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PSI-EXPONENTIAL DICHOTOMY FOR LINEAR DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract. In this article we extend the concept ψ-exponential and ψ-ordinary dichotomies for homogeneous linear differential equations in a Banach space. With these two concepts we prove the existence of ψ-bounded solutions of the appropriate inhomogeneous equation. A roughness of the ψ-dichotomy is also considered.

1. Introduction

The problem of ψ-boundedness and ψ-stability of the solutions of differential equations in finite dimensional Euclidean spaces has been studied by many autors; see for example Akinyele [1], Constantin [6]. In these publications, the function ψ is a scalar continuous function (and increasing, differentiable and bounded in [1], nondecreasing and such that $\psi(t) \geq 1$ on $\mathbb{R}_+$ in [6]). In Diamandescu [8, 9, 10, 11, 12] and Boi [2, 3, 4] the function ψ is a nonnegative continuous diagonal matrix.

Inspired by the famous monographs of Coppel [5], Daleckii and Krein [7] and Massera and Schaeffer [13], considered the important notion of exponential and ordinary dichotomy in detail. Diamandescu [8]-[12] and Boi [2]-[4], introduced and studied the ψ-dichotomy for linear differential equations in finite dimensional Euclidean space.

Here we introduce the concept of ψ-dichotomy for arbitrary Banach spaces instead in finite dimensional Euclidean spaces. Moreover, in our case, $\psi(t)$ is an arbitrary bounded invertible linear operator, instead of the restriction to be a nonnegative diagonal matrix.

Conditions for the existence of ψ-bounded solutions of the homogeneous and the appropriate inhomogeneous equations are proved. A roughness of the ψ-exponential dichotomy is also proved.

2. Preliminaries

Let X be an arbitrary Banach space with norm $|\cdot|$ and identity I. Let $LB(X)$ be the space of all linear bounded operators acting in X with the norm $\|\cdot\|$. Let $J = [0, \infty)$.

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We consider the linear homogenous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \tag{2.1}$$

and the corresponding inhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t), \tag{2.2}$$

where $A(.) : J \to LB(X)$, $f(.) : J \to X$ are strong measurable and Bochner integrable on the finite subintervals of J.

By a solution of (2.2) (or (2.1)) we will understand a continuous function $x(t)$ that is differentiable (in the sense that it is representable in the form $x(t) = \int_a^t y(\tau)d\tau$ of a Bochner integral of a strongly measurable function y) and satisfies (2.2) (or (2.1)) almost everywhere.

By $V(t)$ we will denote the Cauchy operator of (2.1). Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ and let $\psi(.) : J \to RL(X)$ be continuous for any $t \in J$ operator-function.

Definition 2.1. A function $u(.) : J \to X$ is said to be ψ-bounded on J if $\psi(t)u(t)$ is bounded on J.

A function $f(.) : J \to X$ is said to be ψ-integrally bounded on J if it is measurable and there exists a positive constant m such that $\int_t^{t+1} |\psi(\tau)f(\tau)|d\tau \leq m$ for any $t \in J$.

A function $f(.) : J \to X$ is said to be ψ-Bochner integrable on J if it is measurable and $\int_J |\psi(\tau)f(\tau)|d\tau < \infty$.

Let $C_\psi(X)$ denote the Banach space of all ψ-bounded and continuous functions with values in X with the norm

$$|||f|||_{C_\psi} = \sup_{t\in J} |\psi(t)f(t)|.$$

Let $M_\psi(X)$ denote the Banach space of all ψ-integrally bounded functions with values in X with the norm

$$|||f|||_{M_\psi} = \sup_{t\in J} \int_t^{t+1} |\psi(s)f(s)|\mathrm{d}s.$$

Let $L_\psi(X)$ denote the Banach space of all ψ-Bochner integrable on J functions with values in X with the norm

$$|||f|||_{L_\psi} = \int_J |\psi(s)f(s)|\mathrm{d}s.$$

Definition 2.2. The equation (2.1) is said to has a ψ-exponential dichotomy on J if there exist a pair mutually complementary projections P_1 and $P_2 = I - P_1$ and positive constants N_1, N_2, ν_1, ν_2 such that

$$\|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \leq N_1e^{-\nu_1(t-s)} \quad (0 \leq s \leq t) \tag{2.3}$$

$$\|\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\| \leq N_2e^{-\nu_2(s-t)} \quad (0 \leq t \leq s) \tag{2.4}$$

Equation (2.1) is said to has a ψ-ordinary dichotomy on J if (2.3) and (2.4) hold with $\nu_1 = \nu_2 = 0$.

Remark 2.3. For $\psi(t) = I$ for all $t \in J$ we obtain the notion of exponential and ordinary dichotomy in [5, 7, 13].

