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PROPERTIES OF SOLUTIONS OF FOURTH-ORDER NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this article, we consider the existence and uniqueness of global solutions for a fourth-order nonlinear evolution equation which models the formation of facets and corners in the course of kinetically controlled crystal growth. Moreover, the existence of global attractor in H^2 and H^k $(k \geq 0)$ space is also considered.

1. Introduction

In the study of the formation of facets and corners in the course of kinetically controlled crystal growth [2], there arises the fourth-order nonlinear evolution equation

$$h_t + m\nabla^2 h + \nu\nabla^4 h = \frac{1}{2}(\nabla h)^2 + h_{xx}[ah_x^2 + bh_y^2] + h_{yy}[bh_x^2 + ah_y^2] + ch_{xy}h_xh_y.$$

Such equation is derived for the faceting of crystal surfaces with unstable orientations when there is no surface growth. The linear damping coefficient $\nu > 0$ characterizes the stabilizing effect of the additional energy of edges and determines their widths. The coefficient $m > 0$ characterize the linear faceting instability of the thermodynamically unstable surface, and the coefficients of the nonlinear terms determine the stable orientations of the appearing facets and the symmetry of the faceted structure. The coefficients a, b and c characterizing the stable orientation of facets are also taken to be positive.

In this article, we consider the 1D case of the above equation

$$u_t + \nu u_{xxxx} + mu_{xx} - \frac{1}{2}(u_x)^2 - a(u_x)^2u_{xx} = 0, \quad \text{in } Q_T, \tag{1.1}$$

with the Newmann boundary conditions

$$u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = u_{xxx}(1,t) = 0, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad \text{in } (0,1), \tag{1.3}$$

where $Q_T = (0,1) \times (0,T)$, ν, m and a are also positive constants.

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This article is organized as follows. In the next section, we establish the existence and uniqueness of global weak solution in the space $H^{4,1}(Q_T)$; In Section 3, by uniform a priori estimates methods, we show the existence of the global attractor in the space $H^2(0,1)$; In the last section, based on the iteration technique and regularity estimates for the semigroups, we study the existence of global attractor for problem (1.1)-(1.3) in a more generalized space $H^k(0,1)$, where $0 \le k < \infty$.

For notational convenience, we denote by $\|\cdot\|$ the norm of $L^2(0,1)$ with the usual inner product $(\cdot,\cdot)$, $\|\cdot\|_p$ denotes the norm of $L^p(0,1)$ for $1 \le p \le +\infty$ ($\|\cdot\|_2 = \|\cdot\|$), $\|\cdot\|_Y$ denotes the norm of any Banach space Y. In the following, C, C_i, C_i', $(i = 1, 2 \cdots)$ will represent generic positive constants that may change from line to line even if in the same inequality.

2. Existence and uniqueness of global solutions

In this section, we consider the existence and uniqueness of global weak solutions of the problem (1.1)-(1.3). First of all, we define

$$\begin{gathered}
L^\infty(0,1) = \{v; \|v\|_{L^\infty} = \text{ess,}\sup_{x\in(0,1)} |v| < +\infty\},\\
H_E^2(0,1) = \{v \in H^2(0,1); v_x(0,t) = v_x(1,t) = 0\},\\
H_E^4(0,1) = \{v \in H^4(0,1); v_x(x,t) = v_{xxx}(x,t) = 0, x = 0,1\},\\
H^{4,1}(Q_T) = \{v; v_t \in L^2(Q_T), v, v_x, v_{xx}, v_{xxx}, v_{xxxx} \in L^2(Q_T)\}.
\end{gathered}$$

Definition 2.1. A function $u(x,t)$ is called a weak solution to problem (1.1)-(1.3), if $u \in H^{4,1}(Q_T)$, and it satisfies

$$\iint_{Q_T} u_t v\,dx\,dt + \iint_{Q_T} (\nu u_{xxxx} + m u_{xx} - \frac{1}{2}u_x^2 - a u_x^2 u_{xx}) v\,dx\,dt = 0, \quad \forall v \in L^2(Q_T).$$

From the classical approach, it is not difficult to conclude that (1.1)-(1.3) admits a unique solution local in time. So, to obtain the result on the global solution, it is sufficient to make a priori estimates.

Theorem 2.2. *Assume that $u_0 \in H_E^2(0,1)$ and $T > 0$, then problem (1.1)-(1.3) admits one and only one solution $u \in H^{4,1}(Q_T)$.*

Proof. Multiplying both sides of (1.1) by u, then integrating resulting relation with respect to x over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|u_{xx}\|^2 - m\|u_x\|^2 - \frac{1}{2}\int_0^1 u_x^2 u\,dx - a\int_0^1 u_x^2 u_{xx} u\,dx = 0.$$

Note that

$$\begin{gathered}
a\int_0^1 u_x^2 u_{xx} u\,dx = -\frac{a}{3}\|u_x\|_4^4,\\
\frac{1}{2}\int_0^1 u_x^2 u\,dx \le \frac{a}{3}\|u_x\|_4^4 + \frac{3}{16a}\|u\|^2,\\
m\|u_x\|^2 = -m(u_{xx}, u) \le \frac{\nu}{2}\|u_{xx}\|^2 + \frac{m^2}{2\nu}\|u\|^2.
\end{gathered}$$

Summing up, we derive that

$$\frac{d}{dt}\|u\|^2 + \nu\|u_{xx}\|^2 \le (\frac{3}{8a} + \frac{m^2}{\nu})\|u\|^2. \tag{2.1}$$

