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EXISTENCE OF SOLUTIONS FOR A NEUMANN PROBLEM INVOLVING THE $p(x)$-LAPLACIAN

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Abstract. We study the existence and multiplicity of weak solutions for a parametric Neumann problem driven by the $p(x)$-Laplacian. Under a suitable condition on the behavior of the potential at 0^+, we obtain an interval such that when a parameter λ is in this interval, our problem admits at least one nontrivial weak solution. We show the multiplicity of solutions for potentials satisfying also the Ambrosetti-Rabinowitz condition. Moreover, if the right-hand side f satisfies the Ambrosetti-Rabinowitz condition, then we obtain the existence of two nontrivial weak solutions.

1. Introduction

In this article we are interested in the multiplicity of weak solutions of the Neumann problem

$$\begin{gathered} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $p \in C(\bar{\Omega})$, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator, a belongs to $L^\infty(\Omega)$ and $a_- := \operatorname{ess\,inf}_\Omega a(x) > 0$, λ is a positive parameter and ν is the outward unit normal to $\partial\Omega$. In this context we assume that $p \in C(\bar{\Omega})$ satisfies the condition

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty, \tag{1.2}$$

and that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(F1) there exist $a_1, a_2 \in [0, +\infty[$ and $q \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ for each $x \in \bar{\Omega}$, such that

$$|f(x,t)| \le a_1 + a_2|t|^{q(x)-1}$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \ge N. \end{cases} \tag{1.3}$$

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In recent years there has been an increasing interest in the study of variational problems and elliptic equations with variable exponent. We refer to [16, 18, 21] and references therein for general properties of the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. Many authors investigated the existence and multiplicity of solutions for problems involving the $p(x)$-Laplacian, with Neumann boundary conditions. We refer to [3, 14] for the existence of infinitely many solutions and to [8, 9, 10, 12, 17, 19, 20] for results concerning the existence of a finite number of them. Since in this paper we are interested in the latter case, we want to say something more about the results obtained in the last years. We observe that the solutions (three in most cases) are obtained as critical points of a suitable functional I and the main tool for achieving the existence of such points is a critical point result due to Ricceri [23] or some variants of it.

One of the first paper devoted to this topic is [19], where $f(x,t) = |t|^{q(x)-2}t - t$, with $2 < q(x) < p^-$ and $p(x) > N$. Later, Xiayang Shi and Xuanhao Ding in [25] extend the results of [19] to Carathéodory functions f satisfying a growth condition of type (F1), but with $1 < q(x) \leq q^+ < p^-$ and once again $p^- > N$.

A two parameter problem was studied first in [17] and then in [9], where f and g are continuous and satisfy our condition (F1) but with a more restrictive assumption for the variable exponents q and p. However, we emphasize that in both papers the authors need some additional hypotheses on the potentials F and G. For instance, in [17] we have a growth r for F and G, with $1 < r^- < r^+ < p^-$. Furthermore, to obtain their results they strengthen the hypotheses on F, for which they need sign assumptions. Also in [8], the authors have two parameters rather than one, but they deal with $p^- > N \geq 2$ (we do not have such restriction). In [12] the nonlinear term is $f + \lambda g$ with f and g continuous functions verifying our growth condition (F1) with respect to the second variable, but with the restrictions $p^+ < p^*(x)$ and $p^+ < q^-$.

Finally, Liu [15] takes $\lambda = 1$. Under a regularity assumption on $f(x,\cdot)$ and standard growth conditions on $f_u(x,u)$, he shows the existence of three nontrivial solutions: one positive, one negative and the third is nodal. In this paper we obtain multiplicity results for (1.1) weakening the assumptions present in most of the papers cited above. In fact we deal with a Carathéodory function and we avoid the restriction $p^- > N$ for the exponent p. Furthermore, we have no relation between q and p except for the standard $q(x) < p^*(x)$. We point out that the elliptic case has been investigated in [4]. The paper is arranged as follows: in Section 2 we list some auxiliary results that we need to prove our main theorems that are exposed in Section 3. Finally, in Section 4 we give some examples of functions verifying assumptions requested in our main results.

