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SOLVABILITY OF DEGENERATE ANISOTROPIC ELLIPTIC SECOND-ORDER EQUATIONS WITH L^1-DATA

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Abstract. In this article, we study the Dirichlet problem for degenerate anisotropic elliptic second-order equations with L^1-right-hand sides on a bounded open set of $\mathbb{R}^n$ ($n \geqslant 2$). These equations are described with a set of exponents and of a set of weighted functions. The exponents characterize the rates of growth of the coefficients of the equations with respect to the corresponding derivatives of the unknown function, and the weighted functions characterize degeneration or singularity of the coefficients of the equations with respect to the spatial variable. We prove theorems on the existence of entropy solutions, T-solutions, W-solutions, and weighted weak solutions of the problem under consideration.

1. Introduction

In the previous twenty years, the investigations on the existence and properties of solutions to nonlinear equations and variational inequalities with L^1-data, or measures as data, have been developed intensively. As is generally known, an effective approach to the solvability of second-order equations in divergence form with L^1-right-hand sides was proposed in [6]. Then closely related research has been developed for nondegenerate isotropic nonlinear second-order equations with L^1-data, and measures as data, involving entropy and renormalized solutions [2, 7, 8, 9, 10, 12, 16, 18, 19].

As for the solvability of nonlinear elliptic second-order equations with anisotropy and degeneracy (with respect to the spatial variable), we note the following works. The existence of a weak (distributional) solution to the Dirichlet problem for a model nondegenerate anisotropic equation with right-hand side measure was established in [11]. The existence of weak solutions for a class of nondegenerate anisotropic equations with locally integrable data in $\mathbb{R}^n$ ($n \geqslant 2$) was proved in [4]. An analogous existence result concerning the Dirichlet problem for a system of nondegenerate anisotropic equations with measure data was obtained in [5]. Moreover, in [27], the existence of weak solutions to the Dirichlet problem for nondegenerate anisotropic equations with right-hand sides from Lebesgues spaces close to L^1 was established. Solvability of the Dirichlet problem for degenerate isotropic equations

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with L^1-data and measures as data was studied in [1, 3, 13, 15, 28]. We remark that in [1, 13], the existence of entropy solutions to the given problem was proved in the case of L^1-data. In [3], the existence of a renormalized solution of the problem for the same case was established. In [3, 15, 28], the existence of distributional solutions of the problem was obtained in the case of right-hand side measures.

In this article, we study the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides in a bounded open set Ω of $\mathbb{R}^n$ $(n \geqslant 2)$. This class is described by a set of exponents $q_1, \dots, q_n$ and of a set of weighted functions $\nu_1, \dots, \nu_n$. The exponents q_i characterize the rates of growth of the coefficients of the equations with respect to the corresponding derivatives of unknown function. The functions ν_i characterize degeneration or singularity of the coefficients of the equations with respect to the spatial variable. This is the most general situation in comparison with the above-mentioned works: the nondegenerate isotropic case means that $\nu_i \equiv 1$ and $q_i = q_1$, $i = 1, \dots, n$, the nondegenerate anisotropic case means that $\nu_i \equiv 1$, $i = 1, \dots, n$, and q_i, $i = 1, \dots, n$, are generally different, and the degenerate isotropic case means that $\nu_i = \nu_1$, $i = 1, \dots, n$, as in [3, 13, 15, 28] or ν_i, $i = 1, \dots, n$, are generally different as in [1] but $q_i = q_1$, $i = 1, \dots, n$.

Our initial assumptions on the exponents q_i and the functions ν_i are as follows: $q_i \in (1, n)$, $\nu_i : \Omega \to \mathbb{R}$, $\nu_i \geqslant 0$ in Ω, $\nu_i > 0$ a.e. in Ω, $\nu_i \in L^1_{\rm loc}(\Omega)$ and $(1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega)$. Considering such kinds of solutions to the given problem as entropy solutions, T-solutions, W-solutions and weighted weak solutions, we prove the corresponding existence results. In so doing, the theorem on the existence and uniqueness of an entropy solution does not require additional conditions on q_i and ν_i, while the existence of other kinds of solutions is established under additional conditions on the numbers q_i and the exponents of increased summability (that should be assumed) of functions $1/\nu_i$ and ν_i.