Definition 2.4. Equation (2.1) is said to have a ψ-bounded growth on J if for some fixed $h > 0$ there exists a constant $C \geq 1$ such that every solution $x(t)$ of (2.1) satisfies

$$|\psi(t)x(t)| \leq C|\psi(s)x(s)| \quad (0 \leq s \leq t \leq s+h) \tag{2.5}$$

3. Main results

Lemma 3.1. *Equation* (2.1) *has a ψ-exponential dichotomy on J with positive constants ν_1 and ν_2 if and only if there exist a pair mutually complementary projections P_1 and $P_2 = I - P_1$ and positive constants $M, \tilde{N}_1, \tilde{N}_2$ such that following inequalities are fulfilled*

$$|\psi(t)V(t)P_1\xi| \leq \tilde{N}_1 e^{-\nu_1(t-s)}|\psi(s)V(s)P_1\xi| \quad (\xi \in X, 0 \leq s \leq t) \tag{3.1}$$

$$|\psi(t)V(t)P_2\xi| \leq \tilde{N}_2 e^{-\nu_2(s-t)}|\psi(s)V(s)P_2\xi| \quad (\xi \in X, 0 \leq t \leq s) \tag{3.2}$$

$$\|\psi(t)V(t)P_1V^{-1}(t)\psi^{-1}(t)\| \leq M \quad (t \geq 0) \tag{3.3}$$

Proof. Let (2.1) have a ψ-exponential dichotomy on J. Then for any $x \in X$ from (2.3) it follows that

$$|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)x| \leq N_1 e^{-\nu_1(t-s)}|x| \quad (0 \leq s \leq t)$$

For $x = \psi(s)V(s)P_1\xi$ we obtain (3.1). The proof of(3.2) is analogous. Obviously the inequality (3.3) holds.

Now vice versa. Let (3.1), (3.2) and (3.3) are fulfilled. For any $x \in X$ we can choose $\xi = V^{-1}(s)\psi^{-1}(s)x$ and from (3.1) we obtain

$$\begin{aligned} |\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)x| &\leq \tilde{N}_1 e^{-\nu_1(t-s)}|\psi(s)V(s)P_1V^{-1}(s)\psi^{-1}(s)x| \\ &\leq M\tilde{N}_1 e^{-\nu_1(t-s)}|x| \quad (0 \leq s \leq t) \end{aligned}$$

Hence estimate (2.3) holds with $N_1 = M\tilde{N}_1$. The proof of (2.4) is analogous. $\square$

Let us explain in detail the importance of Lemma 3.1, which obviously can be taken as definition for ψ-exponential dichotomy on J instead of Definition 2.2.

The pair mutually complementary projections P_1 and $P_2 = I - P_1$ exists if and only if for some $t_0 \in J$ the space X decomposes into a direct sum of two closed subspaces $X = X_1 + X_2$.

Let us introduce the subspaces $X_k(t) = V(t)V^{-1}(t_0)X_k$ $(k = 1, 2,\ t \in J)$. Then $X_1(t_0) = X_1$ and $X_2(t_0) = X_2$. The projection functions corresponding to the subspaces $X_k(t)$ are

$$P_k(t) = V(t)P_kV^{-1}(t) \quad (k = 1, 2;\ t \in J).$$

And from the estimates (3.1) and (3.2) it follows, that the complemented subspace $X_1(t_0)$ is exactly the subspace of all initial values $x_1^0 \in X_1(t_0)$ such that the solutions $x_1(t) = V(t)V^{-1}(t_0)x_1^0$ starting at moment t_0 from the subspace $X_1(t_0)$ are ψ-bounded on J.

From the existence of the pair mutually complementary projections P_1 and $P_2 = I - P_1$, it follows also the existence of the projection functions

$$Q_k(t) = \psi(t)V(t)P_kV^{-1}(t)\psi^{-1}(t), \quad (k = 1, 2;\ t \in J)$$

which induce the decomposition of the spaces X into direct sums of closed subspaces $X = Q_1(t)X + Q_2(t)X = \tilde{X}_1(t) + \tilde{X}_2(t)$

The condition (3.3) for uniform bondedness of the projections $Q_k(t)$ ($k = 1, 2;\ t \in J$) is equivalent (see [7]) to the requirement, that the angular distance between the subspaces $\tilde{X}_1(t)$ and $\tilde{X}_2(t)$ cannot become arbitrary small under a variation of t. More precisely there must exist a constant $\gamma > 0$ such that

$$Sn(\tilde{X}_1(t), \tilde{X}_2(t)) \geq \gamma \quad (t \in J) \tag{3.4}$$

where the angular distance Sn between two subspaces Y_1 and Y_2 of a space Y is defined as

$$Sn(Y_1, Y_2) = \inf_{y_k \in Y_k, |y_k|=1, (k=1,2)} |y_1 + y_2| \tag{3.5}$$

The subspaces $\tilde{X}_k(t)$ and projection functions $Q_k(t)$, ($k = 1, 2;\ t \in J$) are introduced by us explicitly to fit the concept of the ψ-boundedness and ψ-dichotomy in an arbitrary Banach space. For $\psi(t) = I$ ($t \in J$) (i.e. for the exponential dichotomy in [7, 13]) $\tilde{X}_k(t) \equiv X_k(t)$ and $Q_k(t) \equiv P_k(t)$ ($k = 1, 2, t \in J$).