2. Preliminaries

Here and in the sequel, we assume that $p \in C(\bar{\Omega})$ satisfies condition (1.2). The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx < +\infty\}.$$

On $L^{p(x)}(\Omega)$ we consider the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)} dx \leq 1\Big\}.$$

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}. \tag{2.1}$$

With such norms, $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

The following result generalizes the well-known Sobolev embedding theorem.

Theorem 2.1 ([13, Proposition 2.5])**.** *Assume that $p \in C(\bar{\Omega})$ with $p(x) > 1$ for each $x \in \bar{\Omega}$. If $r \in C(\bar{\Omega})$ and $1 < r(x) < p^*(x)$ for all $x \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ where p^* is the critical exponent related to p defined in* (1.3).

In the sequel, we will denote by k_r the best constant for which one has

$$\|u\|_{L^{r(x)}(\Omega)} \le k_r \|u\| \tag{2.2}$$

for all $u \in W^{1,p(x)}(\Omega)$.

If we assume that

(H1) $a \in L^\infty(\Omega)$, with $a_- := \operatorname{ess\,inf}_\Omega a(x) > 0$,

then on $W^{1,p(x)}(\Omega)$ it is possible to consider the norm

$$\|u\|_a = \inf\Big\{\sigma > 0 : \int_\Omega \Big(\big|\frac{\nabla u(x)}{\sigma}\big|^{p(x)} + a(x)\big|\frac{u(x)}{\sigma}\big|^{p(x)}\Big)\,dx \le 1\Big\},$$

which is equivalent to that introduced in (2.1) (see [8]). In particular, if for $\alpha > 0$ and $h \in C(\bar{\Omega})$ with $1 < h^-$, we put

$$[\alpha]^h := \max\{\alpha^{h^-}, \alpha^{h^+}\}$$
$$[\alpha]_h := \min\{\alpha^{h^-}, \alpha^{h^+}\},$$

then it is easy to verify that

$$[\alpha]^{1/h} = \max\{\alpha^{1/h^-}, \alpha^{1/h^+}\}, \quad [\alpha]_{\frac{1}{h}} = \min\{\alpha^{1/h^-}, \alpha^{1/h^+}\}.$$

Now, starting from the definition of $\|\cdot\|_a$ and $\|\cdot\|_{L^{p(x)}(\Omega)}$ and using standard arguments, the following estimate is obtained

$$\frac{[a_-]_{1/p}}{1 + [a_-]_{1/p}} \|u\|_{W^{1,p(x)}(\Omega)} \le \|u\|_a \le (1 + \|a\|_\infty)^{1/p^-} \|u\|_{W^{1,p(x)}(\Omega)} \tag{2.3}$$

for each $u \in W^{1,p(x)}(\Omega)$.

Remark 2.2. If Ω is an open convex subset of $\mathbb{R}^N$ and the variable exponents r and p verify conditions $r^+ < p^{-*}$ and $p^- \ne N$, then it is possible to provide an upper estimate for the constant k_r in (2.2). We recall that in [7] (see Remark 3.4), an upper bound for the constant of the embedding $W^{1,h}(\Omega) \hookrightarrow L^q(\Omega)$ with $q \in [1, h^*[$ has been obtained when Ω is an open convex set of $\mathbb{R}^N$ and $h \ne N$. Precisely, denoted by $\tilde{k}_{h,q}$ such constant, one has

$$\|u\|_{L^q(\Omega)} \le \tilde{k}_{h,q} \|u\|_{a,W^{1,h}(\Omega)} \tag{2.4}$$

for each $u \in W^{1,h}(\Omega)$ where

$$\|u\|_{a,W^{1,h}(\Omega)} = \Big(\int_\Omega |\nabla u(x)|^h \, dx + \int_\Omega a(x)|u(x)|^h \, dx \Big)^{1/h}$$

and $\tilde{k}_{h,q}$ depends on the diameter of Ω, on the measure of Ω and on a_-. Now, if $p^- \neq N$ and $r^+ < p^{-*}$, starting from (2.4) with $q = r^+$ and $h = p^-$, for each $u \in W^{1,p(x)}(\Omega)$, it results

$$\|u\|_{L^{r^+}(\Omega)} \leq \tilde{k}_{p^-,r^+} \|u\|_{a,W^{1,p^-}(\Omega)} . \tag{2.5}$$

