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# INFINITE SEMIPOSITONE PROBLEMS WITH INDEFINITE WEIGHT AND ASYMPTOTICALLY LINEAR GROWTH FORCING-TERMS

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Abstract. In this work, we study the existence of positive solutions to the singular problem

$$-\Delta_p u = \lambda m(x) f(u) - u^{-\alpha} \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\lambda$ is positive parameter, $\Omega$ is a bounded domain with smooth boundary, $0 < \alpha < 1$, and $f : [0, \infty] \to \mathbb{R}$ is a continuous function which is asymptotically p-linear at $\infty$. The weight function is continuous satisfies $m(x) > m_0 > 0$, $\|m\|_\infty < \infty$. We prove the existence of a positive solution for a certain range of $\lambda$ using the method of sub-supersolutions.

## 1. Introduction

In this article, we consider the positive solution to the boundary-value problem

$$\begin{gathered} -\Delta_p u = \lambda m(x) f(u) - \frac{1}{u^\alpha} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda$ is positive parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < \alpha < 1$, and $f : [0, \infty] \to \mathbb{R}$ is a continuous function which is asymptotically p-linear at $\infty$. The weight function $m(x)$ satisfies $m(x) \in C(\Omega)$ and $m(x) > m_0 > 0$ for $x \in \Omega$ and also $\|m\|_\infty = l < \infty$. We prove the existence of a positive solution for a certain range of $\lambda$.

We consider problem (1.1) under the following assumptions.

- (H1) There exist $\sigma_1 > 0$, $k > 0$ and $s_0 > 1$ such that $f(s) \geq \sigma_1 s^{p-1} - k$ for all $s \in [0, s_0]$;
- (H2) $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$ for some $\sigma > 0$.

Let $F(u) := \lambda m(x) f(u) - \frac{1}{u^\alpha}$. The case when $F(0) < 0$ (and finite) is referred to in the literature as a semipositone problem. Finding a positive solution for a semipositone problem is well known to be challenging (see [2, 10]). Here we consider the more challenging case when $\lim_{u\to 0^+} F(u) = -\infty$. which has received attention

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very recently and is referred to as an infinite semipositone problem. However, most of these studies have concentrated on the case when the nonlinear function satisfies a sublinear condition at $\infty$ (see [11, 12, 14]). The only paper to our knowledge dealing with an infinite semipositone problem with an asymptotically linear nonlinearity is [7], where the author is restricted to the case $p = 2$. Also here the existence of a positive solution is focused near $\lambda_1/\sigma$ , where $\lambda_1$ is the first eigenvalue of$-\Delta$. See also [1, 15], where asymptotically linear nonlinearities have been discussed in the case of a nonsingular semipositone problem and an infinite positone problem. Recently, in the case when $m(x) = 1$ problem (1.1) has been studied by Shivaji et al [8]. The purpose of this paper is to improve the result of [8] with weight $m$. We shall establish our existence results via the method of sub and super-solutions.

**Definition 1.1.** We say that $(\psi)$ (resp. $Z$) in $W^{1,p}(\Omega)\cap C(\overline{\Omega})$ are called a sub-solution (resp. super-solution) of (1.1), if $\psi$ satisfies

$$\begin{gathered}\int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x).\nabla w(x)dx \le \int_\Omega (\lambda m(x)f(\psi) - \frac{1}{\psi^\alpha})w(x)dx \quad \forall w\in W,\\ \psi>0 \quad \text{in } \Omega,\\ \psi=0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

and

$$\begin{gathered}\int_\Omega |\nabla Z|^{p-2}\nabla Z.\nabla w(x)dx \ge \int_\Omega (\lambda m(x)f(Z) - \frac{1}{Z^\alpha})w(x)dx \quad \forall w\in W,\\ Z>0 \quad \text{in } \Omega,\\ Z=0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.3}$$

where $W = \{\xi \in C_0^\infty(\Omega) : \xi \ge 0 \text{in } \Omega\}$

The following lemma was established by Cui [3].