In this connection, we observe that in the nondegenerate anisotropic case our additional conditions for the existence of W-solutions are equivalent to a two-sided bound for q_i which coincides with that given in [4, 5]. Moreover, we note that, unlike the present article, in [13], the existence of entropy solutions was proved under the assumption that the involved weighted function belongs to an appropriate Muckenhoupt class. We also remark that in the case where $q_i = q_1$ and $\nu_i = \nu_1$, $i = 1, \dots, n$, our conditions for the existence of T-solutions are reduced to such requirements on the summability of the functions $1/\nu_1$ and ν_1 as in [28]. At last, we observe that in [1], in the case where the functions ν_i, $i = 1, \dots, n$, are generally different and $q_i = q_1$, $i = 1, \dots, n$, the existence of entropy solutions was established under some implicit hypotheses on $\nu_1, \dots, \nu_n$.

This article is organized as follows. In Section 2, we describe a weighted anisotropic Sobolev space and a set of functions which are used in the sequel. In Section 3, we formulate the problem in question, consider different kinds of its solutions and give the statements of the main results. Section 4 is devoted to the proofs of these results. Observe that the proofs are based on the use of some results of [20, 21, 22] on the existence and properties of solutions of second-order variational inequalities with L^1-right-hand sides and sufficiently general constraints. Finally, in Section 5, we consider particular cases concerning the exponents q_i and the weighted functions ν_i, and give examples where conditions of the main theorems are satisfied.

For completeness we note that an extensive bibliography on the existence and properties of solutions of second-order variational inequalities with L^1-data and measure data one can find in [22].

As far as the solvability of nonlinear elliptic high-order equations with anisotropy, degeneracy and L^1-data is concerned, we refer the reader for instance to [23, 24, 25, 26] where classes of elliptic equations of fourth and higher order with coefficients, satisfying appropriate strengthened coercivity conditions, were considered.

In [14], a class of nondegenerate anisotropic nonlinear elliptic equations of arbitrary even order with L^1-data was considered, and the solvability of the Dirichlet problem in the corresponding energy space was established. However, this was made under a condition on the involved parameters which provides the imbedding of the energy space into the space of bounded functions.

2. Preliminaries

Let $n \in \mathbb{N}$, $n \geqslant 2$, Ω be a bounded open set of $\mathbb{R}^n$, and let for every $i \in \{1, \dots, n\}$ we have $q_i \in (1, n)$. We set $q = \{q_i : i = 1, \dots, n\}$.

For $i \in \{1, \dots, n\}$, let ν_i be nonnegative functions on Ω such that $\nu_i > 0$ a.e. in Ω,

$$\nu_i \in L^1_{\rm loc}(\Omega), \quad \Big(\frac{1}{\nu_i}\Big)^{1/(q_i-1)} \in L^1(\Omega). \tag{2.1}$$

We set $\nu = \{\nu_i : i = 1, \dots, n\}$. We denote by $W^{1,q}(\nu, \Omega)$ the set of all functions $u \in L^1(\Omega)$ such that for every $i \in \{1, \dots, n\}$ there exists the weak derivative $D_i u$ and $\nu_i |D_i u|^{q_i} \in L^1(\Omega)$.

Let $\|\cdot\|_{1,q,\nu}$ be the mapping from $W^{1,q}(\nu, \Omega)$ into $\mathbb{R}$ such that for every function $u \in W^{1,q}(\nu, \Omega)$,

$$\|u\|_{1,q,\nu} = \int_\Omega |u| dx + \sum_{i=1}^n \Big(\int_\Omega \nu_i |D_i u|^{q_i} dx \Big)^{1/q_i}.$$

The mapping $\|\cdot\|_{1,q,\nu}$ is a norm in $W^{1,q}(\nu, \Omega)$, and, in view of the second inclusion of (2.1), the set $W^{1,q}(\nu, \Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,q,\nu}$. Moreover, by the first inclusion of (2.1), we have $C_0^\infty(\Omega) \subset W^{1,q}(\nu, \Omega)$.

We denote by $\mathring{W}^{1,q}(\nu, \Omega)$ the closure of the set $C_0^\infty(\Omega)$ in the space $W^{1,q}(\nu, \Omega)$. Obviously, the set $\mathring{W}^{1,q}(\nu, \Omega)$ is a Banach space with respect to the norm induced by the norm $\|\cdot\|_{1,q,\nu}$. We observe that $C_0^1(\Omega) \subset \mathring{W}^{1,q}(\nu, \Omega)$.