Taking into account that (see for instance [18, Theorem 2.8]) $L^{p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ and $L^{r^+}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ with continuous embeddings and that the constants of such embeddings do not exceed $1 + |\Omega|$, one has

$$\begin{aligned}
\|u\|^{p^-}_{a,W^{1,p^-}(\Omega)} &\leq \|\nabla u\|^{p^-}_{L^{p^-}(\Omega)} + \|a\|_\infty \|u\|^{p^-}_{L^{p^-}(\Omega)} \\
&\leq (1 + |\Omega|)^{p^-} \|\nabla u\|^{p^-}_{L^{p(x)}(\Omega)} + \|a\|_\infty (1 + |\Omega|)^{p^-} \|u\|^{p^-}_{L^{p(x)}(\Omega)} \\
&\leq (1 + |\Omega|)^{p^-} (1 + \|a\|_\infty) \|u\|^{p^-}_{W^{1,p(x)}(\Omega)}
\end{aligned}$$

and so

$$\|u\|_{a,W^{1,p^-}(\Omega)} \leq (1 + |\Omega|)(1 + \|a\|_\infty)^{1/p^-} \|u\|_{W^{1,p(x)}(\Omega)} . \tag{2.6}$$

On the other hand, one has

$$\|u\|_{L^{r(x)}(\Omega)} \leq (1 + |\Omega|) \|u\|_{L^{r^+}(\Omega)} . \tag{2.7}$$

Starting from conditions (2.5), (2.6), (2.7) and (2.3), we obtain

$$\|u\|_{L^{r(x)}(\Omega)} \leq \tilde{k}_{p^-,r^+} (1 + |\Omega|)^2 (1 + \|a\|_\infty)^{1/p^-} \frac{1 + [a_-]_{1/p}}{[a_-]_{1/p}} \|u\|_a ,$$

and so

$$k_r \leq \tilde{k}_{p^-,r^+} (1 + |\Omega|)^2 (1 + \|a\|_\infty)^{1/p^-} \frac{1 + [a_-]_{1/p}}{[a_-]_{1/p}} . \tag{2.8}$$

Remark 2.3. Arguing as in the previous remark, if we denote by k_1 the best constant of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$, then we obtain

$$k_1 \leq \tilde{k}_{p^-,1} (1 + |\Omega|)(1 + \|a\|_\infty)^{1/p^-} \frac{1 + [a_-]_{1/p}}{[a_-]_{1/p}} . \tag{2.9}$$

We recall that, fixed $\lambda > 0$, a point $u \in W^{1,p(x)}(\Omega)$ is a weak solution to (1.1) if

$$\int_\Omega \Big(|\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) + a(x)|u|^{p(x)-2} uv \Big) \, dx = \lambda \int_\Omega f(x, u(x)) v(x) dx$$

holds for each $v \in W^{1,p(x)}(\Omega)$. To obtain one or more solutions to (1.1), fixed $\lambda > 0$, we denote by I_λ the energy functional

$$I_\lambda(\cdot) := \Phi(\cdot) - \lambda \Psi(\cdot),$$

where $\Phi, \Psi : W^{1,p(x)}(\Omega) \to \mathbb{R}$ are defined as follows

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + a(x)|u|^{p(x)} \Big) \, dx,$$

$$\Psi(u) = \int_\Omega F(x, u(x)) dx$$

for each $u \in W^{1,p(x)}(\Omega)$ and

$$F(x,\xi) := \int_0^\xi f(x,t)\,dt$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$. When I_λ is C^1 its critical points are weak solutions to (1.1). Similar arguments to those used in [19] and in [12] imply that Φ is sequentially weakly lower semi-continuous and is a C^1 functional in $W^{1,p(x)}(\Omega)$, with the derivative given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \Big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + a(x)|u|^{p(x)-2}uv \Big)\,dx,$$

for any $u, v \in W^{1,p(x)}(\Omega)$. Moreover (see [8, Lemma 3.1]), Φ' is an homeomorphism. Finally we recall that in [8, Proposition 2.2] it was shown that Φ is in close relation with the norm $\|\cdot\|_a$. In fact, we have the following result.

