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HYERS-ULAM STABILITY FOR GEGENBAUER DIFFERENTIAL EQUATIONS

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ABSTRACT. Using the power series method, we solve the non-homogeneous Gegenbauer differential equation

$$(1-x^2)y''(x)+n(n-1)y(x)=\sum_{m=0}^{\infty}a_m x^m.$$

Also we prove the Hyers-Ulam stability for the Gegenbauer differential equation.

1. Introduction

Let Y be a normed linear space and I be an open subinterval of $\mathbb{R}$. If for any function $f : I \to Y$ satisfying the differential inequality

$$\big\|a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)+h(x)\big\|\le\varepsilon$$

for all $x \in I$ and for some $\varepsilon \ge 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)+h(x)=0$$

such that $\|f(x)-f_0(x)\| \le K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ depends on ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$). We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer the reader to [2, 3, 7].

Apparently Obłoza [12, 13] was the first author who investigated the Hyers-Ulam stability of linear differential equations. Here, we cite a result by Alsina and Ger [1]: If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x)-y(x)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x)-f_0(x)| \le 3\varepsilon$ for any $x \in I$. This result by Alsina and Ger was generalized by Takahasi, Miura and Miyajima [16]. They proved that the Hyers-Ulam stability holds for the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [10, 11, 15]).

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Using the conventional power series method, the author investigated the general solution of the inhomogeneous linear first-order differential equation

$$y'(x) - \lambda y(x) = \sum_{m=0}^{\infty} a_m (x-c)^m,$$

where λ is a complex number and the convergence radius of the power series is positive. This result was applied for proving an approximation property of exponential functions in a neighborhood of c (see [6] and [4, 5]).

Throughout this article, we assume that ρ_1 is a positive real number or infinity. In Section 2, using an idea from [6], we investigate the general solution of the inhomogeneous Gegenbauer differential equation

$$\left(1-x^2\right)y''(x) + n(n-1)y(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.1}$$

where the power series has a radius of convergence greater than or equal to ρ_1. Moreover, we prove the Hyers-Ulam stability of the Gegenbauer differential equation (2.1) in a certain class of analytic functions.

2. General solution of (1.1)

For an integer $n \geq 2$, the second-order ordinary differential equation

$$\left(1-x^2\right)y''(x) + n(n-1)y(x) = 0 \tag{2.1}$$

is a kind of the ultraspherical or Gegenbauer differential equation and has a general solution of the form $y(x) = C_1 J_n(x) + C_2 H_n(x)$, where we denote by $J_n(x)$ and $H_n(x)$ the Gegenbauer functions which are expressed by using the Legendre functions of the first and second kind as follows:

$$J_n(x) = \frac{P_{n-2}(x) - P_n(x)}{2n-1}, \quad H_n(x) = \frac{Q_{n-2}(x) - Q_n(x)}{2n-1}.$$

The Gegenbauer differential equation (2.1) is encountered in hydrodynamics when describing axially symmetric Stokes flows [14]. We recall that ρ_1 is a positive real number or infinity.

Theorem 2.1. *Let n be an integer greater than 1 and let ρ_1 be the radius of convergence of power series $\sum_{m=0}^{\infty} a_m x^m$. Define $\rho := \min\{1, \rho_1\}$. Then every solution $y : (-\rho, \rho) \to \mathbb{C}$ of the inhomogeneous Gegenbauer differential equation (1.1) can be expressed as*

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \tag{2.2}$$

where the coefficients c_m's are given by

$$c_{2m} = \sum_{k=0}^{m-1} \frac{(2k)!a_{2k}}{(2m)!} \prod_{i=k+1}^{m-1} (2i-n)(2i+n-1),$$

$$c_{2m+1} = \sum_{k=0}^{m-1} \frac{(2k+1)!a_{2k+1}}{(2m+1)!} \prod_{i=k+1}^{m-1} (2i-n+1)(2i+n)$$

for each $m \in \mathbb{N}$ and $y_h(x)$ is a solution of the Gegenbauer differential equation (2.1).