By Gronwall's inequality, we obtain

$$\|u\|^2 \le e^{(\frac{3}{8a}+\frac{m^2}{\nu})t}\|u_0\|^2 \le C, \quad \forall t \in (0,T). \tag{2.2}$$

Integrating (2.1) over $(0,T)$, using (2.2), we deduce that

$$\int_0^T \|u_{xx}\|^2 dt \le \frac{1}{\nu}\Big((\frac{3}{8a}+\frac{m^2}{\nu})\int_0^T \|u\|^2 dt + \|u_0\|^2\Big) \le C. \tag{2.3}$$

Multiplying both sides of equation (1.1) by $-u_{xx}$, then integrating with respect to x over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \nu\|u_{xxx}\|^2 - m\|u_{xx}\|^2 + \frac{1}{2}\int_0^1 u_x^2 u_{xx} dx + a\int_0^1 u_x^2 u_{xx}^2 dx = 0.$$

Note that

$$\int_0^1 u_x^2 u_{xx} dx = -2\int_0^1 u_x^2 u_{xx} dx = 0,$$

$$m\|u_{xx}\|^2 = -m(u_{xxx}, u_x) \le \frac{\nu}{2}\|u_{xxx}\|^2 + \frac{m^2}{2\nu}\|u_x\|^2.$$

Then, summing up, we derive that

$$\frac{d}{dt}\|u_x\|^2 + \nu\|u_{xxx}\|^2 \le \frac{m^2}{\nu}\|u_x\|^2. \tag{2.4}$$

By Gronwall's inequality, we obtain

$$\|u_x\|^2 \le e^{\frac{m^2}{\nu}t}\|u_{x0}\|^2 \le C, \quad \forall t \in (0,T). \tag{2.5}$$

Integrating (2.4) over $(0,T)$, using (2.5), we deduce that

$$\int_0^T \|u_{xxx}\|^2 dt \le \frac{1}{\nu}\Big(\frac{m^2}{\nu}\int_0^T \|u_x\|^2 dt + \|u_{x0}\|^2\Big) \le C. \tag{2.6}$$

Here, using Sobolev's embedding theorem, by (2.2) and (2.5), we have

$$\|u\|_\infty = \sup_{x\in[0,1]} |u(x,t)| \le C, \quad \forall t \in (0,T). \tag{2.7}$$

Multiplying both sides of (1.1) by u_{xxxx}, then integrating with respect to x over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \nu\|u_{xxxx}\|^2 - m\|u_{xxx}\|^2 - \frac{1}{2}\int_0^1 u_x^2 u_{xxxx} dx - a\int_0^1 u_x^2 u_{xx} u_{xxxx} dx = 0.$$

Using Nirenberg's inequality, we have

$$\|u_x\|^4 \le C_1'\|u_{xxxx}\|^{1/12}\|u_x\|^{11/12} + C_2'\|u_x\|,$$

$$\|u_x\|_8 \le C_1'\|u_{xxxx}\|^{1/8}\|u_x\|^{7/8} + C_2'\|u_x\|,$$

$$\|u_{xx}\|_4 \le C_1'\|u_{xxxx}\|^{5/12}\|u_x\|^{7/12} + C_2'\|u_x\|.$$

Then

$$\frac{1}{2}\int_0^1 u_x^2 u_{xxxx} dx \le \frac{\nu}{12}\|u_{xxxx}\|^2 + \frac{3}{2\nu}\|u_x\|_4^4 \le \frac{\nu}{6}\|u_{xxxx}\|^2 + C_1,$$

$$a\int_0^1 u_x^2 u_{xx} u_{xxxx} dx \le \frac{\nu}{12}\|u_{xxxx}\|^2 + \frac{3a^2}{2\nu}\|u_x\|_8^8 + \frac{3a^2}{2\nu}\|u_{xx}\|_4^4 \le \frac{\nu}{6}\|u_{xxxx}\|^2 + C_2,$$

$$m\|u_{xxx}\|^2 = -m(u_{xxxx}, u_{xx}) \le \frac{\nu}{6}\|u_{xxxx}\|^2 + \frac{3m^2}{2\nu}\|u_{xx}\|^2.$$

Summing up, we derive that

$$\frac{d}{dt}\|u_{xx}\|^2 + \nu\|u_{xxxx}\|^2 \le \frac{3m^2}{\nu}\|u_{xx}\|^2 + 2C_1 + 2C_2. \tag{2.8}$$

By Gronwall's inequality, we deduce that

$$\|u_{xx}\|^2 \le e^{3m^2t/\nu}\|u_{xx0}\|^2 + \frac{2}{\nu}(C_1 + C_2) \le C, \quad \forall t \in (0,T). \tag{2.9}$$

Integrating (2.8) over $(0,T)$, using (2.9), we deduce that

$$\int_0^T \|u_{xxxx}\|^2 dt \le \frac{1}{\nu}\Big(\frac{3m^2}{\nu}\int_0^T \|u_{xx}\|^2 dt + 2(C_1 + C_2)T + \|u_{xx0}\|^2\Big) \le C. \tag{2.10}$$

By Sobolev's imbedding theorem, from (2.2), (2.5), (2.9) it follows that

$$\|u_x\|_\infty = \sup_{x\in[0,1]} |u_x(x,t)| \le C, \quad \forall t \in (0,T). \tag{2.11}$$

The a priori estimates (2.7), (2.6) and (2.10) complete the proof of global existence. Because the proof of the uniqueness of the solution is easy, we omit it here. The proof is complete. □

3. Global attractor in $H^2(0,1)$

The dynamic properties of diffusion equations such as the global attractors and global asymptotic behavior of solutions are important for the study of diffusion model. There are many studies on the existence of global attractors for diffusion equations, such as [1, 3, 4, 10] and so on. In this section, we are interested in the existence of global attractors in the space $H^2(0,1)$ for problem (1.1)-(1.3).