Further, for every $k > 0$, let $T_k : \mathbb{R} \to \mathbb{R}$ be the function such that

$$T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k \operatorname{sign} s & \text{if } |s| > k. \end{cases}$$

By analogy with known results for nonweighted Sobolev spaces (see for instance [17, Chapter 2]) we have: if $u \in \mathring{W}^{1,q}(\nu, \Omega)$ and $k > 0$, then $T_k(u) \in \mathring{W}^{1,q}(\nu, \Omega)$ and for every $i \in \{1, \dots, n\}$, $D_i T_k(u) = D_i u \cdot 1_{\{|u|<k\}}$ a.e. in Ω.

We denote by $\mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$, $T_k(u) \in \mathring{W}^{1,q}(\nu, \Omega)$. Clearly, $\mathring{W}^{1,q}(\nu, \Omega) \subset \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$. For every $u : \Omega \to \mathbb{R}$ and for every $x \in \Omega$ we set

$$k(u, x) = \min\{l \in \mathbb{N} : |u(x)| \leqslant l\}.$$

Definition 2.1. Let $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ and $i \in \{1, \dots, n\}$. Then $\delta_i u : \Omega \to \mathbb{R}$ is the function such that for every $x \in \Omega$, $\delta_i u(x) = D_i T_{k(u,x)}(u)(x)$.

Definition 2.2. If $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$, then $\delta u : \Omega \to \mathbb{R}^n$ is the mapping such that for every $x \in \Omega$ and for every $i \in \{1, \dots, n\}$, $(\delta u(x))_i = \delta_i u(x)$.

Now we give several propositions which will be used in the next sections.

Proposition 2.3. *Let $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ and $i \in \{1, \dots, n\}$. Then for every $k > 0$ we have $D_i T_k(u) = \delta_i u \cdot 1_{\{|u|<k\}}$ a.e. in Ω.*

The proof of this proposition is analogous to the proof of the corresponding result given in [18] for the nonweighted case.

Proposition 2.4. *Let $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ and $w \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$. Then $u - w \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$, and for every $i \in \{1, \dots, n\}$ and for every $k > 0$ we have*

$$D_i T_k(u - w) = \delta_i u - D_i w \quad \text{a.e. in } \{|u - w| < k\}.$$

Proposition 2.5. *Let $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ and $|\delta u| \in L^1(\Omega)$. Then $u \in \mathring{W}^{1,1}(\Omega)$ and for every $i \in \{1, \dots, n\}$ we have $D_i u = \delta_i u$ a.e. in Ω.*

The proofs of the two propositions above can be found in [20].

3. Statement of main results

Let $c_1, c_2 > 0$, $g_1, g_2 \in L^1(\Omega)$, $g_1, g_2 \geqslant 0$ in Ω, and for every $i \in \{1, \dots, n\}$, let $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory functions. We suppose that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^n (1/\nu_i)^{1/(q_i-1)}(x) |a_i(x, \xi)|^{q_i/(q_i-1)} \leqslant c_1 \sum_{i=1}^n \nu_i(x) |\xi_i|^{q_i} + g_1(x), \tag{3.1}$$

$$\sum_{i=1}^n a_i(x, \xi)\xi_i \geqslant c_2 \sum_{i=1}^n \nu_i(x)|\xi_i|^{q_i} - g_2(x). \tag{3.2}$$

Moreover, we assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^n [a_i(x, \xi) - a_i(x, \xi')](\xi_i - \xi'_i) > 0. \tag{3.3}$$

Note that the following assertions hold: if $u, w \in \mathring{W}^{1,q}(\nu, \Omega)$ and $i \in \{1, \dots, n\}$, then

$$a_i(x, \nabla u) D_i w \in L^1(\Omega); \tag{3.4}$$

if $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$, $w \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, $k > 0$, $l \geqslant k + \|w\|_{L^\infty(\Omega)}$ and $i \in \{1, \dots, n\}$, then

$$a_i(x, \delta u) D_i T_k(u - w) = a_i(x, \nabla T_l(u)) D_i T_k(u - w) \quad \text{a.e. in } \Omega; \tag{3.5}$$

if $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$, $w \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, $k > 0$ and $i \in \{1, \dots, n\}$, then

$$a_i(x, \delta u) D_i T_k(u - w) \in L^1(\Omega). \tag{3.6}$$

Assertion (3.4) is established with the use of (3.1). Assertion (3.5) is proved by means of Propositions 2.3 and 2.4. Assertion (3.6) is derived from Proposition 2.4 and assertions (3.4) and (3.5).