Proof. Since each solution of (1.1) can be expressed as a power series in x, we put $y(x) = \sum_{m=0}^{\infty} c_m x^m$ in (1.1) to obtain

$$\begin{aligned}
&\big(1 - x^2\big) y''(x) + n(n-1) y(x) \\
&= \sum_{m=0}^{\infty} \big[(m+2)(m+1)c_{m+2} - (m-n)(m+n-1)c_m\big] x^m \\
&= \sum_{m=0}^{\infty} a_m x^m,
\end{aligned}$$

from which we obtain the recurrence formula

$$(m+2)(m+1)c_{m+2} - (m-n)(m+n-1)c_m = a_m \tag{2.3}$$

for all $m \in \mathbb{N}_0$.

Now we prove that the formula

$$\begin{aligned}
c_{2m} = &\sum_{k=0}^{m-1} \frac{(2k)! a_{2k}}{(2m)!} \prod_{i=k+1}^{m-1} (2i-n)(2i+n-1) \\
&+ \frac{c_0}{(2m)!} \prod_{i=0}^{m-1} (2i-n)(2i+n-1)
\end{aligned} \tag{2.4}$$

holds for any $m \in \mathbb{N}$: If we set $m = 1$ in (2.4), then we obtain $2c_2 + n(n-1)c_0 = a_0$ which coincides with (2.3) when $m = 0$. We assume that the formula (2.4) is true for some $m \in \mathbb{N}$. Then, it follows from (2.3) and the induction hypothesis that

$$\begin{aligned}
c_{2m+2} &= \frac{a_{2m}}{(2m+2)(2m+1)} + \frac{(2m-n)(2m+n-1)}{(2m+2)(2m+1)} c_{2m} \\
&= \frac{a_{2m}}{(2m+2)(2m+1)} + \sum_{k=0}^{m-1} \frac{(2k)! a_{2k}}{(2m+2)!} \prod_{i=k+1}^{m} (2i-n)(2i+n-1) \\
&\quad + \frac{c_0}{(2m+2)!} \prod_{i=0}^{m} (2i-n)(2i+n-1) \\
&= \sum_{k=0}^{m} \frac{(2k)! a_{2k}}{(2m+2)!} \prod_{i=k+1}^{m} (2i-n)(2i+n-1) \\
&\quad + \frac{c_0}{(2m+2)!} \prod_{i=0}^{m} (2i-n)(2i+n-1),
\end{aligned}$$

which can be obtained provided we replace m in (2.4) with $m+1$. Hence, we conclude that the formula (2.4) is true for all $m \in \mathbb{N}$. Similarly, we can prove the validity of the formula

$$\begin{aligned}
c_{2m+1} = &\sum_{k=0}^{m-1} \frac{(2k+1)! a_{2k+1}}{(2m+1)!} \prod_{i=k+1}^{m-1} (2i-n+1)(2i+n) \\
&+ \frac{c_1}{(2m+1)!} \prod_{i=0}^{m-1} (2i-n+1)(2i+n)
\end{aligned} \tag{2.5}$$

for all $m \in \mathbb{N}$.

Indeed, we can set $c_0 = c_1 = 0$ in (2.4) and (2.5). Under this assumption, we have

$$
\begin{aligned}
c_{2m} &= \sum_{k=0}^{m-1} \frac{(2k)! a_{2k}}{(2m)!} \prod_{i=k+1}^{m-1} (2i-n)(2i+n-1) \\
&= \sum_{k=0}^{[n/2]-1} \frac{(2k)! a_{2k}}{(2m)!} \prod_{i=k+1}^{m-1} (2i-n)(2i+n-1) \\
&\quad + \sum_{k=[n/2]}^{m-1} \frac{(2k)! a_{2k}}{(2m)!} \prod_{i=k+1}^{m-1} (2i-n)(2i+n-1) \\
&= \sum_{k=0}^{[n/2]-1} \frac{(2k)! a_{2k}}{(2m)!} \Big(\prod_{i=k+1}^{[n/2]} (2i-n)(2i+n-1) \Big) \Big(\prod_{i=[n/2]+1}^{m-1} (2i-n)(2i+n-1) \Big) \\
&\quad + \sum_{k=[n/2]}^{m-1} \frac{(2k)! a_{2k}}{(2m)!} \prod_{i=k+1}^{m-1} (2i-n)(2i+n-1).
\end{aligned}
$$