By Theorem 2.2, we can also obtain $u(x,t) \in L^\infty(0,T;H^2(0,1))$. Define the operator semigroup $\{S(t)\}_{t\ge0}$ in $H^2(0,1)$ space as

$$S(t)u_0 = u(t), \quad \forall u_0 \in H_E^2(0,1),\ t \ge 0, \tag{3.1}$$

where $u(t)$ is the solution of (1.1)-(1.3) corresponding to initial value u_0.

Notice that the total mass is conserved for all time; we let

$$\mathcal{U} = \big\{u \in H_E^2(0,1) : \int_0^1 u\,dx = 0\big\}. \tag{3.2}$$

It is sufficient to see that the restriction of $\{S(t)\}$ on the affined space $\mathcal{U}$ is a well defined semigroup.

Now, we give the result on the existence of global attractor for problem (1.1)-(1.3) in $H^2(0,1)$.

Theorem 3.1. *Assume that ν is sufficiently large, then the semiflow associated with the solution u of* (1.1)-(1.3) *possesses in $\mathcal{U}$ a global attractor $\mathcal{A}$ which attracts all the bounded sets in $\mathcal{U}$.*

To prove Theorem 3.1, we establish some a priori estimates for the solution u of (1.1)-(1.3). In this section we always assume that $\{S(t)\}_{t\ge0}$ is the semigroup generated by the weak solutions of equation (1.1) with initial data $u_0 \in H_E^2(0,1)$. We have the following lemmas.

Lemma 3.2. *For initial data u_0 varying in a bounded set $B \subset \mathcal{U}$, there exists a $t_0(B) > 0$ such that*

$$\|u(t)\|_{H^2(0,1)} \le C, \quad t \ge t_0(B).$$

which implies that $\{S(t)\}_{t\ge0}$ has a bounded absorbing set in $\mathcal{U}$.

Proof. We prove this lemma in the following three steps.
Step 1. Based on Poincaré's inequality, we have

$$\|u\|^2 \le \frac{1}{2}\|u_x\|^2, \tag{3.3}$$

Hölder's inequality gives

$$\|u_x\|^2 \le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_{xx}\|^2. \tag{3.4}$$

Adding (3.3) and (3.4) together gives

$$\|u\|^2 \le \frac{1}{3}\|u_{xx}\|^2. \tag{3.5}$$

Using (2.1) and (3.5), we immediately obtain the following inequality

$$\frac{d}{dt}\|u\|^2 + 3\nu\|u\|^2 \le (\frac{3}{8a} + \frac{m^2}{\nu})\|u\|^2;$$

that is,

$$\frac{d}{dt}\|u\|^2 + (3\nu - \frac{3}{8a} - \frac{m^2}{\nu})\|u\|^2 \le 0,$$

where ν is sufficiently large, which satisfies $3\nu - \frac{3}{8a} - \frac{m^2}{\nu} > 0$. By Gronwall's inequality, we get

$$\|u\|^2 \le e^{-(3\nu - \frac{3}{8a} - \frac{m^2}{\nu})t}\|u_0\|^2. \tag{3.6}$$

Thus, for initial data in any bounded set $B \subset \mathcal{U}$, there is a uniform time $t_1(B)$ depending on B such that for $t \ge t_1(B)$,

$$\|u\|^2 \le C. \tag{3.7}$$

Step 2. By (3.3) and (3.4), we can also obtain

$$\|u_x\|^2 \le \frac{2}{3}\|u_{xx}\|^2. \tag{3.8}$$

Adding (2.4) and (3.8) gives

$$\frac{d}{dt}\|u_x\|^2 + \frac{3}{2}\nu\|u_x\|^2 \le \frac{m^2}{\nu}\|u_x\|^2;$$

that is,

$$\frac{d}{dt}\|u_x\|^2 + (\frac{3\nu}{2} - \frac{m^2}{\nu})\|u_x\|^2 \le 0,$$

where ν is sufficiently large, which satisfies $\frac{3\nu}{2} - \frac{m^2}{\nu} > 0$. By Gronwall's inequality, we deduce that

$$\|u_x\|^2 \le e^{-(\frac{3\nu}{2} - \frac{m^2}{\nu})t}\|u_{x0}\|^2. \tag{3.9}$$