Let $f \in L^1(\Omega)$, and consider the Dirichlet problem

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, \nabla u) = f \quad \text{in } \Omega, \tag{3.7}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.8}$$

Definition 3.1. An entropy solution of problem (3.7), (3.8) is a function $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ such that for every function $w \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$ and for every $k \geqslant 1$,

$$\int_\Omega \Big\{ \sum_{i=1}^{n} a_i(x, \delta u) D_i T_k(u - w) \Big\} dx \leqslant \int_\Omega f\, T_k(u - w) dx.$$

Theorem 3.2. *There exists a unique entropy solution of problem* (3.7)*,* (3.8)*.*

Definition 3.3. A T-solution of problem (3.7), (3.8) is a function $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ such that:

(i) for every $i \in \{1, \dots, n\}$, $a_i(x, \delta u) \in L^1(\Omega)$;
(ii) for every function $w \in C_0^1(\Omega)$,

$$\int_\Omega \Big\{ \sum_{i=1}^{n} a_i(x, \delta u) D_i w \Big\} dx = \int_\Omega f w \, dx.$$

The next theorem shows that under additional conditions on q and ν the entropy solution of problem (3.7), (3.8) is a T-solution of the same problem. For the statement of this and further results we need the following numbers depending on the set q. We define

$$\overline{q} = \Big(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{q_i} \Big)^{-1}$$

and for every $m \in \mathbb{R}^n$ such that $m_i > 0$, $i = 1, \dots, n$, we set

$$p_m = n \Big(\sum_{i=1}^{n} \frac{1 + m_i}{m_i q_i} - 1 \Big)^{-1}.$$

Observe that if $m \in \mathbb{R}^n$ and for every $i \in \{1, \dots, n\}$, $m_i \geqslant 1/(q_i - 1)$, then $p_m > 1$. Moreover, if $m \in \mathbb{R}^n$ and for every $i \in \{1, \dots, n\}$ we have $m_i \geqslant 1/(q_i - 1)$ and $1/\nu_i \in L^{m_i}(\Omega)$, then the space $\mathring{W}^{1,q}(\nu, \Omega)$ is continuously imbedded into the space $L^{p_m}(\Omega)$. This fact follows from [22, Proposition 2.8]. In turn, the mentioned proposition was established with the use of an imbedding result for the non-weighted anisotropic case [29].

Theorem 3.4. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ such that the following conditions are satisfied:*

$$m_i \geqslant 1/(q_i - 1), \quad 1/\nu_i \in L^{m_i}(\Omega) \quad \forall i \in \{1, \dots, n\}; \tag{3.9}$$

$$\sigma_i > 0, \quad \frac{1}{\sigma_i} < 1 - \frac{(q_i - 1)\overline{q}}{p_m(\overline{q} - 1)}, \quad \nu_i \in L^{\sigma_i}(\Omega) \quad \forall i \in \{1, \dots, n\}. \tag{3.10}$$

Let u be the entropy solution of problem (3.7)*,* (3.8)*. Then u is a T-solution of problem* (3.7)*,* (3.8)*.*

From Theorems 3.2 and 3.4 we deduce the following result.

Corollary 3.5. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ such that conditions* (3.9) *and* (3.10) *are satisfied. Then there exists a T-solution of problem* (3.7)*,* (3.8)*.*

As we see, T-solutions of the given problem belong to function set $\mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$, and in general such solutions do not have weak derivatives. Now let us consider a kind of solutions having weak derivatives.

Definition 3.6. A W-solution of problem (3.7), (3.8) is a function $u \in \mathring{W}^{1,1}(\Omega)$ such that:

(i) for every $i \in \{1,\dots,n\}$, $a_i(x,\nabla u) \in L^1(\Omega)$;
(ii) for every function $w \in C_0^1(\Omega)$,

$$\int_\Omega \Big\{\sum_{i=1}^n a_i(x,\nabla u) D_i w\Big\} dx = \int_\Omega fw\,dx.$$

Proposition 3.7. *Let $u \in \mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$. Then u is a W-solution of problem* (3.7)*,* (3.8) *if and only if u is a T-solution of problem* (3.7)*,* (3.8) *and $|\delta u| \in L^1(\Omega)$.*

For the proof of this result it suffices to use Propositions 2.3 and 2.5 along with the fact that $D_i T_k(w) = D_i w \cdot 1_{\{|w|<k\}}$ a.e. in Ω if $w \in \mathring{W}^{1,1}(\Omega)$, $k > 0$ and $i \in \{1,\dots,n\}$.