Hence, since $|2i-n||2i+n-1| < 2i(2i-1)$ for $i > [n/2]$, we obtain

$$
\begin{aligned}
|c_{2m}| &\le \sum_{k=0}^{[n/2]-1} \frac{(2k)! |a_{2k}|}{(2m)!} \Big(\prod_{i=k+1}^{[n/2]} |2i-n||2i+n-1| \Big) \Big(\prod_{i=[n/2]+1}^{m-1} (2i)(2i-1) \Big) \\
&\quad + \sum_{k=[n/2]}^{m-1} \frac{(2k)! |a_{2k}|}{(2m)!} \prod_{i=k+1}^{m-1} (2i)(2i-1) \\
&= \sum_{k=0}^{[n/2]-1} \frac{(2k)! |a_{2k}|}{(2m)!} \alpha_n(k) \prod_{i=[n/2]+1}^{m-1} (2i)(2i-1) \\
&\quad + \sum_{k=[n/2]}^{m-1} \frac{(2k)! |a_{2k}|}{(2m)!} \prod_{i=k+1}^{m-1} (2i)(2i-1),
\end{aligned}
$$

where $\alpha_n(k) := \prod_{i=k+1}^{[n/2]} |2i-n||2i+n-1|$ for $k \in \{0, 1, \dots, [n/2]-1\}$. Moreover, taking into account that $\prod_{i=k+1}^{m-1} (2i)(2i-1) = (2m-2)!/(2k)!$, we have

$$
|c_{2m}| \le \sum_{k=0}^{[n/2]-1} \frac{\alpha_n(k)|a_{2k}|}{2m(2m-1)} + \sum_{k=[n/2]}^{m-1} \frac{|a_{2k}|}{2m(2m-1)} \le \frac{1}{m} \sum_{k=0}^{m-1} \frac{\alpha_n |a_{2k}|}{2(2m-1)}, \tag{2.6}
$$

for all $m \in \mathbb{N}$, where $\alpha_n := \max\{\alpha_n(0), \alpha_n(1), \dots, \alpha_n([n/2]-1), 1\}$. Similarly, we obtain

$$
|c_{2m+1}| \le \frac{1}{m} \sum_{k=0}^{m-1} \frac{\beta_n |a_{2k+1}|}{2(2m+1)} \tag{2.7}
$$

for any $m \in \mathbb{N}$, where $\beta_n := \max\{\beta_n(0), \beta_n(1), \dots, \beta_n([n/2]-1), 1\}$ and $\beta_n(k) := \prod_{i=k+1}^{[n/2]} |2i-n+1||2i+n|$ for $k \in \{0, 1, \dots, [n/2]-1\}$.

It follows from (2.6), (2.7), and [9, Problem 8.8.1 (p)] that

$$
\limsup_{m\to\infty} |c_{2m}| \le \limsup_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \frac{\alpha_n |a_{2k}|}{2(2m-1)} \le \limsup_{m\to\infty} \frac{\alpha_n |a_{2m-2}|}{2(2m-1)} \le \limsup_{m\to\infty} |a_{2m-2}|
$$

and

$$\limsup_{m\to\infty} |c_{2m+1}| \le \limsup_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \frac{\beta_n |a_{2k+1}|}{2(2m+1)} \le \limsup_{m\to\infty} \frac{\beta_n |a_{2m-1}|}{2(2m+1)} \le \limsup_{m\to\infty} |a_{2m-1}|$$

which imply that the radius ρ_2 of convergence of the power series $\sum_{m=2}^{\infty} c_m x^m$ is not less than the radius ρ_1 of the power series $\sum_{m=0}^{\infty} a_m x^m$.