Proposition 2.4. *Let $u \in W^{1,p(x)}(\Omega)$. Then*

(j) *If $\|u\|_a < 1$ then $\frac{1}{p^+}\|u\|_a^{p^+} \le \Phi(u) \le \frac{1}{p^-}\|u\|_a^{p^-}$.*
(jj) *If $\|u\|_a > 1$ then $\frac{1}{p^+}\|u\|_a^{p^-} \le \Phi(u) \le \frac{1}{p^-}\|u\|_a^{p^+}$.*

We observe that Ψ can be defined in the space $L^{q(x)}(\Omega)$. In fact, from [18, Theorems 4.1 and 4.2], we know that the growth condition (F1) imposed on f guarantees that the Nemytsky operator N_f defined by $N_f(u) = f(\cdot, u(\cdot))$ maps $L^{q(x)}(\Omega)$ in $L^{q'(x)}(\Omega)$ where $q'(x) = \frac{q(x)}{q(x)-1}$ and that is continuous and bounded. Before studying the regularity properties of Ψ, we introduce the functional $J : L^{q'(x)}(\Omega) \to (L^{q(x)}(\Omega))^*$ defined as

$$J(h)(w) := \int_\Omega h(x)w(x)\,dx$$

for each $h \in L^{q'(x)}(\Omega)$, $w \in L^{q(x)}(\Omega)$. From [11, Theorem 3.4.6], we know that J is an isomorphism from $L^{q'(x)}(\Omega)$ to $(L^{q(x)}(\Omega))^*$.

Lemma 2.5. *Under assumption* (F1) *Ψ is a continuously Gâteaux differentiable functional with*

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

for each $u, v \in W^{1,p(x)}(\Omega)$ and Ψ' is a compact operator.

Proof. In a standard way we obtain

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

from each $u, v \in W^{1,p(x)}(\Omega)$. If $\{u_n\} \to u$ in $W^{1,p(x)}(\Omega)$ then, for Theorem 2.1, $\{u_n\} \to u$ in $L^{q(x)}(\Omega)$. Thanks to the properties of the Nemytsky operator, one has $\{N_f(u_n)\} \to N_f(u)$ in $L^{q'(x)}(\Omega)$ and so $\{J(N_f(u_n))\} \to J(N_f(u))$ in $(L^{q(x)}(\Omega))^*$. This condition leads to $\{J(N_f(u_n))\} \to J(N_f(u))$ in $(W^{1,p(x)}(\Omega))^*$ and, taking into account that

$$J(N_f(u))(\cdot) = \Psi'(u)(\cdot)$$

for each $u \in W^{1,p(x)}(\Omega)$, we obtain the continuity of Ψ'. If we suppose that $\{u_n\} \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, then, thanks to the compactness of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow$

$L^{q(x)}(\Omega)$, $\{u_n\} \to u$ in $L^{q(x)}(\Omega)$ (up to a subsequence). This ensures the continuity of Ψ' on $L^{q(x)}(\Omega)$ and so its compactness. $\square$

To conclude this section we introduce two abstract results obtained by Bonanno in [5] and [6] that will allow us to obtain multiple solutions to (1.1). Before to recall them we give the following definition.

Definition 2.6. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(P.S.)^{[r]}$) if any sequence $\{u_n\}$ in X such that

- (α) $\{I(u_n)\}$ is bounded;
- (β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;
- (γ) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;

has a convergent subsequence.

The following abstract result is a particular case of [5, Theorem 5.1].

Theorem 2.7 ([6])**.** *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that:*

- (A1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,
- (A2) *for each $\lambda \in]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(P.S.)^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}[$, there is $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ such that $I'_\lambda(x_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, r[)$.

Remark 2.8. [5, Proposition 2.1] guarantees that if Φ is a sequentially weakly lower semicontinuous, coercive, continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse and Ψ is a Gâteaux differentiable function whose Gâteaux derivative is compact then the functional $\Phi - \Psi$ satisfies the $(P.S.)^{[r]}$ condition for each $r \in \mathbb{R}$.

