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GROWTH AND OSCILLATION OF DIFFERENTIAL POLYNOMIALS GENERATED BY COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. The main purpose of this article is to study the controllability of solutions to the linear differential equation

$$f^{(k)} + A(z)f = 0 \quad (k \geqslant 2).$$

We study the growth and oscillation of higher-order differential polynomials with meromorphic coefficients generated by solutions of the above differential equation.

1. Introduction and main results

In this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory [6, 13]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r,f)$ is the Nevanlinna characteristic function of f.

Definition 1.1 ([7, 13])**.** Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined as

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

Definition 1.2 ([6, 10])**.** The type of a meromorphic function f of order ρ $(0 < \rho < \infty)$ is defined as

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^\rho}.$$

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Definition 1.3 ([4, 13])**.** Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of $f(z)$ is defined as

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N(r, \frac{1}{f})}{\log r},$$

where $N(r, \frac{1}{f})$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}(r, \frac{1}{f})}{\log r},$$

where $\overline{N}(r, \frac{1}{f})$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.

For $k \geqslant 2$, consider the complex linear differential equation

$$f^{(k)} + A(z)f = 0 \tag{1.1}$$

and the differential polynomial

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f, \tag{1.2}$$

where A and d_j $(j = 0, 1, \dots, k)$ are meromorphic functions in the complex plane.

Chen [5] studied the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients and obtained the following result.

Theorem 1.4 ([5])**.** *For all non-trivial solutions f of*

$$f'' + A(z)f = 0, \tag{1.3}$$

the following statements hold:

(i) *If A is a polynomial with* $\deg A = n \geqslant 1$*, then*
$$\lambda(f - z) = \rho(f) = \frac{n+2}{2}.$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then*
$$\lambda(f - z) = \rho(f) = \infty,$$
$$\lambda_2(f - z) = \rho_2(f) = \rho(A).$$

After him, Wang, Yi and Cai [12] generalized the precedent theorem for the differential polynomial g_f with constant coefficients as follows.

Theorem 1.5 ([12])**.** *For all non-trivial solutions f* (1.3)*, the following statements hold:*

(i) *If A is a polynomial with* $\deg A = n \geqslant 1$*, then*
$$\lambda(g_f - z) = \rho(f) = \frac{n+2}{2}.$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then*
$$\lambda(g_f - z) = \rho(f) = \infty,$$
$$\lambda_2(g_f - z) = \rho_2(f) = \rho(A).$$

Theorem A has been generalized from entire to meromorphic solutions for higher order differential equations by Liu Ming-Sheng and Zhang Xiao-Mei [11] as follows:

Theorem 1.6 ([11]). *Suppose that $k \geqslant 2$ and $A(z)$ is a transcendental meromorphic function satisfying*

$$\delta(\infty, A) = \liminf_{r \to +\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0, \quad \rho(A) = \rho < +\infty.$$

Then every meromorphic solution $f \not\equiv 0$ of (1.1) satisfies that $f, f', f'', \dots, f^{(k)}$ have infinitely many fixed points and

$$\overline{\lambda}(f^{(j)} - z) = \rho(f) = +\infty, \quad (j = 0, 1, \dots, k),$$
$$\overline{\lambda}_2(f^{(j)} - z) = \rho_2(f) = \rho \quad (j = 0, 1, \dots, k).$$

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G} \subset \mathbb{C}$. If $\mathbf{G} = \mathbb{C}$, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{C})$. Special case of such differential subfield

$$\mathcal{L}_{p+1,\rho} = \{g \text{ meromorphic: } \rho_{p+1}(g) < \rho\},$$

where ρ is a positive constant. Laine and Rieppo [9] investigated the fixed points and iterated order of the second order differential equation (1.3) and obtained the following result.

Theorem 1.7 ([9]). *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, and let f be a transcendental meromorphic solution of equation (1.3). Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

Then $\rho_{p+1}(f) = \rho_p(A) = \rho$. Moreover, let

$$P[f] = P(f, f', \dots, f^{(m)}) = \sum_{j=0}^{m} p_j f^{(j)} \tag{1.4}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does vanish identically. Then for the fixed points of $P[f]$, we have $\overline{\lambda}_{p+1}(P[f] - z) = \rho$, provided that neither $P[f]$ nor $P[f] - z$ vanishes identically.