Lemma 3.2. *Equation* (2.1) *has ψ-bounded growth on J if and only if there exist positive constants $K \geq 1$ and $\alpha > 0$ such that*

$$\|\psi(t)V(t)V^{-1}(s)\psi^{-1}(s)\| \leq Ke^{\alpha(t-s)} \quad (0 \leq s \leq t) \tag{3.6}$$

Proof. Let us suppose that (2.1) has ψ-bounded growth; i.e. (2.5) holds. Let $t \geq s$ be two arbitrary positive numbers. Setting $n = [\frac{t-s}{h}]$ and $\eta = \frac{t-s}{h}$ we have $n \leq \eta \leq n + 1$. Then

$$\begin{aligned} |\psi(t)x(t)| &= |\psi(\eta h + s)x(\eta h + s)| \leq C|\psi(nh + s)x(nh + s)| \leq \dots \\ &\leq C^{n+1}|\psi(s)x(s)| \leq C^{\eta+1}|\psi(s)x(s)| \quad (0 \leq s \leq t) \end{aligned}$$

We can take $K = C$ and $\alpha = h^{-1} \ln C$. Obviously, $C^{\eta+1} = Ke^{\alpha(t-s)}$ and we have the estimate

$$|\psi(t)x(t)| \leq Ke^{\alpha(t-s)}|\psi(s)x(s)|.$$

For an arbitrary vector $\xi \in X$ we consider the solution $x(t)$ of (2.1) with $x(0) = V^{-1}(s)\psi^{-1}(s)\xi$. Therefore,

$$|\psi(t)V(t)V^{-1}(s)\psi^{-1}(s)\xi| \leq Ke^{\alpha(t-s)}|\xi|$$

is fulfilled for any $\xi \in X$. Hence the estimate (3.6) holds.

Vice versa - suppose that (3.6) holds. From $x(t) = V(t)V^{-1}(s)x(s)$ and the estimate (3.6) we obtain

$$|x(t)| \leq Ke^{\alpha(t-s)}|x(s)|$$

for some $K \geq 1$ and $\alpha > 0$. Then we can take $C = Ke^{\alpha h}$. Obviously $C \geq 1$. Hence (2.1) has ψ-bounded growth. □

Remark 3.3. The proof shows that the condition for ψ-bounded growth (and for bounded growth) of (2.1) is independent from the choice of h.

Remark 3.4. In the famous monograph by Coppel [5, p. 9], nessesary and sufficient condition for bounded growth are formulated with $K, \alpha \in \mathbb{R}$, which is an typing error. By Boi [2, Lemma 2.4] necessary and sufficient conditions for ψ-bounded growth are formulated with $K, \alpha > 0$, which is also wrong. The only correct necessary and sufficient condition for bounded and ψ-bounded growth which is independent from the choice of h must be formulated with $K \geq 1, \alpha > 0$.

Lemma 3.5. *If* (2.1) *has ψ-bounded growth on J, then* (3.3) *is a consequence of* (3.1) *and* (3.2).

Proof. Let suppose that (2.1) has ψ-bounded growth. Let $m \geq 0$. Then, using Lemma 3.2 we have the estimate

$$\|\psi(t+m)V(t+m)V^{-1}(t)\psi^{-1}(t)\| \leq Ke^{\alpha m} \tag{3.7}$$

with $K \geq 1$ and $\alpha > 0$.

Let us consider, for an arbitrary fixed $t \in J$, a pair unit vectors $y_k(t) \in \tilde{X}_k(t)$ $(k = 1, 2)$.

$$y_k(t) = \psi(t)V(t)P_kV^{-1}(t)\psi^{-1}(t)\omega_k \quad (\omega_k \in X, |y_k(t)| = 1, k = 1, 2)$$

Let $\xi_k = V^{-1}(t)\psi^{-1}(t)\omega_k$. From (3.1), (3.2) and (3.7) we obtain

$$|\psi(t+m)V(t+m)P_1\xi_1| \leq \tilde{N}_1e^{-\nu_1 m}|\psi(t)V(t)P_1\xi_1| = \tilde{N}_1e^{-\nu_1 m}, \tag{3.8}$$

$$|\psi(t+m)V(t+m)P_2\xi_2| \geq \tilde{N}_2^{-1}e^{\nu_2 m}|\psi(t)V(t)P_2\xi_2| = \tilde{N}_2^{-1}e^{\nu_2 m} \tag{3.9}$$

From

$$\begin{aligned}
&|\psi(t+m)V(t+m)(P_1\xi_1 + P_2\xi_2)| = \\
&= |\psi(t+m)V(t+m)V^{-1}(t)\psi^{-1}(t)\psi(t)V(t)(P_1\xi_1 + P_2\xi_2)| \\
&\leq \|\psi(t+m)V(t+m)V^{-1}(t)\psi^{-1}(t)\| \; |\psi(t)V(t)P_1\xi_1 + \psi(t)V(t)P_2\xi_2| \\
&\leq Ke^{\alpha m}|\psi(t)V(t)P_1\xi_1 + \psi(t)V(t)P_2\xi_2| \\
&= Ke^{\alpha m}|y_1(t) + y_2(t)|
\end{aligned}$$

we conclude that

$$\begin{aligned}
&|y_1(t) + y_2(t)| \geq \\
&\geq K^{-1}e^{-\alpha m}|\psi(t+m)V(t+m)P_1\xi_1 + \psi(t+m)V(t+m)P_2\xi_2| \\
&\geq K^{-1}e^{-\alpha m}(|\psi(t+m)V(t+m)P_2\xi_2)| - |\psi(t+m)V(t+m)P_1\xi_1|) \\
&\geq K^{-1}e^{-\alpha m}(\tilde{N}_2^{-1}e^{\nu_2 m} - \tilde{N}_1e^{-\nu_1 m}) = \gamma_m
\end{aligned}$$