Thus, for initial data in any bounded set $B \subset \mathcal{U}$, there is a uniform time $t_2(B)$ depending on B such that for $t \ge t_2(B)$,

$$\|u_x\|^2 \le C. \tag{3.10}$$

Step 3. For (2.8), applying the regularity theorem of elliptic operator, we have

$$\frac{d}{dt}\|u_{xx}\|^2 + \nu C'(\|u_{xx}\|^2 + \|u_{xxx}\|^2) \le \frac{3m^2}{\nu}\|u_{xx}\|^2 + 2(C_1 + C_2),$$

which means

$$\frac{d}{dt}\|u_{xx}\|^2 + (\nu C' - \frac{3m^2}{\nu})\|u_{xx}\|^2 \le 2(C_1 + C_2), \tag{3.11}$$

where ν is sufficiently large, which satisfies $\nu C' - \frac{3m^2}{\nu} > 0$. Then, using Gronall's inequality, we derive that

$$\|u_{xx}\|^2 \le e^{(\nu C' - \frac{3m^2}{\nu})t}\|u_{xx0}\|^2 + \frac{2\nu(C_1+C_2)}{\nu^2 C' - 3m^2}. \tag{3.12}$$

Thus, for initial data in any bounded set $B \subset \mathcal{U}$, there is a uniform time $t_3(B)$ depending on B such that for $t \ge t_3(B)$,

$$\|u_{xx}\|^2 \le \frac{4\nu(C_1+C_2)}{\nu^2 C' - 3m^2}. \tag{3.13}$$

By Sobolev's embedding theorem, we have

$$\|u_x(x,t)\|_\infty \le C.$$

Combining (3.7), (3.10) and (3.13) , we complete the proof. □

In the following, we prove the precompactness of the orbit in $\mathcal{U}$.

Lemma 3.3. *For initial data u_0 varying in a bounded set $B \subset \mathcal{U}$, there exists a $T(B) > 0$ such that*

$$\|u(t)\|_{H^3(0,1)} \le C, \quad \forall t \ge T > 0,$$

which implies that $\bigcup_{t\ge T} u(t)$ is relatively compact in $\mathcal{U}$.

Proof. The uniform bound of H^2-norm of $u(x,t)$ has been obtained in Lemma 3.2. In what follows we derive the estimate on H^3-norm.

Differentiating (1.1) with respect to x, then multiplying by u_{xxxxx} and integrating on $(0,1)$, using the boundary conditions, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{xxx}\|^2 + \nu\|u_{xxxxx}\|^2 - m\|u_{xxxx}\|^2\\ &- \frac{1}{2}\int_0^1 (u_x^2)_x u_{xxxxx}dx - a\int_0^1 (u_x^2 u_{xx})_x u_{xxxxx}dx = 0.\end{aligned}$$

Note that

$$m\|u_{xxxx}\|^2 = -m\int_0^1 u_{xxx}u_{xxxxx}dx \le \frac{\nu}{3}\|u_{xxxxx}\|^2 + \frac{3m^2}{4\nu}\|u_{xxx}\|^2,$$

and

$$\begin{aligned}\frac{1}{2}\int_0^1 (u_x^2)_x u_{xxxxx}dx &= \int_0^1 u_x u_{xx} u_{xxxxx}dx \le \|u_x\|_\infty\|u_{xx}\|\|u_{xxxxx}\|\\ &\le \frac{\nu}{3}\|u_{xxxxx}\|^2 + \frac{3}{4\nu}\|u_x\|_\infty^2\|u_{xx}\|^2 \le \frac{\nu}{3}\|u_{xxxxx}\|^2 + C_3.\end{aligned}$$

On the other hand, Nirenberg's inequality gives

$$\|u_{xx}\|_4 \le C_1'\|u_{xxxxx}\|^{\frac{1}{12}}\|u_{xx}\|^{\frac{11}{12}} + C_2'\|u_{xx}\|.$$

Hence

$$\begin{aligned}a\int_0^1 (u_x^2 u_{xx})_x u_{xxxxx}dx &= 2a\int_0^1 u_x u_{xx}^2 u_{xxxxx}dx + a\int_0^1 u_x^2 u_{xxx}u_{xxxxx}dx\\ &\le 2a\|u_x\|_\infty\|u_{xx}\|_4^2\|u_{xxxxx}\| + a\|u_x\|_\infty^2\|u_{xxx}\|\|u_{xxxxx}\|\\ &\le \frac{\nu}{6}\|u_{xxxxx}\|^2 + C\|u_{xx}\|_4^4 + C_4\|u_{xxx}\|^2 + C\\ &\le \frac{\nu}{3}\|u_{xxxxx}\|^2 + C_4\|u_{xxx}\|^2 + C_5.\end{aligned}$$

Summing up, we have

$$\frac{d}{dt}\|u_{xxx}\|^2 \le (\frac{3m^2}{2\nu} + 2C_4)\|u_{xxx}\|^2 + 2C_3 + 2C_5. \tag{3.14}$$

Integrating (3.11) between t and $t+1$, using (3.13), we have

$$\int_t^{t+1} \|u_{xxx}\|^2 dt \le C. \tag{3.15}$$

Due to (3.14), (3.15), and the uniform Gronwall inequality in [10], we obtain that

$$\|u_{xxx}\|^2 \le C, \quad \forall t \ge 1.$$

The proof is complete. □

Proof of Theorem 3.1. By Lemmas 3.2-3.3 and [10, Theorem I.1.1], we conclude that $\mathcal{A} = \omega(\mathcal{B})$, the ω-limit set of absorbing set $\mathcal{B}$ is a global attractor in $\mathcal{U}$. By lemma 3.3, this global attractor is a bounded set in $H^3(0,1)$. The proof is complete. □

4. Attractor in $H^k(0,1)$

We introduce the following spaces:

$$\begin{gathered} H = \{u \in L^2(0,1) : \int_0^1 u(x,t)dx = 0\}, \\ H_{1/2} = H_E^2(0,1) \cap H = \mathcal{U}, \\ H_1 = H_E^4(0,1) \cap H. \end{gathered} \tag{4.1}$$