Theorem 3.8. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that the following conditions are satisfied:*

$$\frac{\overline{q}}{p_m(\overline{q}-1)} < q_i - 1 - \frac{1}{m_i}, \quad 1/\nu_i \in L^{m_i}(\Omega) \quad \forall i \in \{1,\dots,n\}; \tag{3.11}$$

$$\frac{1}{\sigma_i} < 1 - \frac{(q_i-1)\overline{q}}{p_m(\overline{q}-1)}, \quad \nu_i \in L^{\sigma_i}(\Omega) \quad \forall i \in \{1,\dots,n\}. \tag{3.12}$$

Let u be the entropy solution of problem (3.7)*,* (3.8)*. Then u is a W-solution of problem* (3.7)*,* (3.8)*.*

From Theorems 3.2 and 3.8 we infer the following result.

Corollary 3.9. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that conditions* (3.11) *and* (3.12) *are satisfied. Then there exists a W-solution of problem* (3.7)*,* (3.8)*.*

Now we consider another kind of solutions (in the sense of an integral identity) whose existence requires less additional conditions as compared with W-solutions.

We denote by $\mathring{V}^{1,q}(\nu,\Omega)$ the set of all functions $w \in \mathring{W}^{1,q}(\nu,\Omega) \cap L^\infty(\Omega)$ such that for every $i \in \{1,\dots,n\}$, $\nu_i^{1/q_i} D_i w \in L^\infty(\Omega)$. Obviously, the set $\mathring{V}^{1,q}(\nu,\Omega)$ is nonempty. Moreover, if for every $i \in \{1,\dots,n\}$ we have $\nu_i \in L^\infty_{\rm loc}(\Omega)$, then $C_0^1(\Omega) \subset \mathring{V}^{1,q}(\nu,\Omega)$.

Definition 3.10. A weighted weak solution of problem (3.7), (3.8) is a function $u \in \mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$ such that:

(i) for every $i \in \{1,\dots,n\}$, $\nu_i^{1/q_i} \delta_i u \in L^1(\Omega)$;
(ii) for every $i \in \{1,\dots,n\}$, $(1/\nu_i)^{1/q_i} a_i(x,\delta u) \in L^1(\Omega)$;
(iii) for every function $w \in \mathring{V}^{1,q}(\nu,\Omega)$,

$$\int_\Omega \Big\{\sum_{i=1}^n a_i(x,\delta u) D_i w\Big\} dx = \int_\Omega fw\,dx.$$

Observe that if for every $i \in \{1,\dots,n\}$, $1/\nu_i \in L^\infty(\Omega)$, and u is a weighted weak solution of problem (3.7), (3.8), then $u \in \mathring{W}^{1,1}(\Omega)$. Moreover, if for every $i \in \{1,\dots,n\}$, $\nu_i \equiv 1$, and u is a weighted weak solution of problem (3.7), (3.8), then u is a W-solution of the same problem. These facts are easily established with the use of Proposition 2.5.

Theorem 3.11. *Suppose that there exists $m \in \mathbb{R}^n$ such that the following conditions are satisfied:*

$$m_i \geqslant 1/(q_i - 1), \quad 1/\nu_i \in L^{m_i}(\Omega) \quad \forall i \in \{1,\dots,n\}; \tag{3.13}$$

$$p_m > \frac{\overline{q}}{\overline{q}-1} \max\Big\{\frac{1}{q_i-1},\ q_i - 1\Big\} \quad \forall i \in \{1,\dots,n\}. \tag{3.14}$$

Let u be the entropy solution of problem (3.7), (3.8). *Then u is a weighted weak solution of problem* (3.7), (3.8).

From Theorems 3.2 and 3.11 we obtain the following result.

Corollary 3.12. *Suppose that there exists $m \in \mathbb{R}^n$ such that conditions* (3.13) *and* (3.14) *are satisfied. Then there exists a weighted weak solution of problem* (3.7), (3.8).

From Theorems 3.4, 3.8 and 3.11 we deduce the following result.