Making reference to (3.5) it follows

$$Sn(\tilde{X}_1(t), \tilde{X}_2(t)) \geq \gamma_m$$

Taking m large enough the constant $\gamma_m > 0$ and we can conclude that the angular distance between the subspaces $\tilde{X}_1(t)$ and $\tilde{X}_2(t)$ is bounded from below. By Daleckii and Krein [7, Corollary 1.1, Chapter IV] this is equivalent to the boundedness from above of the corresponding projection function $Q_1(t)$. Hence (3.3) holds and the proof is complete. □

Theorem 3.6. *If the homogeneous equation* (2.1) *has ψ-exponential dichotomy on J, then the inhomogeneous equation* (2.2) *has for every ψ-bounded function $f(t) \in C_\psi(X)$ at least one ψ-bounded solution $x(t) \in C_\psi(X)$. This solution is*

$$x(t) = \int_0^t V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty V(t)P_2V^{-1}(s)f(s)\mathrm{d}s \tag{3.10}$$

Proof. Let us consider the function

$$\begin{aligned}
\tilde{x}(t) &= \int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s \\
&= \int_0^t \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s
\end{aligned}$$

$$- \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s$$

Because (2.2) has a ψ-exponential dichotomy on J, from (2.3), (2.4) and the condition for ψ-boundedness of $f(t)$ (i.e. the existence of a constant c such that $|\psi(t)f(t)| \le c$) we obtain the estimate

$$\begin{aligned}|\tilde{x}(t)| &\le \int_0^t \|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \; |\psi(s)f(s)|\mathrm{d}s \\ &\quad + \int_t^\infty \|\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\| \; |\psi(s)f(s)|\mathrm{d}s \\ &\le c(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2})\end{aligned}$$

Hence $|||\tilde{x}(t)|||_{C_\psi} \le (\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2})|||f(t)|||_{C_\psi}$; i.e. $\tilde{x}(t)$ is bounded on J.

Let $x(t) = \psi^{-1}(t)\tilde{x}(t)$. Obviously $x(t)$ is ψ-bounded on J. Then

$$x(t) = \psi(t)^{-1}\Big(\int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s\Big)$$

We have already proved, that the integrals exist. Then

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= A(t)\int_0^t V(t)P_1V^{-1}(s)f(s)\mathrm{d}s + V(t)P_1V^{-1}(t)f(t) \\ &\quad + V(t)P_2V^{-1}(t)f(t) - A(t)\int_t^\infty V(t)P_2V^{-1}(s)f(s)\mathrm{d}s \\ &= A(t)x(t) + V(t)P_1V^{-1}(t)f(t) + V(t)P_2V^{-1}(t)f(t) \\ &= A(t)x(t) + f(t)\end{aligned}$$

Hence the function

$$x(t) = \int_0^t V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty V(t)P_2V^{-1}(s)f(s)\mathrm{d}s$$

is a ψ-bounded solution of the inhomogeneous equation (2.2) on J. □

Remark 3.7. Let introduce the principal Green function of (2.2) with the projections P_1 and P_2 from the definition for ψ-exponential dichotomy

$$G(t,s) = \begin{cases} V(t)P_1V^{-1}(s) & (t > s) \\ -V(t)P_2V^{-1}(s) & (t < s)\end{cases} \tag{3.11}$$

Clearly G is continuous except at $t = s$ where it has a jump discontinuity. Then the solution (3.10) can be rewritten as

$$x(t) = \int_J G(t,s)f(s)\mathrm{d}s$$

Remark 3.8. Since $J = [0,\infty)$ then every ψ-bounded on J solution of equation (2.2),

$$x(t) = \int_0^\infty G(t,s)f(s)\mathrm{d}s$$

has an initial value

$$x(t) = \int_0^\infty G(0,s)f(s)\mathrm{d}s = -P_2\int_0^\infty V^{-1}(s)f(s)\mathrm{d}s$$

belonging to the subspace X_2.

We obtain the general form of the ψ-bounded solutions on J by adding to the already obtained solution an arbitrary ψ-bounded solution of the homogeneous equation (2.1). These are exactly the solutions that are initially in X_1.

Remark 3.9. The solution (3.10) remains ψ-bounded when the condition for ψ-boundedness of the function $f(t)$ is replaced by the more general condition for its ψ-integrally boundedness