Define the linear operator L and the nonlinear operator G by

$$\begin{gathered} Lu = -\nu u_{xxxx}, \\ Gu = g(u) = -mu_{xx} + \frac{1}{2}(u_x)^2 + a(u_x)^2 u_{xx}. \end{gathered} \tag{4.2}$$

It is easy to check that L given by (4.2) is a sectorial operator and the tractional power operator $(-L)^{1/2}$ is given by $(-L)^{1/2} = \nu^{1/2}\frac{\partial^2}{\partial x^2}$. The space $H_{1/2}$ is the same as (4.1), $H_{\frac{1}{4}}$ is given by $H_{\frac{1}{4}}$ = closure of $H_{1/2}$ in $H^1(\Omega)$ and $H_k = H^{4k} \cap H_1$ for $k \ge 1$.

Based on [6], the solution $u(t,u_0)$ of the problem (1.1) can be written as

$$u(t,u_0) = e^{tL}u_0 + \int_0^t e^{(t-\tau)L} Gu d\tau = e^{tL}u_0 + \int_0^t e^{(t-\tau)L} g(u) d\tau. \tag{4.3}$$

We introduce a result on the sectorial operator L in (4.2), which is important in this section and can be found in [5, 6, 7, 8, 9, 11].

Lemma 4.1. *Assume that $L : H \to H$ is a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $Re\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\kappa(\mathcal{L} = -L)$ we have*

- (C1) *$T(t) : H \to H$ is bounded for all $\kappa \in R^1$ and $t > 0$;*
- (C2) *$T(t)\mathcal{L}^\kappa x = \mathcal{L}^\kappa T(t)x, \forall x \in H$;*
- (C3) *For each $t > 0$, $\mathcal{L}^\kappa T(t) : H \to H$ is bounded, and*

$$\|\mathcal{L}^\kappa T(t)\|_H \le Ct^{-\kappa}e^{-\delta t};$$

where some $\delta > 0$ and $C > 0$ is a constant depending only on κ;

(C4) *The $H-$norm can be defined by* $\|x\|_{H_\kappa} = \|\mathcal{L}^\kappa x\|_H$.

Now, we give the main result of this section.

Theorem 4.2. *Let $u_0 \in H_\kappa(0,1)$ and ν is sufficiently large. Then, for any $\kappa \geq 0$, the semigroup associated with the problem* (1.1)-(1.3) *possesses a global attractor in $H_\kappa(0,1)$, which attracts all the bounded sets in the H_κ-norm.*

To prove Theorem 4.2, we should prove the following two lemmas.

Lemma 4.3. *Let $u_0 \in H_\kappa(0,1)$ and ν is sufficiently large. Then for any $\kappa \geq 0$, the semigroup $S(t)$ generated by the problem* (1.1)-(1.3) *is uniformly compact in H_κ.*

Proof. It suffices to prove that for any bounded set $U \subset H_\kappa$, there exists $C > 0$ such that

$$\|u(t,u_0)\|_{H_\kappa} \leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\kappa,\ \kappa \geq 0. \tag{4.4}$$

For $\kappa = 1/2$, this follows form Theorem 3.1; i.e., for any bounded set $U \subset H_{1/2}$, there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_{1/2}} \leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_{1/2}. \tag{4.5}$$

Then, we shall prove (4.4) for any $\kappa > \frac{1}{2}$, which will be proved in the following steps.

Step 1. We prove that for any bounded set $U \subset H_\kappa$ ($\frac{1}{2} < \kappa < 1$), there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\kappa} \leq C, \quad \forall t \geq 0,\ u_0 \in U,\ \frac{1}{2} < \kappa < 1. \tag{4.6}$$

By the embedding theorem, we have

$$H_{1/2}(0,1) \hookrightarrow W^{1,4}(0,1), \quad H_{1/2}(0,1) \hookrightarrow W^{1,\infty}(0,1).$$

Hence

$$\begin{aligned}
\|g(u)\|_H &= \int_0^1 (-mu_{xx} + \frac{1}{2}u_x^2 + au_x^2 u_{xx})^2 dx \\
&\leq C \int_0^1 (u_{xx}^2 + u_x^4 + u_x^4 u_{xx}^2) dx \\
&\leq C(\|u\|_{H_{1/2}}^2 + \|u\|_{W^{1,4}}^4 + \|u\|_{W^{1,\infty}}^4 \|u\|_{H_{1/2}}^2) \\
&\leq C(\|u\|_{H_{1/2}}^2 + \|u\|_{H_{1/2}}^4 + \|u\|_{H_{1/2}}^6) \leq C.
\end{aligned}$$

which implies that $g : H_{1/2} \to H$ is bounded. Hence,

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &= \|e^{tL}u_0 + \int_0^t e^{(t-\tau)L} g(u) d\tau\|_{H_\kappa} \\
&\leq C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^\kappa e^{(t-\tau)L} g(u)\|_H d\tau \\
&\leq C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^\kappa e^{(t-\tau)L}\| \cdot \|g(u)\|_H d\tau \\
&\leq C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\kappa} e^{-\delta\tau} d\tau \\
&\leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H,
\end{aligned} \tag{4.7}$$

where $0 < \kappa < 1$. Then, (2.4) holds.