The last abstract result that we will use in this paper is the following.

Theorem 2.9 ([6, Theorem 3.2])**.** *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)$ condition and it is unbounded from below. Then, for each

$$\lambda \in]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}[,$$

the functional I_λ admits two distinct critical points.

3. Main results

The first result guarantees the existence of one non trivial solution to problem (1.1).

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (F1) *and*

(F2)
$$\limsup_{t\to 0^+} \frac{\int_\Omega F(x,t)\,dx}{t^{p^-}} = +\infty.$$

Put $\lambda^* = \frac{1}{a_1 k_1 (p^+)^{1/p^-} + \frac{a_2}{q^-}[k_q]^q (p^+)^{\frac{q^+}{p^-}}}$, *where k_1 and k_q are given by* (2.2). *Then for each $\lambda \in]0,\lambda^*[$, problem* (1.1) *admits at least one nontrivial weak solution.*

Proof. Put $X := W^{1,p(x)}(\Omega)$ equipped by norm $\|\cdot\|_a$. We consider the functional
$$I_\lambda(\cdot) := \Phi(\cdot) - \lambda\Psi(\cdot)$$
introduced in the previous section and note that Φ and Ψ satisfy the regularity assumptions required in Theorem 2.7 as well as the condition (A2) for all $r, \lambda > 0$ (see Lemma 2.5 and Remark 2.8). Fixed $\lambda \in]0,\lambda^*[$, we choose $r = 1$ and verify condition (A1) of Theorem 2.7. By (F2) there exists
$$0 < \xi_\lambda < \min\Big\{1, \Big(\frac{p^-}{\|a\|_\infty |\Omega|}\Big)^{1/p^-}\Big\} \tag{3.1}$$
such that
$$\frac{p^- \int_\Omega F(x,\xi_\lambda)\,dx}{\xi_\lambda^{p^-}\|a\|_\infty|\Omega|} > \frac{1}{\lambda}\,. \tag{3.2}$$
We denote by u_λ the function of X defined by $u_\lambda(x) = \xi_\lambda$ for each $x \in \Omega$ and observe that
$$\Phi(u_\lambda) \le \frac{1}{p^-}\|a\|_\infty|\Omega|[\xi_\lambda]^p < 1 \tag{3.3}$$
and
$$\Psi(u_\lambda) = \int_\Omega F(x,\xi_\lambda)\,dx\,.$$
We observe that condition (F1) implies
$$|F(x,t)| \le a_1|t| + \frac{a_2}{q(x)}|t|^{q(x)}$$
for each $(x,t) \in \Omega \times \mathbb{R}$. For each $u \in \Phi^{-1}(]-\infty,1])$ it results
$$\Psi(u) \le a_1\int_\Omega |u(x)|\,dx + \frac{a_2}{q^-}\int_\Omega |u(x)|^{q(x)}\,dx = a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q^-}\rho_q(u)\,.$$
[16, Theorem 1.3] and the embeddings $W^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ ensure
$$a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q^-}\rho_q(u) \le a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q^-}[\|u\|_{L^{q(x)}(\Omega)}]^q \le a_1k_1\|u\|_a + \frac{a_2}{q^-}[k_q\|u\|_a]^q \tag{3.4}$$
Taking into account that for each $u \in \Phi^{-1}(]-\infty,1])$, thanks to Proposition 2.4, one has
$$\|u\|_a \le (p^+)^{1/p^-},$$

conditions (3.2) and (3.4) lead to

$$\begin{aligned}\sup_{\Phi(u)\le 1}\Psi(u) &\le a_1k_1(p^+)^{1/p^-}+\frac{a_2}{q^-}[k_q]^q(p^+)^{\frac{q^+}{p^-}}\\ &=\frac{1}{\lambda^*}<\frac{1}{\lambda}\\ &<\frac{p^-\int_\Omega F(x,\xi_\lambda)\,dx}{\xi_\lambda^{p^-}\|a\|_\infty|\Omega|}<\frac{\Psi(u_\lambda)}{\Phi(u_\lambda)}\end{aligned}\tag{3.5}$$

and so condition (A1) of Theorem 2.7 is verified. Since $\lambda\in\big]\frac{\Phi(u_\lambda)}{\Psi(u_\lambda)},\frac{1}{\sup_{\Phi(u)\le 1}\Psi(u)}\big[$, Theorem 2.7 guarantees the existence of a local minimum point $\bar u$ for the functional I_λ such that

$$0<\Phi(\bar u)<1$$

and so $\bar u$ is a non-trivial weak solution of problem (1.1). □

To establish the existence of two solutions to problem (1.1), we assume that the nonlinear term f satisfies this Ambrosetti-Rabinowitz-type condition