Remark 1.8 ([9, p. 904]). In Theorem 1.7, in order to study $P[f]$, the authors consider $m \leqslant 1$. Indeed, if $m \geqslant 2$, we obtain, by repeated differentiation of (1.3), that $f^{(k)} = q_{k,0} f + q_{k,1} f'$, $q_{k,0}$, $q_{k,1} \in \mathcal{L}_{p+1,\rho}$ for $k = 2, \dots, m$. Substitution into (1.4) yields the required reduction.

The main purpose of this paper is to study the growth and oscillation of the differential polynomial (1.2) generated by meromorphic solutions of equation (1.1). The method used in the proofs of our theorems is simple, and different, from the method in Laine and Rieppo [9]. Before we state our results, we define the sequence of functions $\alpha_{i,j}$ $(j = 0, \dots, k-1)$ by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \text{for } i = 1, \dots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases} \tag{1.5}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, & \text{for } i = 1, \dots, k-1, \\ d_0 - d_k A, & \text{for } i = 0. \end{cases} \tag{1.6}$$

We define also

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \dots & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \dots & \alpha_{k-1,1} \\ \vdots & & & \vdots \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \dots & \alpha_{k-1,k-1} \end{vmatrix} \tag{1.7}$$

and

$$\psi(z) = C_0\varphi + C_1\varphi' + \dots + C_{k-1}\varphi^{(k-1)}, \tag{1.8}$$

where C_j $(j = 0, \dots, k-1)$ are finite order meromorphic functions depending on $\alpha_{i,j}$ and $\varphi \not\equiv 0$ is a meromorphic function with $\rho(\varphi) < \infty$.

Theorem 1.9. *Let $A(z)$ be a meromorphic function of finite order. Let $d_j(z)$ $(j = 0, 1, \dots, k)$ be finite order meromorphic functions that are not all vanishing identically such that $h \not\equiv 0$. If $f(z)$ is an infinite order meromorphic solution of* (1.1) *with $\rho_2(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \infty$$

and

$$\rho_2(g_f) = \rho_2(f) = \rho.$$

Furthermore, if f is a finite order meromorphic solution of (1.1) *such that*

$$\rho(f) > \max\{\rho(A), \rho(d_j)\ (j = 0, 1, \dots, k)\}, \tag{1.9}$$

then

$$\rho(g_f) = \rho(f).$$

Remark 1.10. In Theorem 1.9, if we do not have the condition $h \not\equiv 0$, then the conclusions of Theorem 1.9 cannot hold. For example, if we take $d_k = 1, d_0 = A$ and $d_j \equiv 0$ $(j = 1, \dots, k-1)$, then $h \equiv 0$. It follows that $g_f \equiv 0$ and $\rho(g_f) = 0$. So, if $f(z)$ is an infinite order meromorphic solution of (1.1), then $\rho(g_f) = 0 < \rho(f) = \infty$, and if f is a finite order meromorphic solution of (1.1) such that (1.9) holds, then $\rho(g_f) = 0 < \rho(f)$.

Theorem 1.11. *Under the hypotheses of Theorem 1.9, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite order such that $\psi(z)$ is not a solution of* (1.1)*. If $f(z)$ is an infinite order meromorphic solution of* (1.1) *with $\rho_2(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty,$$
$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho.$$

Furthermore, if f is a finite order meromorphic solution of (1.1) *such that*

$$\rho(f) > \max\{\rho(A), \rho(\varphi), \rho(d_j) \quad (j = 0, 1, \dots, k)\}, \tag{1.10}$$

then

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f).$$

Corollary 1.12. *Let $A(z)$ be a transcendental entire function of finite order and let $d_j(z)$ $(j = 0, 1, \dots, k)$ be finite order entire functions that are not all vanishing identically such that $h \not\equiv 0$. If $f \not\equiv 0$ is a solution of* (1.1)*, then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \infty,$$
$$\rho_2(g_f) = \rho_2(f) = \rho(A) = \rho.$$

Corollary 1.13. *Under the hypotheses of Corollary 1.12, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that $\psi(z) \not\equiv 0$. Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty,$$
$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A).$$

Corollary 1.14. *Let $A(z)$ be a nonconstant polynomial and for $j = 0, 1, \dots, k$ let $d_j(z)$ be nonconstant polynomials that are not all vanishing identically such that $h \not\equiv 0$. If $f \not\equiv 0$ is a solution of* (1.1)*, then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \frac{\deg(A) + k}{k}.$$

Corollary 1.15. *Let $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) > 0$ such that $\delta(\infty, A) = \delta > 0$, and let $f \not\equiv 0$ be a meromorphic solution of* (1.1)*. Suppose, moreover, that either:*