**Lemma 1.2** ([3])**.** *If there exist sub-supersolutions $\psi$ and $Z$, respectively, such that $\psi \le Z$ on $\Omega$, Then* (1.1) *has a positive solution $u$ such that $\psi \le u \le Z$ in $\Omega$.*

In next Section, we will state and prove the existence of a positive solution for a certain range of $\lambda$.

## 2. Main result

Our main result for problem (1.1) reads as follows.

**Theorem 2.1.** *Assume* (H1)–(H2). *If there exist constants $s_0^*(\sigma,\Omega)$, $J(\Omega)$, $\underline{\lambda}$, and $\hat{\lambda} > \underline{\lambda}$ such that if $s_0 \ge s_0^*$ and $m_0\sigma_1/(l\sigma) \ge J$, then* (1.1) *has a positive solution for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$. Here $\mu_1$ is the principal eigenvalue of operator $-\Delta_p$ with zero Dirichlet boundary condition.*

*Proof.* By Anti-maximum principle [13], there exists $\xi = \xi(\Omega) > 0$ such that the solution $z_\mu$ of

$$\begin{gathered}-\Delta_p z - \mu|z|^{p-2}z = -1 \quad \text{in } \Omega,\\ z=0 \quad \text{on } \partial\Omega,\end{gathered}$$

for $\mu \in (\mu_1, \mu_1 + \xi)$, is positive in $\Omega$ and is such that $\frac{\partial z_\mu}{\partial\nu} < 0$ on $\partial\Omega$, where $\nu$ is outward normal vector at $\partial\Omega$.

Since $z_\mu > 0$ in $\Omega$ and $\frac{\partial z_\mu}{\partial \nu} < 0$ there exist $m > 0$, $A > 0$, and $\delta > 0$ such that $|\nabla z_\mu| \geq m$ in $\overline{\Omega}_\delta$ and $z_\mu \geq A$ in $\Omega \setminus \overline{\Omega}_\delta$, where $W = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$.

We prove the existence of a solution by the comparison method [4]. It is easy to see that any sub-solution of

$$\begin{gathered} -\Delta_p u = \lambda m_0 f(u) - \frac{1}{u^\alpha} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

is a sub-solution of (1.1). Also any super-solution of

$$\begin{gathered} -\Delta_p u = \lambda l f(u) - \frac{1}{u^\alpha} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

is a super-solution of (1.1), where $l$ is as defined above.

We first construct a supersolution for(1.1).Let $Z = M_\lambda e_p$, where $M_\lambda \gg 1$ and $e_p$ is the unique positive solution of

$$\begin{gathered} -\Delta_p e_p = 1 \quad \text{in } \Omega, \\ e_p = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

Let $\tilde{f}(s) = \max_{t \in [0,s]} f(t)$. Then $f(s) \leq \tilde{f}(s)$, $\tilde{f}(s)$ is increasing, and

$$\lim_{u \to +\infty} \frac{\tilde{f}(u)}{u^{p-1}} = 0\,.$$

Hence, we can choose $M_\lambda \gg 1$ such that

$$2\sigma \geq \frac{\tilde{f}(M_\lambda \|e_p\|_\infty)}{(M_\lambda \|e_p\|_\infty)^{p-1}}$$

Now let $\hat{\lambda} = 1/\big(2l\sigma \|e_p\|_\infty^{p-1}\big)$. For $\lambda \leq \hat{\lambda}$,

$$-\Delta_p Z = M_\lambda^{p-1} \geq \frac{\tilde{f}(M_\lambda \|e_p\|_\infty)}{2\sigma \|e_p\|_\infty^{p-1}} \geq \lambda l \tilde{f}(M_\lambda e_p) \geq \lambda l f(M_\lambda e_p) \geq \lambda l f(Z) - \frac{1}{Z^\alpha}.$$

Thus, $Z$ is a supersolution of (2.2); therefore $Z$ is a supersolution of (1.1) Define

$$\psi := k_0 z_\mu^{\frac{p}{p-1+\alpha}},$$

where $k_0 > 0$ is such that

$$\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2l\sigma \|e_p\|_\infty^{p-1}}\Big) \leq \min\Big\{\Big(\frac{m^p(1-\alpha)(p-1)p^{p-1}}{(p-1+\alpha)^p}\Big), \Big(\frac{p}{p-1+\alpha}\Big)^{p-1} A\Big\}.$$

