\le 1$ for $k\ge k_2$. By (3.11), there exists $n_0\in\mathbb{N}$ such that $|\Phi_{\lambda_n}(u_n)|\le 2$ for $n\ge n_0$ and $k\ge\max\{k_1,\ k_2\}$. By (F1), (F3) and (3.5), we have

$$\begin{aligned}2 &\ge -\Phi_{\lambda_n}(u_n)\\ &= \frac{1}{2}\Phi'_{\lambda_n}|_{Y_n}(u_n)u_n - \Phi_{\lambda_n}(u_n)\\ &\ge \lambda_n\int_{\Omega_n}\Big[G(t,u_n) - \frac{1}{2}\langle G_u(t,u_n),u_n\rangle\Big]\,\mathrm{d}t\\ &\ge \frac{\lambda_n(2-\mu)}{2}\int_{\Omega_n} G(t,u_n)\,\mathrm{d}t\\ &\ge \frac{d_0\lambda_n(2-\mu)}{4}\int_{\Omega_n}|u_n|\,\mathrm{d}t, \quad \forall n\in\mathbb{N},\end{aligned}$$

where $\Omega_n := \{t\in\mathbb{R} : |u_n(t)|\ge L_0\}$. Consequently,

$$\int_{\Omega_n}|u_n|\,\mathrm{d}t \le \frac{8}{d_0(2-\mu)}, \quad \forall n\in\mathbb{N},\ n\ge n_0. \tag{3.12}$$

For any $n \in N$, define $\omega_n : \mathbb{R} \to \mathbb{R}$ by

$$\omega_n = \begin{cases} 1, & t \in \Omega_n \\ 0, & t \notin \Omega_n. \end{cases}$$

Noticing that $\dim E^- \oplus E^0 < \infty$ and $\dim E^- < \infty$, by the equivalence of the norms in finite-dimensional spaces, there exist two constants $a_1, a_2 > 0$ such that

$$|u_n^- + u_n^0|_1 \le a_1 |u_n^- + u_n^0|_2, \ |u_n^- + u_n^0|_\infty \le a_1 |u_n^- + u_n^0|_2, \tag{3.13}$$

$$\|u_n^- + u_n^0\| \le a_1 |u_n^- + u_n^0|_2, \tag{3.14}$$

$$|u_n^-|_1 \le a_2 |u_n^-|_2, \ |u_n^-|_\infty \le a_2 |u_n^-|_2, \tag{3.15}$$

$$\|u_n^-\| \le a_2 |u_n^-|_2. \tag{3.16}$$

By Lemma 2.1, (3.12) and the Hölder inequality, we have

$$\begin{aligned} |u_n^- + u_n^0|_2^2 &= (u_n^- + u_n^0, u_n)_2 \\ &= (u_n^- + u_n^0, (1-\omega_n)u_n)_2 + (u_n^- + u_n^0, \omega_n u_n)_2 \\ &\le |(1-\omega_n)u_n|_\infty |u_n^- + u_n^0|_1 + |\omega_n u_n|_1 |u_n^- + u_n^0|_\infty \\ &\le a_1 \Big(L_0 + \frac{8}{d_0(2-\mu)}\Big) |u_n^- + u_n^0|_2, \quad \forall n \in \mathbb{N}, \ n \ge n_0. \end{aligned}$$

By (3.14), we obtain that

$$\|u_n^- + u_n^0\| \le a_1^2 \Big(L_0 + \frac{8}{d_0(2-\mu)}\Big), \quad \forall n \in \mathbb{N}. \tag{3.17}$$

Similarly, by Lemma 2.1, (3.15) (3.16) and the Hölder inequality, we have

$$\|u_n^-\| \le a_2^2 \Big(L_0 + \frac{8}{d_0(2-\mu)}\Big), \quad \forall n \in \mathbb{N}, \ n \ge n_0. \tag{3.18}$$

Without loss of generality, we assume that $\|u_n\| \ge 1$. Then by Lemma 2.1, (3.4) (3.17) and (3.18), for all $n \in \mathbb{N}$, $n \ge n_0$, we obtain

$$\begin{aligned} \|u_n\|^2 &= \|u_n^+\|^2 + \|u_n^- + u_n^0\|^2 \\ &= 2\Phi_{\lambda_n}(u_n) + \lambda_n \|u_n^-\|^2 + \|u_n^- + u_n^0\|^2 + 2\lambda_n \int_{\mathbb{R}} G(t, u_n)\, \mathrm{d}t \\ &\le 4 + (a_1^4 + 2a_2^4)\Big(L_0 + \frac{8}{d_0(2-\mu)}\Big)^2 + 8c_2(\tau_{1+v}^{1+v}\|u_n\|^{1+v} + \tau_\mu^\mu \|u_n\|^\mu) \\ &\le \Big(4 + (a_1^4 + 2a_2^4)\Big(L_0 + \frac{8}{d_0(2-\mu)}\Big)^2 + 8c_2\tau_{1+v}^{1+v} + 8c_2\tau_\mu^\mu\Big)\|u_n\|^s, \end{aligned}$$

where $s = \max\{1+v, \mu\}$. By noting that $1 < \mu < 2$ and $\frac{1}{2} \le v < 1$, we have

$$\|u_n\| \le \Big(4 + (a_1^4 + 2a_2^4)\Big(L_0 + \frac{8}{d_0(2-\mu)}\Big)^2 + 8c_2\tau_{1+v}^{1+v} + 8c_2\tau_\mu^\mu\Big)^{\frac{1}{2-s}}, \tag{3.19}$$

where the constant does not depend on L_1.