(F3) there exist $\mu>p^+$ and $\beta>0$ such that

$$0<\mu F(x,\xi)\le \xi f(x,\xi)$$

for each $x\in\Omega$ and for $|\xi|\ge\beta$.

Lemma 3.2. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying* (F1) *and* (F3). *Then, for each $\lambda>0$, I_λ satisfies the (PS)-condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence for I_λ. Then:

$$|I_\lambda(u_n)|\le M \quad \text{for some } M>0 \text{ and all } n\ge 1\,, \tag{3.6}$$

$$I'_\lambda(u_n)\to 0 \quad \text{in } W^{1,p(x)}(\Omega)^*,\ \text{as } n\to\infty\,. \tag{3.7}$$

Due to (3.7) we can find $\overline{n}\in N$, such that

$$\begin{aligned}-I'_\lambda(u_n)(u_n)&=-\int_\Omega\Big(|\nabla u_n|^{p(x)}+a(x)|u_n|^{p(x)}\Big)\,dx+\lambda\int_\Omega f(x,u_n(x))u_n(x)\,dx\\ &\le\|u_n\|_a \quad \text{for all } n\ge\overline{n}\,.\end{aligned}\tag{3.8}$$

We argue by contradiction and we assume that $\{u_n\}$ is unbounded, so we can choose $\overline{n}$ such that $\|u_n\|_a>1$ for any $n\ge\overline{n}$. Our assumptions on f guarantee that we can find a number $A(\beta)>0$ such that for any $n\in\mathbb{N}$ one has:

$$\int_{\{x\in\Omega:|u_n(x)|\le\beta\}}\left(f(x,u_n(x))u_n(x)-\mu F(x,u_n(x))\right)\,dx\ge -A(\beta). \tag{3.9}$$

Gathering (3.6), (3.8), (3.9) and taking into account (jj) of Proposition 2.4, for n large enough we obtain

$$\begin{aligned}
&\mu \cdot M + \|u_n\|_a \\
&\geq \mu I_\lambda(u_n) - I'_\lambda(u_n)(u_n) \\
&= \int_\Omega \frac{\mu - p(x)}{p(x)} \Big(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)} \Big)\, dx \\
&\quad + \lambda \int_{\{x\in\Omega: |u_n(x)| \leq \beta\}} \left(f(x,u_n(x))u_n(x) - \mu F(x,u_n(x)) \right) dx \\
&\quad + \lambda \int_{\{x\in\Omega: |u_n(x)| \geq \beta\}} \left(f(x,u_n(x))u_n(x) - \mu F(x,u_n(x)) \right) dx \\
&\geq (\mu - p^+)\Phi(u_n) - \lambda A(\beta) \\
&\geq \frac{(\mu - p^+)}{p^+} \|u_n\|_a^{p^-} - \lambda A(\beta)\,,
\end{aligned} \tag{3.10}$$

which contradicts the unboundedness of $\{u_n\}$, since $p^- > 1$. So $\{u_n\}$ is bounded, so, taking into account that Ψ' is compact, we obtain the existence of a convergent subsequence. □

Theorem 3.3. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (F1) *and* (F3)*. Then, for each $\lambda \in]0, \lambda^*[$, where λ^* is the constant introduced in the statement of Theorem 3.1, problem* (1.1) *admits at least two distinct weak solutions.*