$$\int_t^{t+1} |\psi(\tau)f(\tau)|d\tau \le m$$

Proof. We have the estimate

$$\begin{aligned}
&|\psi(t)x(t)|\\
&= |\psi(t)\int_J G(t,\tau)f(\tau)\mathrm{d}\tau|\\
&\le \int_J \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\ |\psi(\tau)f(\tau)|\mathrm{d}\tau\\
&= \int_{t\le\tau} \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\ |\psi(\tau)f(\tau)|\mathrm{d}\tau\\
&\quad + \int_{t\ge\tau} \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\ |\psi(\tau)f(\tau)|\mathrm{d}\tau\\
&\le N_2\int_{t\le\tau} e^{-\nu_2(\tau-t)}|\psi(\tau)f(\tau)|\mathrm{d}\tau + N_1\int_{t\ge\tau} e^{-\nu_1(t-\tau)}|\psi(\tau)f(\tau)|\mathrm{d}\tau\\
&\le N_2\int_{s\ge 0} e^{-\nu_2 s}|\psi(t+s)f(t+s)|\mathrm{d}s + N_1\int_{s\le 0} e^{\nu_1(s)}|\psi(t+s)f(t+s)|\mathrm{d}s\\
&\le N_2 m\sum_{k=0}^{\infty} e^{-\nu_2 k} + N_1 m\sum_{k=0}^{\infty} e^{-\nu_1 k}\\
&= \frac{N_2 m}{1-e^{-\nu_2}} + \frac{N_1 m}{1-e^{-\nu_1}}.
\end{aligned}$$

□

As was just shown, the ψ-exponential dichotomy of (2.1) is a sufficient condition for the existence of ψ-bounded solutions of the inhomogeneous equation (2.2) with ψ-bounded or ψ-integrally bounded free term.

Since our phase space is an arbitrary Banach space (i.e. it may be with infinite dimension), in order to explain the extent to which this condition is necessary we must introduce some additional assumptions.

Definition 3.10. The linear manifold X_1 consisting of the initial values x_0 of the solutions of equation (2.1) that are ψ-bounded on J is called the $\mathfrak{Y}_\psi$-set of this equation.

We will assume that X_1 is a complemented subspace; i.e., that it is closed and has a direct complement: $X = X_1 + X_2$.

In the finite-dimensional case this condition is automatically satisfied. In a Hilbert space the second part of the condition is superfluous since an orthogonal complement always exists.

We note, that this condition is essentially contained in the definition of ψ-exponential dichotomy of an equation, because a subspace is complemented if and only if there exists at least one projection that projects the space into this subspace.

Theorem 3.11. *Let $B_\psi(X)$ denote any of the Banach spaces $C_\psi(X)$, $M_\psi(X)$, $L_\psi(X)$. Suppose that equation* (2.2) *has for each function $f(t) \in B_\psi(X)$ at least one solution x that is ψ-bounded on J:*

$$|||x|||_{C_\psi} = \sup_{t\in J} |\psi(t)x(t)| < \infty.$$

Suppose further that the $\mathfrak{Y}_\psi$-set X_1 of equation (2.1) *is a complemented subspace and that X_2 is a complement of it. Then to each function $f(t) \in B_\psi(X)$ there corresponds an unique solution $x(t)$ that is ψ-bounded on J and initially in X_2 : $x(0) \in X_2$.*

This solution satisfies to the estimate

$$|||x|||_{C_\psi} \le K_{B_\psi} |||f|||_{B_\psi},$$

where $K_{B_\psi} > 0$ is a constant not depending on f.

Proof. Suppose $f(t) \in B_\psi(X)$. By hypothesis, there exists a solution $x(t) \in C_\psi(X)$ of equation (2.2). Let P_1 and P_2 be the mutually complementary projections on the subspaces X_1 and X_2.

We denote by $x_1(t)$ the solution of the corresponding homogeneous equation which satisfies the condition $x(0) = P_1x(0)$. This solution is ψ-bounded by definition of the subspace X_1. But then the solution $x_2(t) = x(t) - x_1(t)$ of the inhomogeneous equation for which $x_2(0) = x(0) - P_1x(0) = P_2x(0) \in X_2$ is also ψ-bounded.

The uniqueness follows from the fact that the difference of two such solutions would be bounded by a solution initially in X_2 of the homogeneous equation, which is possible only for the zero solution.

It remains for us to prove the last assertion of the lemma. We consider the space C_1 of all functions $x(t)$ that are solutions of equations of the form

$$x'(t) - A(t)x(t) = f(t)$$

under the conditions $x(0) \in X_2$ and $f(t) \in B_\psi(X)$. It was essentially shown above that the operator $Tx(t) = x'(t) - A(t)x(t)$ effects a one-to-one mapping of the linear space C_1 onto $B_\psi(X)$. If in C_1 we introduce the norm

$$|||x|||_{C_1} = |||x|||_{C_\psi} + |||Tx|||_{B_\psi}$$

the operator Tx automatically turns out to be continuous. If, in addition, the space C_1 turns out to be complete, the inverse operator T^{-1} will also be continuous by Banach's theorem, and the solution $x = T^{-1}f$ of equation (2.2) will then satisfy the estimate

$$|||x|||_{C_\psi} \le |||x|||_{C_1} \le \|T^{-1}\| \; |||f|||_{B_\psi}.$$

Thus it remains to prove the completeness of C_1. Let $\{x_n(t)\}$ be a Cauchy sequence in it. Such a sequence is also a Cauchy sequence in $C_\psi(X)$ and hence has a limit $x(t)$ in it. In this case clearly

$$x(0) = \lim_{n\to\infty} x_n(0) \in X_2.$$

In exactly the same way it follows that the sequence $\{f_n(t)\} = \{Tx_n(t)\}$ has a limit $f(t)$ in $B_\psi(X)$. Therefore for each $t \in J$

$$\begin{aligned} x(t) - x(0) &= \lim_{n\to\infty} \int_0^\infty x'_n(\tau)\mathrm{d}\tau \\ &= \lim_{n\to\infty} \int_0^\infty (f_n(\tau) + A(\tau)x_n(\tau))\mathrm{d}\tau \\ &= \int_0^\infty (f(\tau) + A(\tau)x(\tau))\mathrm{d}\tau \end{aligned}$$

which implies that $x(t)$ satisfies the equation $x'(t)-A(t)x(t) = f(t)$. Thus $x(t) \in C_1$ and, as easily seen, $|||x - x_n|||_{C_1} \to 0$ for $n \to 0$, i.e. C_1 is complete. The theorem is proved. □