(i) *all poles of f are uniformly bounded multiplicity, or*
(ii) *$\delta(\infty, f) > 0$.*

Let $d_j(z)$ $(j = 0, 1, \dots, k)$ be finite order meromorphic functions that are not all vanishing identically such that $h \not\equiv 0$. Then the differential polynomial (1.2) *satisfies $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) = \rho(A)$.*

Corollary 1.16. *Under the hypotheses of Corollary 1.15, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite order such that $\psi(z) \not\equiv 0$. Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty,$$
$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A).$$

2. Auxiliary lemmas

Lemma 2.1 ([1, 3])**.** *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \dots + A_1 f' + A_0 f = F \tag{2.1}$$

with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

The following lemma is a special case of the result due to Cao, Chen, Zheng and Tu [2].

Lemma 2.2. *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of* (2.1) *with*

$$\max\{\rho(A_j)\ (j = 0, 1, \dots, k-1), \rho(F)\} < \rho(f) < +\infty,$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

By using similar proofs as in [8, Propositions 5.1 and 5.5], we easily obtain the following lemma.

Lemma 2.3. *For all non-trivial solutions f of* (1.1) *the following statements hold:*

(i) *If A is a polynomial with* $\deg A = n \geqslant 1$*, then*

$$\rho(f) = \frac{n+k}{k}. \tag{2.2}$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then*

$$\rho(f) = \infty \quad \text{and} \quad \rho_2(f) = \rho(A). \tag{2.3}$$

Lemma 2.4 ([1])**.** *Let $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) > 0$ such that $\delta(\infty, A) = \delta > 0$, and let $f \not\equiv 0$ be a meromorphic solution of* (1.1)*. Suppose, moreover, that either:*

(i) *all poles of f are uniformly bounded multiplicity, or*
(ii) $\delta(\infty, f) > 0$*.*

Then $\rho(f) = \infty$ and $\rho_2(f) = \rho(A)$.

We remark that for $k = 2$, Lemma 2.4 was obtained by Laine and Rieppo in [9]. Using the properties of the order of growth and the definition of the type, we easily obtain the following result which we omit the proof.

Lemma 2.5. *Let f and g be meromorphic functions such that $0 < \rho(f), \rho(g) < \infty$ and $0 < \tau(f), \tau(g) < \infty$. Then*

(i) *If $\rho(f) > \rho(g)$, then*

$$\tau(f + g) = \tau(fg) = \tau(f). \tag{2.4}$$

(ii) *If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then*

$$\rho(f + g) = \rho(fg) = \rho(f) = \rho(g). \tag{2.5}$$

Lemma 2.6 ([6])**.** *Let f be a meromorphic function and let $k \geqslant 1$ be an integer. Then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f),$$

where $S(r, f) = O(\log T(r, f) + \log r)$, possibly outside of an exceptional set $E \subset (0, +\infty)$ with finite linear measure. If f is of finite order of growth, then

$$m(r, \frac{f^{(k)}}{f}) = O(\log r).$$

3. Proofs of main results

Proof of Theorem 1.9. Suppose that f is an infinite order meromorphic solution of (1.1) with $\rho_2(f) = \rho$. By (1.1), we have

$$f^{(k)} = -Af \tag{3.1}$$

which implies

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f = d_{k-1} f^{(k-1)} + \cdots + (d_0 - d_k A) f. \tag{3.2}$$

We can rewrite this euqality as

$$g_f = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.3}$$

where $\alpha_{i,0}$ are defined in (1.6). Differentiating both sides of equation (3.3) and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$\begin{aligned} g_f' &= \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} (\alpha_{i,0}' + \alpha_{i-1,0}) f^{(i)} - \alpha_{k-1,0} A f \\ &= \sum_{i=1}^{k-1} (\alpha_{i,0}' + \alpha_{i-1,0}) f^{(i)} + (\alpha_{0,0}' - \alpha_{k-1,0} A) f. \end{aligned} \tag{3.4}$$

We can rewrite the above equality as

$$g_f' = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.5}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0}, & \text{for } i = 1, \dots, k-1, \\ \alpha_{0,0}' - A\alpha_{k-1,0}, & \text{for } i = 0. \end{cases} \tag{3.6}$$