If we define $\rho_3 := \min\{\rho_0, \rho_1, \rho_2\}$, where $\rho_0 = 1$ is the radius of convergence of the general solution to (2.1), then $\rho = \rho_3$. According to [8, Theorem 2.1] and our assumption that $c_0 = c_1 = 0$, every solution $y : (-\rho_3, \rho_3) \to \mathbb{C}$ of the inhomogeneous Gegenbauer differential equation (1.1) can be expressed by (2.2). $\square$

3. Hyers-Ulam stability for (2.1)

Let n be an integer larger than 1 and let ρ_1 be a positive real number larger than 1 or infinity. We denote by $\tilde{C}$ the set of all functions $f : (-1, 1) \to \mathbb{C}$ with the following properties:

(a) $f(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ_1;

(b) There exists a constant $K \ge 0$ such that $\sum_{m=0}^{\infty} |a_m x^m| \le K |\sum_{m=0}^{\infty} a_m x^m|$ for all $x \in (-\rho_1, \rho_1)$, where $a_m = (m+2)(m+1)b_{m+2} - (m-n)(m+n-1)b_m$ for all $m \in \mathbb{N}_0$.

If we define

$$(y_1 + y_2)(x) = y_1(x) + y_2(x) \quad \text{and} \quad (\lambda y_1)(x) = \lambda y_1(x)$$

for all $y_1, y_2 \in \tilde{C}$ and $\lambda \in \mathbb{C}$, then $\tilde{C}$ is a vector space over the complex numbers. We remark that the set $\tilde{C}$ is a vector space.

In the following theorem, we investigate the Hyers-Ulam stability of the Gegenbauer differential equation (2.1) for functions in $\tilde{C}$.

Theorem 3.1. *If a function $y \in \tilde{C}$ satisfies the differential inequality*

$$\big|(1 - x^2) y''(x) + n(n-1) y(x)\big| \le \varepsilon \tag{3.1}$$

for all $x \in (-1, 1)$ and for some $\varepsilon \ge 0$, then there exist constants $C_1, C_2 > 0$ and a solution $y_h : (-1, 1) \to \mathbb{C}$ of the Gegenbauer differential equation (2.1) such that

$$|y(x) - y_h(x)| \le C_1 |x| \ln \frac{1 + |x|}{1 - |x|} + C_2 \Big(\ln \frac{1 + |x|}{1 - |x|} - 2|x| \Big)$$

for any $x \in (-1, 1)$.

Proof. According to (a), $y(x)$ can be expressed as $y(x) = \sum_{m=0}^{\infty} b_m x^m$ and it follows from (a) and (b) that

$$\begin{aligned} &(1 - x^2) y''(x) + n(n-1) y(x) \\ &= \sum_{m=0}^{\infty} \big[(m+2)(m+1) b_{m+2} - (m-n)(m+n-1) b_m\big] x^m \\ &= \sum_{m=0}^{\infty} a_m x^m \end{aligned} \tag{3.2}$$

for all $x \in (-1,1)$. By considering (3.1) and (3.2), we have

$$\Big|\sum_{m=0}^{\infty} a_m x^m\Big| \le \varepsilon$$

for any $x \in (-1,1)$. This inequality, together with (b), yields that

$$\sum_{m=0}^{\infty} \big|a_m x^m\big| \le K\Big|\sum_{m=0}^{\infty} a_m x^m\Big| \le K\varepsilon \tag{3.3}$$

for all $x \in (-1,1)$.

Now, it follows from Theorem 2.1, (3.2), and (3.3) that there exists a solution $y_h : (-1,1) \to \mathbb{C}$ of the Gegenbauer differential equation (2.1) such that

$$|y(x) - y_h(x)| \le \Big|\sum_{m=2}^{\infty} c_m x^m\Big| \le \sum_{m=1}^{\infty} |c_{2m}||x|^{2m} + \sum_{m=1}^{\infty} |c_{2m+1}||x|^{2m+1}$$

for all $x \in (-1,1)$. By (2.6) and (2.7), we moreover have

$$\begin{aligned}
&|y(x) - y_h(x)| \\
&\le \alpha_n \sum_{m=1}^{\infty} \frac{|x|^{2m}}{2(2m-1)} \frac{1}{m} \sum_{k=0}^{m-1} |a_{2k}| + \beta_n \sum_{m=1}^{\infty} \frac{|x|^{2m+1}}{2(2m+1)} \frac{1}{m} \sum_{k=0}^{m-1} |a_{2k+1}|
\end{aligned} \tag{3.4}$$

for all $x \in (-1,1)$. (See the proof of Theorem 2.1 for the definitions of α_n and β_n).