Proof. We choose $r = 1$, $X = W^{1,p(x)}(\Omega)$ and apply Theorem 2.9 to the functionals Φ and Ψ introduced before. Clearly, Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. From Lemma 3.2 we know that our functional $I_\lambda(\cdot) := \Phi(\cdot) - \lambda\Psi(\cdot)$ satisfies the $(P.S.)$ condition for each $\lambda > 0$. By integrating condition (F3), we can find $a_3 > 0$ such that

$$F(x,\xi) \geq a_3 \xi^\mu$$

for each $|\xi| \geq \beta_1 > \beta$. Fixed $k > \max\{\beta_1, 1\}$, we consider the function $\bar{u} \equiv k \in X$ and we observe that, for each $t > 1$ it results

$$I_\lambda(t\bar{u}) \leq \frac{1}{p^-} \|a\|_\infty |\Omega| t^{p^+} k^{p^+} - \lambda a_3 |\Omega| t^\mu k^\mu\,.$$

Since $\mu > p^+$, this condition implies that I_λ is unbounded from below. Finally, fixed $\lambda \in]0, \lambda^*[$ and taking into account (3.5), we have

$$0 < \lambda < \frac{1}{\sup_{u\in\Phi^{-1}(]-\infty,1[)} \Psi(u)}$$

and so, the functional I_λ admits two distinct critical points that are weak solutions to problem (1.1). □

Remark 3.4. We observe that, if $f(x,0) \neq 0$, then Theorem 3.3 ensures the existence of two non trivial weak solutions for problem (1.1).

Remark 3.5. Taking into account Remark 2.2 and Remark 2.3, if Ω is an open convex subset of $\mathbb{R}^N$ and the variable exponents q and p verify conditions $q^+ < {p^-}^*$ and $p^- \neq N$, then it is possible to obtain a precise estimate of parameter λ^* in Theorems 3.1 and 3.3.

4. Examples

Now we give some applications of the previous results.

Example 4.1. Let a_1 and a_2 in $L^\infty(\Omega)$, with $\operatorname{ess\,inf}_{x\in\Omega} a_1(x) > 0$. We consider

$$f(x,t) = a_1(x) + a_2(x)|t|^{q(x)-1}$$

for each $(x,t) \in \Omega \times \mathbb{R}$ where $q \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ for each $x \in \bar{\Omega}$. We observe that condition (F1) of Theorem 3.1 is easily verified. Moreover, by integration we obtain

$$F(x,t) = a_1(x)t + \frac{a_2(x)}{q(x)} t^{q(x)}$$

for each $x \in \Omega$ and $t > 0$. This implies that

$$\lim_{t\to 0^+} \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,t)}{t^{p^-}} = +\infty$$

and so condition (F2) of Theorem 3.1 is satisfied.

Finally, we present an application of Theorem 3.3.

Example 4.2. We take the function f defined by

$$f(x,t) = a + bq(x)|t|^{q(x)-2}t \quad \text{for } (x,t) \in \Omega \times \mathbb{R},$$

where a and b are two positive constants and $p, q \in C(\bar{\Omega})$ satisfy the inequalities $1 < p^+ < q^- \le q(x) < p^*(x)$ for any $x \in \bar{\Omega}$. Fixed $p^+ < \mu < q^-$ and

$$\beta > \max\Big\{ \Big[\frac{a(\mu-1)}{b(q^- - \mu)}\Big]^{\frac{1}{q^- -1}},\ \big(\frac{a}{b}\big)^{\frac{1}{q^- -1}},\ 1 \Big\}.$$

We prove that f fulfills the assumptions requested in Theorem 3.3. Condition (F1) of Theorem 3.3 is easily verified. Taking into account that

$$F(x,t) = at + b|t|^{q(x)} \quad \text{for } (x,t) \in \Omega \times \mathbb{R}, \tag{4.1}$$

and $\beta > (\frac{a}{b})^{\frac{1}{q^- -1}}$, one has

$$F(x,t) \ge -a|t| + b|t|^{q(x)} = |t|(-a + b|t|^{q(x)-1}) > 0$$

for each $x \in \Omega$ and for $|t| \ge \beta$. Moreover, the assumption $\beta > [\frac{a(\mu-1)}{b(q^- -\mu)}]^{\frac{1}{q^- -1}}$ leads to the following inequality

$$b(q(x) - \mu)|t|^{q(x)-1} \ge b(q^- - \mu)\beta^{q^- -1} \ge a(\mu - 1)$$

for each $x \in \Omega$ and $t \ge \beta$. This implies that

$$\mu F(x,t) \le t f(x,t)$$

holds for each $x \in \Omega$ and $|t| \ge \beta$ and so condition (F3) is verified.