Differentiating both sides of (3.5) and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$\begin{aligned} g_f'' &= \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)} \\ &= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)} \\ &= \alpha_{0,1}' f + \sum_{i=1}^{k-1} (\alpha_{i,1}' + \alpha_{i-1,1}) f^{(i)} - \alpha_{k-1,1} A f \\ &= \sum_{i=1}^{k-1} (\alpha_{i,1}' + \alpha_{i-1,1}) f^{(i)} + (\alpha_{0,1}' - \alpha_{k-1,1} A) f \end{aligned} \tag{3.7}$$

which implies that

$$g_f'' = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{3.8}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha_{i,1}' + \alpha_{i-1,1}, & \text{for } i = 1, \dots, k-1, \\ \alpha_{0,1}' - A\alpha_{k-1,1}, & \text{for } i = 0. \end{cases} \tag{3.9}$$

By using the same method as above we can easily deduce that

$$g_f^{(j)} = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \quad j = 0, 1, \dots, k-1, \tag{3.10}$$

where

$$\alpha_{i,j}=\begin{cases}\alpha'_{i,j-1}+\alpha_{i-1,j-1}, & \text{for } i=1,\dots,k-1,\\ \alpha'_{0,j-1}-A\alpha_{k-1,j-1}, & \text{for } i=0\end{cases} \tag{3.11}$$

and

$$\alpha_{i,0}=\begin{cases}d_i, & \text{for } i=1,\dots,k-1,\\ d_0-d_kA, & \text{for } i=0.\end{cases} \tag{3.12}$$

By (3.3)–(3.12), we obtain the system of equations

$$\begin{gathered} g_f=\alpha_{0,0}f+\alpha_{1,0}f'+\dots+\alpha_{k-1,0}f^{(k-1)},\\ g'_f=\alpha_{0,1}f+\alpha_{1,1}f'+\dots+\alpha_{k-1,1}f^{(k-1)},\\ g''_f=\alpha_{0,2}f+\alpha_{1,2}f'+\dots+\alpha_{k-1,2}f^{(k-1)},\\ \dots\\ g_f^{(k-1)}=\alpha_{0,k-1}f+\alpha_{1,k-1}f'+\dots+\alpha_{k-1,k-1}f^{(k-1)}. \end{gathered} \tag{3.13}$$

By Cramer's rule, and since $h\not\equiv 0$ we have

$$f=\frac{1}{h}\begin{vmatrix} g_f & \alpha_{1,0} & \dots & \alpha_{k-1,0}\\ g'_f & \alpha_{1,1} & \dots & \alpha_{k-1,1}\\ \vdots & & & \vdots\\ g_f^{(k-1)} & \alpha_{1,k-1} & \dots & \alpha_{k-1,k-1}\end{vmatrix}. \tag{3.14}$$

Then

$$f=C_0g_f+C_1g'_f+\dots+C_{k-1}g_f^{(k-1)}, \tag{3.15}$$

where C_j are finite order meromorphic functions depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (3.11).

If $\rho(g_f)<+\infty$, then by (3.15), we obtain $\rho(f)<+\infty$, which is a contradiction. Hence $\rho(g_f)=\rho(f)=+\infty$.

Now, we prove that $\rho_2(g_f)=\rho_2(f)=\rho$. By (3.2) , we have $\rho_2(g_f)\leqslant\rho_2(f)$ and by (3.15), we have $\rho_2(f)\leqslant\rho_2(g_f)$. This yield $\rho_2(g_f)=\rho_2(f)=\rho$.

Furthermore, if f is a finite order meromorphic solution of equation (1.1) such that

$$\rho(f)>\max\{\rho(A),\rho(d_j)\ (j=0,1,\dots,k)\}, \tag{3.16}$$

then

$$\rho(f)>\max\{\rho(\alpha_{i,j}):i=0,\dots,k-1,j=0,\dots,k-1\}. \tag{3.17}$$

By (3.2) and (3.16) we have $\rho(g_f)\leqslant\rho(f)$. Now, we prove $\rho(g_f)=\rho(f)$. If $\rho(g_f)<\rho(f)$, then by (3.15) and (3.17), we obtain

$$\rho(f)\leqslant\max\{\rho(C_j)\ (j=0,\dots,k-1),\rho(g_f)\}<\rho(f)$$

and this is a contradiction. Hence $\rho(g_f)=\rho(f)$. □

Remark 3.1. From (3.15), it follows that the condition $h\not\equiv 0$ is equivalent to the condition $g_f,g'_f,g''_f,\dots,g_f^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite order.