Step 2. We prove that for any bounded set $U \subset H_\kappa$ ($1 \le \kappa < \frac{5}{4}$), there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\kappa} \le C, \quad \forall t \ge 0,\ u_0 \in U,\ 1 \le \kappa < \frac{5}{4}. \tag{4.8}$$

In fact, by the embedding theorems, we derive that

$$H_\kappa(0,1) \hookrightarrow H^3(0,1), \quad H^3(0,1) \hookrightarrow W^{1,\infty}(0,1), \quad H^3(0,1) \hookrightarrow W^{2,4}(0,1),$$

where $\frac{3}{4} \le \kappa < 1$. Then, using (1.1), we obtain

$$\begin{aligned}
\|g(u)\|^2_{\frac{1}{4}} &= \int_0^1 (g(u)_x)^2 dx \\
&= \int_0^1 (-mu_{xxx} + u_x u_{xx} + 2au_x u_{xx}^2 + au_x^2 u_{xxx})^2 dx \\
&\le C \int_0^1 (u_{xxx}^2 + u_x^2 u_{xxx}^2 + u_x^2 u_{xx}^4 + u_x^4 u_{xxx}^2) dx \\
&\le C(\|u\|^2_{H^3} + \|u\|^2_{W^{1,\infty}} \|u\|^2_{H^3} + \|u\|^2_{W^{1,\infty}} \|u\|^4_{W^{2,4}} + \|u\|^4_{W^{1,\infty}} \|u\|^2_{H^3}) \\
&\le C(\|u\|^2_{H_\kappa} + \|u\|^4_{H_\kappa} + \|u\|^6_{H_\kappa} + \|u\|^6_{H_\kappa}) \le C,
\end{aligned} \tag{4.9}$$

which implies that $g : H_\kappa \to H_{\frac{1}{4}}$ is bounded for $\frac{3}{4} \le \kappa < 1$. Hence,

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &= \|e^{tL} u_0 + \int_0^t e^{(t-\tau)L} g(u) d\tau\|_{H_\kappa} \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^\kappa e^{(t-\tau)L} g(u)\|_H d\tau \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{\kappa - \frac{1}{4}} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{\frac{1}{4}}} d\tau \\
&\le C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\epsilon} e^{-\delta\tau} d\tau \\
&\le C, \quad \forall t \ge 0, u_0 \in U \subset H,
\end{aligned}$$

where $\epsilon = \kappa - \frac{1}{4}$ ($0 < \epsilon < 1$). Then, (2.5) holds.

Step 3. We prove that for any bounded set $U \subset H_\kappa$ ($5/4 \le \kappa < 3/2$), there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\kappa} \le C, \quad \forall t \ge 0, \quad u_0 \in U,\ \frac{5}{4} \le \kappa < \frac{3}{2}. \tag{4.10}$$

In fact, by the embedding theorems, we have

$$\begin{gathered}
H_\kappa(0,1) \hookrightarrow H^4(0,1), \quad H^4(0,1) \hookrightarrow W^{1,\infty}(0,1), \quad H^4(0,1) \hookrightarrow W^{2,4}(0,1), \\
H^4(0,1) \hookrightarrow W^{3,4}(0,1), \quad H^4(0,1) \hookrightarrow W^{2,6}(0,1),
\end{gathered}$$

where $1 \le \kappa < 5/4$. Then,

$$\begin{aligned}
&\|g(u)\|_{H_{1/2}}^2 \\
&= \int_0^1 (g(u)_{xx})^2 dx \\
&= \int_0^1 (-mu_{xxxx} + u_x u_{xxxx} + u_{xx}u_{xxx} + 2au_{xx}^3 \\
&\quad + 6au_x u_{xx} u_{xxx} + au_x^2 u_{xxxx})^2 dx \\
&\le C\int_0^1 (u_{xxxx}^2 + u_x^2 u_{xxxx}^2 + u_{xx}^4 + u_{xxx}^4 + u_{xx}^6 + u_x^4 u_{xx}^4 + u_x^4 u_{xxxx}^2) dx \\
&\le C(\|u\|_{H^4}^2 + \|u\|_{W^{1,\infty}}^2 \|u\|_{H^4}^2 + \|u\|_{W^{2,4}}^4 + \|u\|_{W^{3,4}}^4 + \|u\|_{W^{2,6}}^6 \\
&\quad + \|u\|_{W^{1,\infty}}^4 \|u\|_{W^{2,4}}^4 + \|u\|_{W^{1,\infty}}^4 \|u\|_{H^4}^2) \\
&\le C(\|u\|_{H_\kappa}^2 + \|u\|_{H_\kappa}^4 + \|u\|_{H_\kappa}^6 + \|u\|_{H_\kappa}^8) \le C.
\end{aligned} \tag{4.11}$$

which implies that $g : H_\kappa \to H_{1/2}$ is bounded for $\kappa \ge \frac{3}{4}$. Hence,

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &= \|e^{tL}u_0 + \int_0^t e^{(t-\tau)L} g(u) d\tau\|_{H_\kappa} \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{\kappa-\frac{1}{2}} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/2}} d\tau \\
&\le C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\epsilon} e^{-\delta\tau} d\tau \\
&\le C, \quad \forall t \ge 0,\ u_0 \in U \subset H,
\end{aligned} \tag{4.12}$$

where $\epsilon = \kappa - \frac{1}{2}$ $(0 < \epsilon < 1)$. Then, (4.10) holds.