Theorem 3.12. *In order for equation* (2.1) *to has ψ-ordinary dichotomy on J it is necessary and sufficient that its $\mathfrak{Y}_\psi$-set be a complemented subspace and that to each function $f(t) \in L_\psi(X)$ there corresponds at least one ψ-bounded solution on J of the inhomogeneous equation* (2.2).

Proof. The necessity of the second condition follows from Theorem 3.6 and Remark 3.9, because obviously $L_\psi(X) \subset M_\psi(X)$. The necessity of the first was noted in defining the $\mathfrak{Y}$-set.

Now the sufficiency. Let $\xi \in X$ be an arbitrary fixed vector and let us consider the function

$$f(t) = \begin{cases} \psi^{-1}(t)\xi & \text{for } \ s \le t \le s+h \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

where $s \ge 0$ and $h > 0$. Then $f \in L_\psi(X)$ and $|||f|||_{L_\psi} = h|\xi|$. The corresponding solution of (2.2) is

$$x(t) = \int_J G(t,\tau)f(\tau)\mathrm{d}\tau = \int_s^{s+h} G(t,\tau)\psi^{-1}(t)\xi\mathrm{d}\tau.$$

From Theorem 3.11, it follows the estimate

$$|\psi(t)x(t)| = |\int_s^{s+h} \psi(t)G(t,\tau)\psi^{-1}(t)\xi\mathrm{d}\tau| \le K_{L_\psi}h|\xi|.$$

It follows that

$$|\psi(t)G(t,\tau)\psi^{-1}(t)\xi| \le K_{L_\psi}|\xi|.$$

Hence, since ξ is arbitrary,

$$\|\psi(t)G(t,\tau)\psi^{-1}(t)\| \le K_{L_\psi}.$$

Thus (2.3) and (2.4) hold with $N_1 = N_2 = K_{L_\psi}$ and $\nu_1 = \nu_2 = 0$. Obviously (2.3) and (2.4) remains valid also in the excepted case $t = s$. □

Corollary 3.13. *In a finite-dimensional phase space the homogeneous equation* (2.1) *has ψ-ordinary dichotomy on J if and only if there corresponds to each function $f(t) \in L_\psi(X)$ at least one ψ-bounded solution on J of the inhomogeneous equation* (2.2).

Lemma 3.14. *Suppose that* (2.2) *has a ψ-bounded solution for every function $f \in C_\psi$ and let $r = K_{C_\psi}$. Let $x(t)$ be a solution of the corresponding homogeneous equation* (2.1) *and let*

$$x_1(t) = V(t)P_1V^{-1}(t)x(t), \quad x_2(t) = V(t)P_2V^{-1}(t)x(t).$$

If for some fixed $s \ge 0$ is fulfilled $|\psi(t)x_1(t)| \le N|\psi(s)x(s)|$ for $s \le t \le s + r$, then

$$|\psi(t)x_1(t)| \le eN|\psi(s)x(s)|e^{-r^{-1}(t-s)} \quad \text{for } s \le t < \infty.$$

If for some fixed $s \ge 0$ is fulfilled $|\psi(t)x_2(t)| \le N|\psi(s)x(s)|$ for $\max\{0, s-r\} \le t \le s$, then

$$|\psi(t)x_2(t)| \le eN|\psi(s)x(s)|e^{-r^{-1}(s-t)} \quad \text{for } 0 \le t \le s.$$

Proof. Let us take

$$f(t) = \chi(t)x(t)|\psi(t)x(t)|^{-1}$$

where $x(t) = V(t)\xi$ is a nontrivial solution of the homogeneous equation (2.1) and $\chi(t)$ be an arbitrary real valued function such that $0 \le \chi(t) \le 1$ for all $t \ge 0$ and $\chi(t) = 0$ for $f \ge t_1$. Then obviously $f \in C_\psi(X)$ and $|||f|||_{C_\psi} \le 1$. Hence by the arbitrary nature of $\chi(t)$ applying Theorem 3.11 we have with $r = K_{C_\psi}$, the estimate

$$|\psi(t)\int_{t_0}^{t_1} G(t,\tau)x(\tau)|\psi(\tau)x(\tau)|^{-1}\mathrm{d}\tau| \le r \quad (0 \le t_0 \le t_1,\ t \ge 0).$$

Putting $t_1 = t$ and respectively $t_0 = t$ we obtain

$$\begin{gathered}
|\psi(t)V(t)P_1\xi|\int_{t_0}^{t} |\psi(\tau)x(\tau)|^{-1}\mathrm{d}\tau \le r \quad (0 \le t_0 \le t), \\
|\psi(t)V(t)P_2\xi|\int_{t}^{t_1} |\psi(\tau)x(\tau)|^{-1}\mathrm{d}\tau \le r \quad (t \le t_1 \le \infty).
\end{gathered} \tag{3.13}$$