Proof of Theorem 1.11. Suppose that f is an infinite order meromorphic solution of equation (1.1) with $\rho_2(f) = \rho$. Set $w(z) = g_f - \varphi$. Since $\rho(\varphi) < \infty$, then by Theorem 1.9 we have $\rho(w) = \rho(g_f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho$. To prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho$ we need to prove $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho$. By $g_f = w + \varphi$ and (3.15), we obtain

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi(z), \tag{3.18}$$

where

$$\psi(z) = C_0 \varphi + C_1 \varphi' + \cdots + C_{k-1} \varphi^{(k-1)}. \tag{3.19}$$

Substituting (3.18) in (1.1), we obtain

$$C_{k-1} w^{(2k-1)} + \sum_{i=0}^{2k-2} \phi_i w^{(i)} = -\big(\psi^{(k)} + A(z)\psi\big) = H, \tag{3.20}$$

where ϕ_i $(i = 0, \dots, 2k-2)$ are meromorphic functions with finite order. Since $\psi(z)$ is not a solution of (1.1), it follows that $H \not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho$; i. e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho$.

Suppose that f is a finite order meromorphic solution of (1.1) such that (1.10) holds. Set $w(z) = g_f - \varphi$. Since $\rho(\varphi) < \rho(f)$, then by Theorem 1.9 we have $\rho(w) = \rho(g_f) = \rho(f)$. To prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f)$ we need to prove $\overline{\lambda}(w) = \lambda(w) = \rho(f)$. Using the same reasoning as above, we obtain

$$C_{k-1} w^{(2k-1)} + \sum_{i=0}^{2k-2} \phi_i w^{(i)} = -\big(\psi^{(k)} + A(z)\psi\big) = F,$$

where C_{k-1}, ϕ_i $(i = 0, \dots, 2k-2)$ are meromorphic functions with finite order $\rho(C_{k-1}) < \rho(w)$, $\rho(\phi_i) < \rho(w)$ $(i = 0, \dots, 2k-2)$ and

$$\psi(z) = C_0 \varphi + C_1 \varphi' + \cdots + C_{k-1} \varphi^{(k-1)}, \ \rho(F) < \rho(w).$$

Since $\psi(z)$ is not a solution of (1.1), it follows that $F \not\equiv 0$. Then by Lemma 2.2, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(f)$; i. e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f)$. $\square$

Proof of Corollary 1.14. Suppose that $f \not\equiv 0$ is a solution of (1.1). Since A is a nonconstant polynomial, then by Lemma 2.3, we have $\rho(f) = \frac{\deg(A)+k}{k}$, which implies that

$$\rho(f) > \max\{\rho(A), \rho(d_j) \quad (j = 0, 1, \dots, k)\} = 0.$$

Thus, by Theorem 1.9, we obtain $\rho(g_f) = \rho(f) = \frac{\deg(A)+k}{k}$. $\square$

Proof of Corollary 1.15. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.1) such that: (i) all poles of f are uniformly bounded multiplicity, or that (ii) $\delta(\infty, f) > 0$. Then by Lemma 2.4, we have $\rho(f) = \infty$ and $\rho_2(f) = \rho(A)$. Now, by using Theorem 1.9, we obtain $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) = \rho(A)$. $\square$

4. Discussion and applications

In this section, we consider the differential equation

$$f''' + A(z)f = 0, \tag{4.1}$$

where $A(z)$ is a meromorphic function of finite order. It is clear that the difficulty of the study of the differential polynomial generated by solutions lies in the calculation of the coefficients $\alpha_{i,j}$. We explain here that by using our method, the calculation of the coefficients $\alpha_{i,j}$ can be deduced easily. We study for example the growth of the differential polynomial

$$g_f = f''' + f'' + f' + f. \tag{4.2}$$

We have

$$\begin{aligned} g_f &= \alpha_{0,0}f + \alpha_{1,0}f' + \alpha_{2,0}f'',\\ g_f' &= \alpha_{0,1}f + \alpha_{1,1}f' + \alpha_{2,1}f'',\\ g_f'' &= \alpha_{0,2}f + \alpha_{1,2}f' + \alpha_{2,2}f''. \end{aligned} \tag{4.3}$$

By (1.6) we obtain

$$\alpha_{i,0} = \begin{cases} 1, & \text{for } i = 1,2,\\ 1 - A, & \text{for } i = 0. \end{cases} \tag{4.4}$$

Now, by (3.6) we obtain

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0}, & \text{for } i = 1,2\\ \alpha_{0,0}' - A\alpha_{2,0}, & \text{for } i = 0. \end{cases}$$