Corollary 3.13. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that conditions* (3.11) *and* (3.12) *are satisfied. Then the entropy solution of problem* (3.7), (3.8) *is also a T-solution, a W-solution and a weighted weak solution of the same problem.*

4. Proofs

4.1. **Basis for the proofs.** Here we give two results which were established in [20, 21, 22]. They form a basis for the proof of the theorems stated in the previous section.

Theorem 4.1. *Let V be a closed convex set in $\mathring{W}^{1,q}(\nu,\Omega)$ satisfying the conditions:*

$$V \cap L^\infty(\Omega) \neq \emptyset, \tag{4.1}$$

$$\text{if } u, w \in V \text{ and } k > 0, \text{ then } u - T_k(u-w) \in V. \tag{4.2}$$

Then there exists a unique function $u \in \mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$ such that the following assertions hold:

(i) *for every $w \in V \cap L^\infty(\Omega)$ and for every $k \geqslant 1$ we have $w - T_k(w-u) \in V$;*
(ii) *if $w \in V \cap L^\infty(\Omega)$, $k \geqslant 1$ and $l = k + \|w\|_{L^\infty(\Omega)}$, then*

$$\int_\Omega \Big\{\sum_{i=1}^n a_i(x, \nabla T_l(u)) D_i T_k(u-w)\Big\} dx \leqslant \int_\Omega f\, T_k(u-w) dx.$$

We note that conditions (3.2) and (3.3) are essential in the proof of the given theorem.

Proposition 4.2. *Let $m \in \mathbb{R}^n$, and let condition* (3.9) *be satisfied. Let V be a closed convex set in $\mathring{W}^{1,q}(\nu,\Omega)$ satisfying conditions* (4.1) *and* (4.2). *Let $u \in \mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$, and let assertions (i) and (ii) of Theorem 4.1 hold. Then for every $i \in \{1,\dots,n\}$ and for every λ, $0 < \lambda < \frac{q_i p_m(\overline{q}-1)}{p_m(\overline{q}-1)+\overline{q}}$, we have $\nu_i^{1/q_i} \delta_i u \in L^\lambda(\Omega)$.*

4.2. **Proof of Theorem 3.2.** Applying Theorem 4.1 for the case where $V = \mathring{W}^{1,q}(\nu,\Omega)$, we obtain that there exists a unique function $u \in \mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$ such that the following assertion holds: if $w \in \mathring{W}^{1,q}(\nu,\Omega) \cap L^\infty(\Omega)$, $k \geqslant 1$ and $l = k + \|w\|_{L^\infty(\Omega)}$, then

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \nabla T_l(u)) D_i T_k(u-w) \Big\} dx \leqslant \int_\Omega f\, T_k(u-w) dx.$$

This and assertion (3.5) imply that u is the unique entropy solution of problem (3.7), (3.8). The proof is complete.

4.3. **Proof of Theorem 3.4.** First of all, taking into account Proposition 2.4 and assertion (3.5), from Proposition 4.2 we deduce the following result.

Proposition 4.3. *Let $m \in \mathbb{R}^n$, and let condition* (3.9) *be satisfied. Let u be the entropy solution of problem* (3.7), (3.8). *Then for every $i \in \{1,\dots,n\}$ and for every λ, $0 < \lambda < \frac{q_i p_m(\overline{q}-1)}{p_m(\overline{q}-1)+\overline{q}}$, we have $\nu_i^{1/q_i} \delta_i u \in L^\lambda(\Omega)$.*

Now, suppose that there exist $m, \sigma \in \mathbb{R}^n$ such that conditions (3.9) and (3.10) are satisfied, and let u be the entropy solution of problem (3.7), (3.8).

Let us show that for every $i \in \{1,\dots,n\}$, $a_i(x,\delta u) \in L^1(\Omega)$. In fact, let $i \in \{1,\dots,n\}$. By (3.1), we have

$$|a_i(x,\delta u)| \leqslant (c_1+1) \sum_{j=1}^n \nu_i^{1/q_i} |\nu_j^{1/q_j} \delta_j u|^{q_j(q_i-1)/q_i} + \nu_i^{1/q_i} g_1^{(q_i-1)/q_i} \quad \text{a.e. in } \Omega. \tag{4.3}$$