Step 4. We prove that for any bounded set $U \subset H_\kappa$ $(3/2 \le \alpha < 7/4)$, there exists a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\kappa} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H,\ \frac{3}{2} \le \kappa < \frac{7}{4}. \tag{4.13}$$

Based on the following embedding theorems, we deduce that

$$\begin{gathered}
H_\kappa(0,1) \hookrightarrow H^5(0,1), \quad H^5(0,1) \hookrightarrow W^{1,\infty}(0,1), \quad H^5(0,1) \hookrightarrow W^{2,4}(0,1), \\
H^5(0,1) \hookrightarrow W^{4,4}(0,1), \quad H^5(0,1) \hookrightarrow W^{2,8}(0,1), \quad H^5(0,1) \hookrightarrow W^{3,2}(0,1), \\
H^5(0,1) \hookrightarrow W^{3,4}(0,1), \quad H^5(0,1) \hookrightarrow W^{1,8}(0,1),
\end{gathered}$$

where $5/4 \le \alpha < 3/2$. Then

$$\begin{aligned}
&\|g(u)\|_{H_{3/4}}^2 \\
&= \int_0^1 (g(u))_{xxx}^2 dx \\
&= \int_0^1 (-mu_{xxxxx} + u_x u_{xxxxx} + 2u_{xx}u_{xxxx} + u_{xxx}^2 + 12au_{xx}^2 u_{xxx} \\
&\quad + 6au_x u_{xxx}^2 + 8au_x u_{xx} u_{xxxx} + au_x^2 u_{xxxxx})^2 dx \\
&\le C\int_0^1 (u_{xxxxx}^2 + u_x^2 u_{xxxxx}^2 + u_{xx}^4 + u_{xxxx}^4 + u_{xxx}^2 + u_{xx}^8 + u_{xxx}^4 \\
&\quad + u_x^2 u_{xxx}^4 + u_{xxxx}^4 + u_x^8 + u_{xx}^8 + u_x^4 u_{xxxxx}^2) dx
\end{aligned}$$

$$
\begin{aligned}
&\le C(\|u\|_{H^5}^2 + \|u\|_{W^{1,\infty}}^2\|u\|_{H^5}^2 + \|u\|_{W^{2,4}}^4 + \|u\|_{W^{4,4}}^4 + \|u\|_{W^{2,8}}^8 + \|u\|_{W^{3,2}}^2 \\
&\quad + \|u\|_{W^{3,4}}^4 + \|u\|_{W^{1,\infty}}^2\|u\|_{W^{3,4}}^4 + \|u\|_{W^{1,8}}^8 + \|u\|_{W^{1,\infty}}^4\|u\|_{H^5}^2) \\
&\le C(\|u\|_{H_\kappa}^2 + \|u\|_{H_\kappa}^4 + \|u\|_{H_\kappa}^8 + \|u\|_{H_\kappa}^6) \le C.
\end{aligned}
$$

which implies that $g : H_\kappa \to H_{3/4}$ is bounded for $\kappa \ge 1$. Hence,

$$
\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &= \|e^{tL}u_0 + \int_0^t e^{(t-\tau)L}g(u)d\tau\|_{H_\kappa} \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{\kappa-\frac{3}{4}}e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{3/4}}d\tau \\
&\le C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\epsilon}e^{-\delta\tau}d\tau \\
&\le C, \quad \forall t \ge 0,\ u_0 \in U \subset H,
\end{aligned} \tag{4.14}
$$

where $\epsilon = \kappa - \frac{3}{4}$ $(0 < \epsilon < 1)$. Then, (2.8) holds.

In the same way as in the proof of (4.13), by iteration we can prove that for any bounded set $U \subset H_\kappa$ $(\kappa \ge 0)$ there exists a constant $C > 0$ such that (4.4) holds; i.e., for all $\kappa \ge 0$ the semigroup $S(t)$ generated by problem (1.1) is uniformly compact in H_κ. The proof is complete. □

Lemma 4.4. *Let $u_0 \in H_\kappa(0,1)$ and ν is sufficiently large. Then for any $\kappa \ge 0$, the problem* (1.1)-(1.3) *has a bounded absorbing set in H_κ.*

Proof. It suffices to prove that for any bounded set $U \subset H_\kappa$ $(\kappa \ge 0)$, there exist $T > 0$ and a constant $C > 0$ independent of u_0, such that

$$
\|u(t,u_0)\|_{H_\kappa} \le C, \quad \forall t \ge T,\ u_0 \in U \subset H_\kappa. \tag{4.15}
$$

For $\kappa = 1/2$, this follows from Theorem 3.1. So we shall prove (4.15) for any $\kappa > 1/2$. We prove it in the following steps:

Step 1. We prove that for any $\frac{1}{2} < \kappa < 1$, problem (1.1)-(1.3) has a bounded absorbing set in H_κ. By (4.3), we deduce that

$$
u(t,u_0) = e^{(t-T)L}u(T,u_0) + \int_T^t e^{(t-\tau)L}g(u)d\tau. \tag{4.16}
$$

Suppose that B is a bounded absorbing set of problem (1.1)-(1.3), which satisfies $B \subset H_{1/2}$, we also assume $T_0 > 0$ such that

$$
u(t,u_0) \in B, \quad \forall t > T_0,\ u_0 \in U \subset H_\kappa,\ \kappa > \frac{1}{2}. \tag{4.17}
$$