Hence

$$\begin{aligned} \alpha_{0,1} &= \alpha_{0,0}' - A\alpha_{2,0} = -A' - A,\\ \alpha_{1,1} &= \alpha_{1,0}' + \alpha_{0,0} = 1 - A,\\ \alpha_{2,1} &= \alpha_{2,0}' + \alpha_{1,0} = 1. \end{aligned} \tag{4.5}$$

Finally, by (3.9) we have

$$\alpha_{i,2} = \begin{cases} \alpha_{i,1}' + \alpha_{i-1,1}, & \text{for} i = 1,2,\\ \alpha_{0,1}' - A\alpha_{2,1}, & \text{for } i = 0. \end{cases}$$

So, we obtain

$$\begin{aligned} \alpha_{0,2} &= \alpha_{0,1}' - A\alpha_{2,1} = -A'' - A' - A,\\ \alpha_{1,2} &= \alpha_{1,1}' + \alpha_{0,1} = -2A' - A,\\ \alpha_{2,2} &= \alpha_{2,1}' + \alpha_{1,1} = 1 - A. \end{aligned} \tag{4.6}$$

Hence

$$\begin{aligned} g_f &= (1-A)f + f' + f'',\\ g_f' &= (-A' - A)f + (1-A)f' + f'',\\ g_f'' &= (-A'' - A' - A)f + (-2A' - A)f' + (1-A)f'' \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} h &= \begin{vmatrix} 1-A & 1 & 1\\ -A' - A & 1-A & 1\\ -A'' - A' - A & -2A' - A & 1-A \end{vmatrix}\\ &= 3A' - A - AA' - AA'' + A^2 - A^3 + 2(A')^2 + 1. \end{aligned} \tag{4.8}$$

Suppose that $h \not\equiv 0$, by simple calculations we have

$$f = \frac{Ag_f'' + (-1-2A')g_f' + (1-A+2A'+A^2)g_f}{h} \tag{4.9}$$

and by different conditions on the solution f we can ensure that

$$\rho(g_f) = \rho(f''' + f'' + f' + f) = \rho(f).$$

Turning now to the problem of oscillation, for that we consider a meromorphic function $\varphi(z) \not\equiv 0$ of finite order. From (4.9), we obtain

$$f = \frac{Aw'' + (-1-2A')w' + (1-A+2A'+A^2)w}{h} + \psi(z), \tag{4.10}$$

where $w = g_f - \varphi$ and

$$\psi(z) = \frac{A\varphi'' + (-1-2A')\varphi' + (1-A+2A'+A^2)\varphi}{h}. \tag{4.11}$$

Hence

$$f = \frac{A}{h}w'' + C_1 w' + C_0 w + \psi, \tag{4.12}$$

where

$$C_1 = -\frac{1+2A'}{h}, \quad C_0 = \frac{1-A+2A'+A^2}{h}.$$

Substituting (4.12) into (4.1) , we obtain

$$\frac{A}{h}w^{(5)} + \sum_{i=0}^{4}\phi_i w^{(i)} = -\big(\psi^{(3)} + A(z)\psi\big),$$

where ϕ_i $(i = 0, \dots, 4)$ are meromorphic functions with finite order. Suppose that all meromorphic solutions $f \not\equiv 0$ of (4.1) are of infinite order and $\rho_2(f) = \rho$. If $\psi \not\equiv 0$, then by Lemma 2.1, we obtain

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = +\infty, \tag{4.13}$$

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho. \tag{4.14}$$

Suppose that f is a meromorphic solution of (4.1) of finite order such that

$$\rho(f) > \max\{\rho(A), \rho(\varphi)\}.$$

If $\psi^{(3)} + A(z)\psi \not\equiv 0$, then by Lemma 2.2, we obtain

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f).$$

Finally, we can state the following two results.

Theorem 4.1. *Let $A(z)$ be a transcendental entire function of finite order satisfying $0 < \rho(A) < \infty$ and $0 < \tau(A) < \infty$, and let $d_j(z)$ $(j = 0, 1, 2, 3)$ be finite order entire functions that are not all vanishing identically such that*

$$\max\{\rho(d_j)\ (j = 0, 1, 2, 3)\} < \rho(A).$$

If f is a nontrivial solution of (4.1) *, then the differential polynomial*

$$g_f = d_3 f^{(3)} + d_2 f'' + d_1 f' + d_0 f \tag{4.15}$$

satisfies

$$\rho(g_f) = \rho(f) = \infty,$$
$$\rho_2(g_f) = \rho_2(f) = \rho(A).$$

Theorem 4.2. *Under the hypotheses of Theorem 4.1, let $\varphi(z) \not\equiv 0$ be an entire function with finite order. If f is a nontrivial solution of* (4.1)*, then the differential polynomial $g_f = d_3 f^{(3)} + d_2 f'' + d_1 f' + d_0 f$ with $d_3 \not\equiv 0$ satisfies*
$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty, \tag{4.16}$$
$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A). \tag{4.17}$$