Since E is embedded compactly into L^p for $1 \le p \le \infty$, by a standard argument, we obtain that $\{u_n\}_{n=1}^\infty$ possesses a strong convergent subsequence in E for each $k \ge \max\{k_1, k_2\}$. Hence, by Proposition 2.2, system (3.6) possesses infinitely many homoclinic solutions. By Lemma 3.3 and (3.11), we know that $\Phi_{\lambda_n}(u^k_{\lambda_n})$ is bounded uniformly for $\forall k \ge \max\{k_1, k_2\}$. Set

$$M := \tau_\infty\Big(4 + (a_1^4 + 2a_2^4)\Big(L_0 + \frac{8}{d_0(2-\mu)}\Big)^2 + 8c_2\tau_{1+v}^{1+v} + 8c_2\tau_\mu^\mu\Big)^{\frac{1}{2-s}}.$$

By (3.19) we obtain $\|u^k\| \le M$, $\forall k \ge \max\{k_1, k_2\}$, where u^k is the limit of $\{u_n^k\}_{n=1}^\infty$. Therefore, there exists a constant $M > 0$ independent of L_1 such that $|u^k|_\infty \le M$, $\forall k \ge \max\{k_1, k_2\}$. Combining this with $F(t,u) = G(t,u)$ for $|u| \le L_1$, we know that system (1.1) possesses infinitely many homoclinic solutions if $L_1 \ge M$. The proof is complete. □

Proof of Theorem 1.3. Let $M_0 > 0$, and let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ and $C > 0$ be such that $\chi(y) \equiv 1$, if $y \le M_0$; $\chi(y) \equiv 0$, if $y \ge M_0 + 1$; and $|\chi'(y)| < C$, if $y \in (M_0, M_0+1)$. Set

$$G(t,u) := \chi(|u|)F(t,u) + |u|(1 - \chi(|u|)). \tag{3.20}$$

Then $G \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ and

$$|G(t,u)| \le a_3(1 + |u|),$$

for some $a_3 > 0$. Let

$$\widetilde{\varphi}(u) = \frac{1}{2}\int_{\mathbb{R}}(|\dot{u}|^2 + \langle L(t)u, u\rangle)\mathrm{d}t - \int_{\mathbb{R}} G(t,u)\,\mathrm{d}t.$$

Then $\widetilde{\varphi} \in C^1(E, \mathbb{R})$ and the nontrivial critical points of $\widetilde{\varphi}$ in E are the homoclinic solutions of system

$$\ddot{u} - L(t)u + G_u(t,u) = 0, \quad \forall t \in \mathbb{R}. \tag{3.21}$$

Let

$$\begin{aligned}\psi(u) &= \frac{1}{2}\int_{\mathbb{R}}(|\dot{u}|^2 + \langle L(t)u, u\rangle)\mathrm{d}t - \chi(|u|)\int_{\mathbb{R}} G(t,u)\,\mathrm{d}t\\ &= \frac{1}{2}\|u\|^2 - \chi(|u|)\int_{\mathbb{R}} G(t,u)\,\mathrm{d}t.\end{aligned}$$

Then, $\psi \in C^1(E,\mathbb{R})$. For $\|u\| \ge \tau_\infty^{-1}(M_0+1)$, we have $\psi(u) = \frac{1}{2}\|u\|^2$, which implies that $\psi(u) \to \infty$ as $\|u\| \to \infty$. Hence ψ is coercive on E. Then $\psi(u)$ is bounded from below and, by noticing Lemma 2.1, it satisfies the (PS) condition. By (3.20), it is easy to see that $\psi(u)$ is even and $\psi(\theta) = 0$. This shows that (A_1) holds. By (F4), for any $\varepsilon > 0$, there exists $\delta > 0$, such that $F(t,u) \ge \varepsilon^{-1}|u|^2$, $|u| \le \delta$. For any given k, let $E_k := \operatorname{span}\{e_1, \dots, e_k\}$. Then there exists a constant η_k such that $|u|_2 \ge \eta_k\|u\|$ for $u \in E_k$. Therefore, for any $u \in E_k$ with

$$\|u\| = \rho < \min\{\tau_\infty^{-1}M_0, \tau_\infty^{-1}\delta, 2\varepsilon^{-1}\eta_k\},$$

where ε is small enough, we have

$$\begin{aligned}\psi(u) &= \frac{1}{2}\|u\|^2 - \chi(|u|)\int_{\mathbb{R}} G(t,u)\,\mathrm{d}t\\ &\le \frac{1}{2}\|u\|^2 - \varepsilon^{-1}\eta_k^2\|u\|^2 < 0.\end{aligned}$$

Then $A := \{u \in E_k : \|u\| = \rho\} \subset \{u \in X : \psi(u) < 0\}$. By Remark 2.4, we have that $\gamma(A) = k$ and $\gamma(\{u \in X : \psi(u) < 0\}) \ge \gamma(A) = k$. Setting $A_k = \{u \in X : \psi(u) < 0\}$, then $A_k \in \Gamma_k$ and $\sup_{u\in\Gamma_k}\psi(u) < 0$. This shows that ($A_2$) holds. Hence, by Proposition 2.5, we obtain that ψ admits a sequence of nontrivial solutions $\{u_k\}$ such that $\lim_{k\to\infty} u_k = \theta$. Then there exists $k_1 > 0$ such that $\|u_k\| \le \tau_\infty^{-1} M_0$ for $k \ge k_1$. Since $\widetilde{\varphi} = \psi$ for $|u| \le M_0$, we know that $\widetilde{\varphi}$ possesses infinitely many nontrivial nontrivial critical points $\{u_k\}$ for $k \ge k_1$. Therefore, (3.21) possesses infinitely many nontrivial solutions. That is, system (1.1) has infinitely many solutions by noting that $F(t,u) = G(t,u)$ for $|u| \le M_0$. The proof is completed. □

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