Remark 4.3. We observe that the function f in Example 4.2 satisfies the condition $f(x,0) \ne 0$. This implies that problem (1.1) admits at least two non trivial distinct solutions (see Remark 3.4).

References

[1] Bonanno, G.; Chinnì, A.; *Discontinuous elliptic problems involving the $p(x)$-Laplacian*, Math. Nachr., **284**, n.5,6 (2011), 639–652.

[2] Bonanno, G.; Chinnì, A.; *Multiple solutions for elliptic problems involving the p(x)-Laplacian*, Le Matematiche, **66** (2011), 105–113.

[3] Bonanno, G.; Chinnì, A.; *Existence results of infinitely many solutions for p(x)-Laplacian elliptic Dirichlet problems*, Complex Variables and Elliptic Equations,**57**, n. 11 (2012), 1233-246.

[4] Bonanno, G.; Chinnì, A.; *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent*, preprint.

[5] Bonanno, G.; *A critical point theorem via the Ekeland variational principle*, Nonlinear Analysis, **75** (2012), 2992–3007.

[6] Bonanno, G.; *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal., **1** (2012), 205–220.

[7] Bonanno, G.; Molica Bisci, G.; Rădulescu, V. D.; *Weak solutions and energy estimates for a class of nonlinear elliptic Neumann problems*, Advanced Nonlinear Studies, 13, no. 2 (2013), 373–390.

[8] Cammaroto, F.; Chinnì, A.; Di Bella, B.; *Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian*, Nonlinear Anal., **71** (2009), 4486–4492.

[9] Chao, Ji; *Remarks on the existence of three solutions for the p(x)-Laplacian equations*, Nonlinear Anal., **74** (2011), 2908–2915.

[10] Chinnì, A.; Livrea, R. *Multiple solutions for a Neumann-type differential inclusion problem involving the $p(\cdot)$-Laplacian*, Discrete and Continuous Dynamical Systems(Series S), **5 (4)** (2012), 753–764.

[11] Diening, L.; Harjulehto, P.; Hästö, P.; Ružička, M.; *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, **2017**, Springer-Verlag, Heidelberg (2011).

[12] Fan, X. L.; Deng, S. G.; *Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations*, Nonlinear Anal., **67** (2007), 3064–3075.

[13] Fan, X. L.; Zhang, Q. H.; *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal., **52** (2003), 1843–1852.

[14] Fan, X. L.; Ji, C.; *Existence of infinitely many solutions for a Neumann problem involving the p(x)-Laplacian*, J. Math. Anal. Appl., **334** (2007), 248–260.

[15] Liu, D.; *Existence of multiple solutions for a $p(x)$-Laplace equations*, Electron. J. Differential Equations, Vol. 2010(2010) **33**, 1-11.

[16] Fan, X. L.; Zhao, D.; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., **263** (2001), 424–446.

[17] Liu, Q.; *Existence of three solutions for $p(x)$-Laplacian equations*, Nonlinear Anal., **68** (2008), 2119–2127.

[18] Kováčik, O.; Rákosník, J.; *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math., **41** (1991), 592–618.

[19] Mihăilescu, M.; *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear Anal., **67** (2007), 1419–1425.

[20] Moschetto, D. S.; *A quasilinear Neumann problem involving the $p(x)$-Laplacian*, Nonlinear Analysis, **71** (2009), 2739–2743.

[21] Musielak, J.; *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034**, Springer, Berlin (1983).

[22] Rabinowitz, P.H.; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, C.B.M.S., A.M.S., **65** (1986).

[23] Ricceri, B.; *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401–410.

[24] Ružička, M.; *Electrorheological fluids: modeling and mathematical theory,* Springer-Verlag, Berlin 2000.

[25] Ding, X.; Shi, X.; *Existence and multiplicity of solutions for a general p(x)-Laplacian Neumann problem*, Nonlinear Analysis, **70** (2009), 3715-3720.

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