Proof of Theorem 4.1. Suppose that f is a nontrivial solution of (4.1). Then by Lemma 2.3,
$$\rho(f) = \infty,\ \rho_2(f) = \rho(A).$$
First, we suppose that $d_3 \not\equiv 0$. By the same reasoning as before we obtain that
$$h = \begin{vmatrix} H_0 & H_1 & H_2 \\ H_3 & H_4 & H_5 \\ H_6 & H_7 & H_8 \end{vmatrix},$$
where $H_0 = d_0 - d_3 A$, $H_1 = d_1$, $H_2 = d_2$, $H_3 = d_0' - (d_2 + d_3')A - d_3 A'$, $H_4 = d_0 + d_1' - d_3 A$, $H_5 = d_1 + d_2'$, $H_6 = d_0'' - (d_1 + 2d_2' + d_3'')A - (d_2 + d_3')A' - d_3 A''$, $H_7 = 2d_0' + d_1'' - (d_2 + 2d_3')A - 2d_3 A'$, $H_8 = d_0 + 2d_1' + d_2'' - d_3 A$. Then
$$\begin{aligned}
h = \Big(&3d_0d_1d_2 + 3d_0d_1d_3' + 3d_0d_2d_2' - 6d_0d_3d_1' + 3d_1d_2d_1' + 3d_1d_3d_0' \\
&+ d_0d_2d_3'' - 2d_0d_3d_2'' + d_1d_2d_2'' + d_1d_3d_1'' + d_2d_3d_0'' + 2d_0d_2'd_3' + 2d_1d_1'd_3' - 4d_2d_0'd_3' \\
&+ 2d_2d_1'd_2' + 2d_3d_0'd_2' - d_1d_2'd_3'' + d_1d_3'd_2'' + d_2d_1'd_3'' - d_2d_1''d_3' - d_3d_1'd_2'' \\
&+ d_3d_2'd_1'' - d_1^3 - 3d_0^2d_3 - 2d_1(d_2')^2 - 3d_1^2d_2' - 2d_3(d_1')^2 - d_2^2d_1'' - d_1^2d_3'' - 3d_2^2d_0'\Big)A \\
&+ \Big(2d_0d_2d_3' + 2d_0d_3d_2' - d_1d_2d_2' + 2d_1d_3d_1' - 4d_2d_3d_0' + d_1d_3d_2'' \\
&- d_2d_3d_1'' - 2d_1d_2'd_3' + 2d_2d_1'd_3' + 3d_0d_1d_3 + d_0d_2^2 - d_1^2d_2 + d_2^2d_1' - 2d_1^2d_3'\Big)A' \\
&+ \Big(d_2d_3d_1' + d_0d_2d_3 - d_1d_3d_2' - d_1^2d_3\Big)A'' + \Big(2d_2d_3d_3' - 3d_1d_3^2 + 2d_2^2d_3 - 2d_3^2d_2'\Big)AA' \\
&+ \Big(d_2^3 - 3d_1d_2d_3 - 3d_1d_3d_3' - 3d_2d_3d_2' - d_2d_3d_3'' - 2d_3d_2'd_3' \\
&+ 3d_0d_3^2 + 3d_3^2d_1' + 2d_2(d_3')^2 + 3d_2^2d_3' + d_3^2d_2''\Big)A^2 \\
&- d_3^3A^3 + 2d_2d_3^2(A')^2 - d_2d_3^2AA'' - 3d_0d_1d_0' - d_0d_1d_1'' - d_0d_2d_0'' - 2d_0d_0'd_2' \\
&+ d_1d_0''d_2' + d_2d_0'd_1'' - d_2d_1'd_0'' + d_0^3 + 2d_0(d_1')^2 + 3d_0^2d_1' + 2d_2(d_0')^2 \\
&+ d_1^2d_0'' + d_0^2d_2'' - 2d_1d_0'd_1' + d_0d_1'd_2'' - d_0d_2'd_1'' - d_1d_0'd_2''.
\end{aligned}$$
Finally, if $d_3 \equiv 0$, $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$, we have $h = d_3 \not\equiv 0$. Hence $h \not\equiv 0$.