Replacing ξ by $P_1\xi$, respectively $P_2\xi$, it follows by integration that

$$\begin{gathered}
\int_{t_0}^{s} |\psi(\tau)V(\tau)P_1\xi|^{-1}\mathrm{d}\tau \le e^{-r^{-1}(t-s)} \int_{t_0}^{t} |\psi(\tau)V(\tau)P_1\xi|^{-1}\mathrm{d}\tau \quad (t_0 \le s \le t), \\
\int_{s}^{t_1} |\psi(\tau)V(\tau)P_1\xi|^{-1}\mathrm{d}\tau \le e^{-r^{-1}(s-t)} \int_{t}^{t_1} |\psi(\tau)V(\tau)P_1\xi|^{-1}\mathrm{d}\tau \quad (t \le s \le t_1).
\end{gathered} \tag{3.14}$$

Replacing t_0 by s and s by $s + r$ in the first inequality (3.14) and using the first assumption of the lemma, for $t \ge s + r$, we obtain

$$rN^{-1}|\psi(s)x(s)|^{-1} \le \int_{s}^{s+r} |\psi(\tau)x_1(\tau)|^{-1}\mathrm{d}\tau \le ee^{-r^{-1}(t-s)} \int_{s}^{t} |\psi(\tau)x_1(\tau)|^{-1}\mathrm{d}\tau$$

Using the first inequality (3.13), for $t \ge s + r$, we have

$$|\psi(t)x_1(t)| \le r\Big(\int_{s}^{t} |\psi(\tau)x_1(\tau)|^{-1}\mathrm{d}\tau\Big)^{-1} \le eN|\psi(s)x(s)|e^{-r^{-1}(t-s)}$$

Since obviously the same inequality holds for $s \le t \le s + r$, the first assertion of the lemma is proved.

The proof of the second assertion of the lemma is similar, using the second assumption of it and replacing s by $s - r$ and t_1 by s in the second inequality (3.14). $\square$

Theorem 3.15. *For equation* (2.1) *to be ψ-exponential dichotomous on J it is necessary and sufficient that its $\mathfrak{Y}_\psi$-set be a complemented subspace and that to each function $f(t) \in M_\psi(X)$ there corresponds at least one ψ-bounded solution on J of the inhomogeneous equation* (2.2).

Proof. The necessity of the second condition follows from Theorem 3.6 and Remark 3.9, while the necessity of the first was noted in defining the $\mathfrak{Y}$-set.

Now the sufficiency. Let the $\mathfrak{Y}_\psi$-set of the homogeneous equation (2.1) be a complemented subspace and suppose that to each function $f(t) \in M_\psi(X)$ there corresponds at least one ψ-bounded solution on J of the inhomogeneous equation (2.2). Since $C_\psi(X) \subset M_\psi(X)$ and $L_\psi(X) \subset M_\psi(X)$ the equation (2.2) has a ψ-bounded solution on J for every $f \in C_\psi(X)$ and for every $f \in L_\psi(X)$ too.

By Theorem 3.12 and its proof (2.3) and (2.4) hold with $N_1 = N_2 = K_{L_\psi}$ and $\nu_1 = \nu_2 = 0$. Hence the conditions of Lemma 3.14 are fulfilled with $N = K_{L_\psi}$ for every solution $x(t)$ of (2.1) and for every $s \geq 0$. Applying Lemma 3.14 we obtain (2.3) and (2.4) with $N_1 = N_2 = eK_{L_\psi}$ and $\nu_1 = \nu_2 = {K_{C_\psi}}^{-1}$. The theorem is proved. $\square$

Corollary 3.16. *In a finite-dimensional phase space the homogeneous equation* (2.1) *is ψ-exponential dichotomous on J if and only if there corresponds to each function $f(t) \in M_\psi(X)$ at least one ψ-bounded solution on J of the inhomogeneous equation* (2.2).

Theorem 3.17. *Suppose that* (2.1) *has ψ-bounded growth. For equation* (2.1) *to be ψ-exponential dichotomous on J it is necessary and sufficient that its $\mathfrak{Y}_\psi$-set be a complemented subspace and that to each function $f(t) \in C_\psi(X)$ there corresponds at least one ψ-bounded solution on J of the inhomogeneous equation* (2.2).

Proof. The necessity of the second condition follows from Theorem 3.6, while the necessity of the first was noted in defining the $\mathfrak{Y}$-set.