It is easy to check that

$$
\|e^{tL}\| \le Ce^{-\lambda_1^2 t},
$$

where $\lambda_1 > 0$ is the first eigenvalue of the equation

$$
\begin{gathered}
-\nu^{1/2}u_{xx} = \lambda u, \\
u_x(0,t) = u_x(1,t) = 0.
\end{gathered} \tag{4.18}
$$

Thus, for any given $T > 0$ and $u_0 \in U \subset H_\kappa$ $(\kappa > 1/2)$, we deduce that

$$
\lim_{t\to\infty} \|e^{(t-T)L}u(T,u_0)\|_{H_\kappa} = 0. \tag{4.19}
$$

Using (4.16), (4.17) and (4.19), we have

$$\begin{aligned}\|u(t,u_0)\|_{H_\kappa} &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + \int_{T_0}^t \|(-L)^\kappa e^{(t-T)L}\|\cdot\|g(u)\|_H d\tau\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_{T_0}^t \|(-L)^\kappa e^{(t-T)L}\|\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_0^{T-T_0} \tau^{-\kappa}e^{-\delta\tau}d\tau\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C,\end{aligned} \tag{4.20}$$

where $C>0$ is a constant independent of u_0. Then (4.15) holds for all $1/2<\kappa<1$.
Step 2. We shall show that for any $1\le\kappa<5/4$, problem (1.1)-(1.3) has a bounded absorbing set in H_κ. Using (4.16) and (4.9), we deduce that

$$\begin{aligned}\|u(t,u_0)\|_{H_\kappa} &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + \int_{T_0}^t \|(-L)^{\kappa-\frac14} e^{(t-\tau)L}\|\cdot\|g(u)\|_{H_{\frac14}} d\tau\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_{T_0}^t \|(-L)^{\kappa-\frac14} e^{(t-\tau)L}\|dx\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_0^{T-T_0} \tau^{-(\kappa-\frac14)}e^{-\delta\tau}d\tau\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C,\end{aligned}$$

where $C>0$ is a constant independent of u_0. Then, (2.9) holds for all $1\le\kappa<5/4$.
Step 3. We shall show that for any $5/4\le\kappa<3/2$, problem (1.1)-(1.3) has a bounded absorbing set in H_κ. Using (4.16) and (4.11), we deduce that

$$\begin{aligned}\|u(t,u_0)\|_{H_\kappa} &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + \int_{T_0}^t \|(-L)^{\kappa-\frac12} e^{(t-\tau)L}\|\cdot\|g(u)\|_{H_{1/2}} d\tau\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_{T_0}^t \|(-L)^{\kappa-\frac12} e^{(t-\tau)L}\|dx\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_0^{T-T_0} \tau^{-(\kappa-\frac12)}e^{-\delta\tau}d\tau\\ &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C,\end{aligned}$$

where $C>0$ is a constant independent of u_0. Then (2.9) holds for all $5/4\le\kappa<3/2$.

By the iteration method, we have that (4.15) holds for all $\kappa>1/4$. The proof is complete. □

Proof of Theorem 4.2. By Lemma 4.3 and Lemma 4.4, we immediately conclude that the statement of the theorem. □

Remark 4.5. Since the tools used work for the periodic boundary values, the results of this article are also valid for equation (1.1) with the periodic boundary conditions in the sense [10], That is, for any $u_0\in H^k_{\rm per}(0,1)$, there exists a global unique weak solution $u(x,t)$, a global attractor in H^k $(0\le k<\infty)$ space for equation (1.1) under the initial value condition (1.3) and the periodic boundary conditions

$$\varphi|_{x=0} = \varphi|_{x=1},$$

for u and the derivatives of u at least of order ≤ 3.

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References

[1] T. Dlotko; *Global attractor for the Cahn-Hilliard equation in H^2 and H^3*, J. Differential Equations, 113 (1994), 381-393.

[2] A. A. Golovin, S. H. Davis, A. A. Nepomnyashchy; *Model for faceting in a kinetically controlled crystal growth*, Physical Review E, 59(1) (1999), 803-825.

[3] B. Guo, B. Wang; *Long-time behaviour of the solutions for the multidimensional Kolmogorov-Spieqel-Sivashinsky equation*, Acta Math. Sin. (Engl. Ser.), 18 (2002), 579-596.

[4] J. K. Hale; *Asymptotic Behaviour of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[5] H. Luo, Q. Zhang; *Regularity of global attractor for the fourth-order reaction-diffusion equation*, Commun. Nonlinear Sci. Numer. Simul., 3824-3831.

[6] T. Ma, S. H. Wang; *Stability and Bifurcation of Nonlinear Evolution Equations*, Science Press, Beijing, 2006, (in Chinese).

[7] L. Song, Y. Zhang, T. Ma; *Global attractor of the Cahn-Hilliard euqation in H^k spaces*, J. Math. Anal. Appl., 355(2009), 53–62.

[8] L. Song, Y. He, Y. Zhang; *The existence of global attractors for semilinear parabolic equation in H^k space*, Nonlinear Anal., 68(2008), 3541–3549.

[9] L. Song, Y. Zhang, T. Ma; *Global attractor of a modified Swift-Hohenberg equation in H^k space*, Nonlinear Anal., 72(2010), 183–191.

[10] R. Temam; *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1988.

[11] X. Zhao, B. Liu; *The existence of global attractor for convective Cahn-Hilliard equation*, J. Korean Math. Soc., 49(2)(2012), 359-378.

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