By $d_3 \not\equiv 0$, $A \not\equiv 0$ and Lemma 2.5, we have $\rho(h) = \rho(A)$, hence $h \not\equiv 0$. For the cases (i) $d_3 \equiv 0$, $d_2 \not\equiv 0$; (ii) $d_3 \equiv 0, d_2 \equiv 0$ and $d_1 \not\equiv 0$, by using a similar reasoning as above we obtain $h \not\equiv 0$. By $h \not\equiv 0$, we obtain
$$f = \frac{1}{h} \begin{vmatrix} g_f & d_1 & d_2 \\ g_f' & d_0 + d_1' - d_3 A & d_1 + d_2' \\ g_f'' & 2d_0' + d_1'' - (d_2 + 2d_3')A - 2d_3 A' & d_0 + 2d_1' + d_2'' - d_3 A \end{vmatrix},$$
which we can write
$$f = \frac{1}{h}(D_0 g_f + D_1 g_f' + D_2 g_f''), \tag{4.18}$$

where

$$\begin{aligned}D_0 = \Big(&d_1d_2 - 2d_0d_3 + 2d_1d_3' + d_2d_2' - 3d_3d_1' - d_3d_2'' + 2d_2'd_3'\Big)A \\ &+ \Big(2d_1d_3 + 2d_3d_2'\Big)A' + A^2d_3^2 + 3d_0d_1' - 2d_1d_0' + d_0d_2'' - d_1d_1'' \\ &- 2d_0'd_2' + d_1'd_2'' - d_2'd_1'' + d_0^2 + 2(d_1')^2,\\ D_1 = \Big(&d_1d_3 - 2d_2d_3' - d_2^2\Big)A + d_2d_1'' - d_0d_1 - 2d_1d_1' + 2d_2d_0' - d_1d_2'',\\ D_2 &= d_2d_3A + d_1^2 - d_2d_1' + d_1d_2' - d_0d_2.\end{aligned}$$

If $\rho(g_f) < +\infty$, then by (4.18), we obtain $\rho(f) < +\infty$, and this is a contradiction. Hence $\rho(g_f) = \rho(f) = +\infty$.

Now, we prove that $\rho_2(g_f) = \rho_2(f) = \rho(A)$. By (4.15), we obtain $\rho_2(g_f) \leqslant \rho_2(f)$ and by (4.18) we have $\rho_2(f) \leqslant \rho_2(g_f)$. This yield $\rho_2(g_f) = \rho_2(f) = \rho(A)$. □

Proof of Theorem 4.2. Suppose that f is a nontrivial solution of (4.1). By setting $w = g_f - \varphi$ in (4.18), we have

$$f = \frac{1}{h}(D_0w + D_1w' + D_2w'') + \psi, \tag{4.19}$$

where

$$\psi = \frac{D_2\varphi'' + D_1\varphi' + D_0\varphi}{h}. \tag{4.20}$$

Since $d_3 \not\equiv 0$, then $h \not\equiv 0$. It follows from Theorem 4.1 that g_f is of infinite order and $\rho_2(g_f) = \rho(A)$. Substituting (4.19) into (4.1), we obtain

$$\frac{D_2}{h}w^{(5)} + \sum_{i=0}^{4}\phi_iw^{(i)} = -\big(\psi^{(3)} + A(z)\psi\big),$$

where ϕ_i $(i = 0, \dots, 4)$ are meromorphic functions with finite order. First, we prove that $\psi \not\equiv 0$. Suppose that $\psi \equiv 0$, then by (4.20) we obtain

$$D_2\varphi'' + D_1\varphi' + D_0\varphi = 0 \tag{4.21}$$

and by Lemma 2.5, we have

$$\rho(D_0) > \max\{\rho(D_1), \rho(D_2)\}. \tag{4.22}$$

By (4.21), we can write

$$D_0 = -\big(D_2\frac{\varphi''}{\varphi} + D_1\frac{\varphi'}{\varphi}\big).$$

Since $\rho(\varphi) < \infty$, by Lemma 2.6 we obtain

$$T(r, D_0) \leqslant T(r, D_1) + T(r, D_2) + O(\log r).$$

Then

$$\rho(D_0) \leqslant \max\{\rho(D_1), \rho(D_2)\},$$

which is a contradiction with (4.22). It is clear now that $\psi \not\equiv 0$ cannot be a solution of (4.1) because $\rho(\psi) < \infty$. Then, by Lemma 2.1 we

$$\overline{\lambda}(w) = \lambda(w) = \overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty,$$

$$\overline{\lambda}_2(w) = \lambda_2(w) = \overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A).$$

□

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