In view of (a) and (b), the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is ρ_1 which is larger than 1. This fact implies that

$$\sum_{m=0}^{\infty} |a_m| = \sum_{k=0}^{\infty} |a_{2k}| + \sum_{k=0}^{\infty} |a_{2k+1}| < \infty,$$

which again implies that

$$\lim_{k\to\infty} |a_{2k}| = 0, \quad \lim_{k\to\infty} |a_{2k+1}| = 0.$$

According to [9, Theorem 2.8.6], the sequences $\big\{|a_{2k}|\big\}$ and $\big\{|a_{2k+1}|\big\}$ are $(C,1)$ summable to 0; i.e.,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} |a_{2k}| = 0, \quad \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} |a_{2k+1}| = 0.$$

Thus, there exists a constant $C > 0$ such that

$$\frac{1}{m} \sum_{k=0}^{m-1} |a_{2k}| \le C, \quad \frac{1}{m} \sum_{k=0}^{m-1} |a_{2k+1}| \le C$$

for any $m \in \mathbb{N}$.

Hence, from (3.4) it follows that

$$|y(x) - y_h(x)| \le \frac{\alpha_n C}{2} \sum_{m=1}^{\infty} \frac{|x|^{2m}}{2m-1} + \frac{\beta_n C}{2} \sum_{m=1}^{\infty} \frac{|x|^{2m+1}}{2m+1} \tag{3.5}$$

for all $x \in (-1,1)$. Since

$$\frac{1}{2} \ln \frac{1+|x|}{1-|x|} = \sum_{m=1}^{\infty} \frac{|x|^{2m-1}}{2m-1} = \sum_{m=0}^{\infty} \frac{|x|^{2m+1}}{2m+1}$$

for $x \in (-1,1)$, it holds that

$$|y(x) - y_h(x)| \le C_1 |x| \ln \frac{1+|x|}{1-|x|} + C_2 \Big(\ln \frac{1+|x|}{1-|x|} - 2|x| \Big)$$

for any $x \in (-1,1)$, where we set

$$C_1 = \frac{\alpha_n C}{4}, \quad C_2 = \frac{\beta_n C}{4},$$

which completes the proof. □

According to the previous theorem, each approximate solution of the Gegenbauer differential equation (2.1) can be well approximated by an exact solution of the Gegenbauer differential equation in a (small) neighborhood of 0.

Corollary 3.2. *If a function $y \in \tilde{C}$ satisfies the differential inequality (3.1) for all $x \in (-1,1)$ and for some $\varepsilon \ge 0$, then there exists a solution $y_h : (-1,1) \to \mathbb{C}$ of the Gegenbauer differential equation (2.1) such that*

$$|y(x) - y_h(x)| = O\big(x^2\big)$$

as $x \to 0$, where $O(\cdot)$ denotes the Landau symbol (big-O).

Proof. According to Theorem 3.1 and (3.5), there exists a solution $y_h : (-1,1) \to \mathbb{C}$ of the Gegenbauer differential equation (2.1) such that

$$|y(x) - y_h(x)| \le \frac{\alpha_n C}{2} |x|^2 \sum_{m=1}^{\infty} \frac{|x|^{2m-2}}{2m-1} + \frac{\beta_n C}{2} |x|^3 \sum_{m=1}^{\infty} \frac{|x|^{2m-2}}{2m+1}$$

for any $x \in (-1,1)$, where we see the proof of Theorem 3.1 for the definition of C, which completes our proof. □

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