Then

$$\nabla \psi = k_0 \Big(\frac{p}{p-1+\alpha}\Big) z_\mu^{\frac{1-\alpha}{p-1+\alpha}} \nabla z_\mu,$$

and

$$\begin{aligned}
-\Delta_p\psi &= -\operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi)\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\operatorname{div}(z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}|\nabla z_\mu|^{p-2}\nabla z_\mu)\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\Big\{(\nabla z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}).|\nabla z_\mu|^{p-2}\nabla z_\mu\\
&\quad + z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\Delta_p z_\mu\Big\}\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\Big\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}z_\mu^{\frac{-\alpha p}{p-1+\alpha}}|\nabla z_\mu|^p\\
&\quad + z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}(1-\mu z_\mu^{p-1})\Big\}\\
&= k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\\
&\quad - \frac{k_0^{p-1}p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{(p-1+\alpha)^p z_\mu^{\frac{\alpha p}{p-1+\alpha}}}.
\end{aligned} \tag{2.3}$$

Now we let $s_0^*(\sigma,\Omega)=k_0\|z_\mu^{\frac{p}{p-1+\alpha}}\|_\infty$. If we can prove that

$$-\Delta_p\psi \le \lambda m_0\sigma_1 k_0^{p-1}z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - \lambda k - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}, \tag{2.4}$$

then (H1) implies that $-\Delta_p\psi \le \lambda m_0 f(\psi) - \frac{1}{\psi^\alpha}$, and $\psi$ will be a subsolution of (1.1).

We will now prove (2.4) by comparing terms in (2.3) and (2.4). Let $\underline{\lambda} = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{m_0\sigma_1}$. For $\lambda \ge \underline{\lambda}$,

$$k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \le \lambda m_0\sigma_1 k_0^{p-1}z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \tag{2.5}$$

Also since $\lambda \le \hat{\lambda} = \frac{1}{2l\sigma\|e_p\|_\infty^{p-1}}$,

$$\begin{aligned}
\lambda k + \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} &\le \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \frac{k}{2l\sigma\|e_p\|_\infty^{p-1}}\\
&= \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}(1+\frac{kk_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2l\sigma\|e_p\|_\infty^{p-1}})\Big]
\end{aligned} \tag{2.6}$$

Now in $\Omega_\delta$, we have $|\nabla z_\mu| \ge m$, and by (2.2),

$$\frac{1}{k_0^{p-1+\alpha}}(1+\frac{kk_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2l\sigma\|e_p\|_\infty^{p-1}})] \le \frac{m^p(1-\alpha)(p-1)p^{p-1}}{(p-1+\alpha)^p}.$$

Hence

$$\lambda k + \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} \le \frac{k_0^{p-1}p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{(p-1+\alpha)^p z_\mu^{\frac{\alpha p}{p-1+\alpha}}} \quad \text{in } \Omega_\delta \tag{2.7}$$

From (2.5), (2.7) it can be seen that (2.4) holds in $\Omega_\delta$.

We will now prove (2.4) holds also in $\Omega \setminus \Omega_\delta$. Since $z_\mu \geq A$ in $\Omega \setminus \Omega_\delta$ and by (2.2) and (2.6) we obtain

$$\lambda k + \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} \leq \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu \leq k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \tag{2.8}$$

in $\Omega \setminus \Omega_\delta$.

From (2.5) and (2.8), (2.4) holds also in $\Omega \setminus \overline{\Omega}_\delta$. Thus $\psi$ is a positive subsolution of(1.1) if $\lambda \in [\underline{\lambda}, \hat{\lambda}]$. We can now choose $M_\lambda \gg 1$ such that $\psi \leq Z$. Let $J(\Omega) = 2\|e_p\|_\infty^{p-1}\mu\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}$. if $\frac{m_0\sigma_1}{l\sigma} \geq J$ it is easy to see that $\underline{\lambda} \leq \hat{\lambda}$, and for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$ we have a positive solution. This completes the proof □

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