Now the sufficiency. Let assume that the equation (2.1) has ψ-bounded growth. From Lemma 3.2 it follows

$$\|\psi(t)V(t)V^{-1}(s)\psi^{-1}(s)\| \leq Ke^{\alpha(t-s)} \quad (0 \leq s \leq t)$$

where $K \geq 1$ and $\alpha > 0$ are constants. Because the initial conditions of Lemma 3.14 are fulfilled, replacing ξ by $V^{-1}(s)\psi^{-1}(s)\xi$ and putting $t_1 = \infty$ in the second inequality (3.13) we obtain for $t \leq s$,

$$\begin{aligned} |\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\xi| &\leq r\Big(\int_t^\infty |\psi(\tau)V(\tau)V^{-1}(s)\psi^{-1}(s)\xi|\Big)^{-1} \\ &\leq r\Big(K^{-1}|\xi|^{-1}\int_t^\infty e^{\alpha(s-\tau)}\Big)^{-1}. \end{aligned}$$

Thus

$$\|\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\| \leq \alpha rK \quad (t \leq s).$$

Analogously, we obtain

$$\|\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\| \leq \alpha rKe^{\alpha(t-s)} \quad (t \geq s)$$

and hence

$$\|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \leq (1+\alpha r)Ke^{\alpha(t-s)} \quad (t \geq s). \tag{3.15}$$

In the same way, from the first inequality (3.13) it follows

$$\|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \le \alpha r K\big(1-e^{-\alpha(t-s)}\big)^{-1} \quad (t>s). \tag{3.16}$$

Let $h=\alpha^{-1}\ln\frac{1+2\alpha r}{1+\alpha r}$. By using (3.16) for $t-s\ge h$ and (3.15) for $t-s\le h$ we obtain

$$\|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \le (1+2\alpha r)K \quad \text{for all } (t\ge s).$$

Now we can apply Lemma 3.14 with $N=(1+2\alpha r)K$ and obtain

$$\|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \le e(1+2\alpha r)Ke^{-r^{-1}(t-s)} \quad (0\le s\le t),$$
$$\|\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\| \le e\alpha rKe^{-r^{-1}(s-t)} \quad (0\le t\le s).$$

Thus (2.1) has a ψ-exponential dichotomy. □

Corollary 3.18. *In a finite-dimensional phase space the homogeneous equation* (2.1) *with ψ-bounded growth is ψ-exponential dichotomous on J if and only if there corresponds to each function $f(t)\in C_\psi(X)$ at least one ψ-bounded solution on J of the inhomogeneous equation* (2.2).

An important property of the ψ-exponential dichotomies is their roughness. That is, they are not destroyed by small perturbations of the coefficient operator. Let consider the perturbed equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (A(t)+B(t))\,x\,. \tag{3.17}$$

Theorem 3.19. *Suppose that the equation* (2.1) *has a ψ-exponential dichotomy on J. If $\delta=\sup_{t\in J}\|\psi(t)B(t)\psi^{-1}(t)\|$ is sufficient small, then the perturbed equation* (3.17) *has also a ψ-exponential dichotomy on J.*

Proof. Let us consider the inhomogeneous equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = (A(t)+B(t))\,x(t)+f(t), \tag{3.18}$$

and introduce the map

$$Tz(t)=\int_J G(t,\tau)\,(B(\tau)z(\tau)+f(\tau))\,\mathrm{d}\tau$$

First we shall prove that T maps C_ψ into itself. Using the same technic and notations as in the proofs of Theorem 3.6 and Remark 3.9, we obtain the estimate

$$\begin{aligned}
|\psi(t)Tz(t)| &= |\psi(t)\int_J G(t,\tau)\,(B(\tau)z(\tau)+f(\tau))\,\mathrm{d}\tau| \le \\
&\le \int_J \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\ \|\psi(\tau)B(\tau)\psi^{-1}(\tau)\|\ |\psi(\tau)z(\tau)|\mathrm{d}\tau \\
&\quad + \int_J \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\ |\psi(\tau)f(\tau)|\mathrm{d}\tau \\
&\le \delta c\Big(\frac{N_1}{\nu_1}+\frac{N_2}{\nu_2}\Big)+\frac{N_2 m}{1-e^{-\nu_2}}+\frac{N_1 m}{1-e^{-\nu_1}}.
\end{aligned}$$

Hence $Tz\in C_\psi$ and $T: C_\psi\to C_\psi$.

Now we will show that the map T is a contraction. Let $z_1,z_2\in C_\psi$. Then

$$|||Tz_1-Tz_2|||_{C_\psi}$$

$$
\begin{aligned}
&\leq |\psi(t) \int_J G(t,\tau)B(\tau)\left(z_1(\tau)-z_2(\tau)\right)\mathrm{d}\tau| \leq \\
&\leq \int_J \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\ \|\psi(\tau)B(\tau)\psi^{-1}(\tau)\|\ |\psi(\tau)(z_1(\tau)-z_1(\tau))|\mathrm{d}\tau \\
&\leq \delta\Big(\frac{N_1}{\nu_1}+\frac{N_2}{\nu_2}\Big)|||z_1-z_2|||_{C_\psi}.
\end{aligned}
$$

By selecting a sufficient small δ we can obtain $\delta\big(\frac{N_1}{\nu_1}+\frac{N_2}{\nu_2}\big)<1$ and the map T will be a contraction.

By the fixed point principle of Banach it follows, that the map T has an unique fixed point. Denoting this point by z we have

$$
z(t)=\int_J G(t,\tau)\big(B(\tau)z(\tau)+f(\tau)\big)\mathrm{d}\tau.
$$

Thus $z(t)$ is a solution of (3.18). Hence the equation (3.18) has for every ψ-integrally bounded function $f(t)$ at least a ψ-bounded solution. From Theorem 3.15 it follows that the equation (3.17) has a ψ-exponential dichotomy. $\square$

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