Using Young's inequality with the exponents q_i and $q_i/(q_i-1)$, we obtain that $\nu_i^{1/q_i} g_1^{(q_i-1)/q_i} \leqslant \nu_i + g_1$. Hence, taking into account that $g_1 \in L^1(\Omega)$ and, by condition (3.10), $\nu_i \in L^1(\Omega)$, we infer that

$$\nu_i^{1/q_i} g_1^{(q_i-1)/q_i} \in L^1(\Omega). \tag{4.4}$$

Next, we fix $j \in \{1,\dots,n\}$ and set

$$\lambda_{ij} = \frac{\sigma_i (q_i-1) q_j}{\sigma_i q_i - 1}\,.$$

Using Young's inequality with the exponents $\sigma_i q_i$ and $\sigma_i q_i/(\sigma_i q_i - 1)$, we obtain

$$\nu_i^{1/q_i} |\nu_j^{1/q_j} \delta_j u|^{q_j(q_i-1)/q_i} \leqslant \nu_i^{\sigma_i} + |\nu_j^{1/q_j} \delta_j u|^{\lambda_{ij}}\,. \tag{4.5}$$

Observe that, by condition (3.10), we have

$$\nu_i \in L^{\sigma_i}(\Omega), \tag{4.6}$$

$$\lambda_{ij} < \frac{q_j p_m(\overline{q}-1)}{p_m(\overline{q}-1)+\overline{q}}\,.$$

Since condition (3.9) is satisfied, from the latter inequality and Proposition 4.3 it follows that $\nu_j^{1/q_j} \delta_j u \in L^{\lambda_{ij}}(\Omega)$. This inclusion along with (4.6) and (4.5) implies that for every $j \in \{1,\dots,n\}$,

$$\nu_i^{1/q_i} |\nu_j^{1/q_j} \delta_j u|^{q_j(q_i-1)/q_i} \in L^1(\Omega). \tag{4.7}$$

From (4.3), (4.4) and (4.7) we deduce that for every $i \in \{1,\dots,n\}$, $a_i(x,\delta u) \in L^1(\Omega)$.

Further, we fix $w \in C_0^1(\Omega)$ and for every $h \in \mathbb{N}$ we set $w_h = T_h(u) - w$. Now let us fix $k \geqslant \|w\|_{L^\infty(\Omega)} + 1$, and let $h \in \mathbb{N}$. Since u is the entropy solution of problem (3.7), (3.8) and $w_h \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, by Definition 3.1, we have

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i T_k(u - w_h) \Big\} dx \leqslant \int_\Omega f\, T_k(u - w_h) dx. \tag{4.8}$$

From Propositions 2.3 and 2.4, it follows that for every $i \in \{1, \dots, n\}$,

$$D_i T_k(u - w_h) = (\delta_i u \cdot 1_{\{|u| \geqslant h\}} + D_i w) \cdot 1_{\{|u - w_h| < k\}} \quad \text{a.e. in } \Omega.$$

Using this fact and (3.2), we obtain

$$\begin{aligned} &\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i T_k(u - w_h) \Big\} dx \\ &\geqslant \int_{\{|u - w_h| < k\}} \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx - \int_{\{|u| \geqslant h\}} g_2 dx. \end{aligned}$$

This and (4.8) imply that for every $h \in \mathbb{N}$,

$$\int_{\{|u - w_h| < k\}} \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx \leqslant \int_\Omega f\, T_k(u - w_h) dx + \int_{\{|u| \geqslant h\}} g_2 dx. \tag{4.9}$$

Observe that for every $h \in \mathbb{N}$, $\operatorname{meas}(\Omega \setminus \{|u - w_h| < k\}) \leqslant \operatorname{meas}\{|u| \geqslant h\}$. Then, taking into account that $\operatorname{meas}\{|u| \geqslant h\} \to 0$ as $h \to +\infty$ and the functions g_2 and $a_i(x, \delta u) D_i w$, $i = 1, \dots, n$, are summable in Ω, we obtain

$$\int_{\{|u - w_h| < k\}} \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx \to \int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx, \tag{4.10}$$

$$\int_{\{|u| \geqslant h\}} g_2 dx \to 0. \tag{4.11}$$

Finally, since $u - w_h \to w$ in Ω and $k \geqslant \|w\|_{L^\infty(\Omega)}$, we have $T_k(u - w_h) \to w$ in Ω. Hence, applying Dominated Convergence Theorem, we obtain

$$\int_\Omega f\, T_k(u - w_h) dx \to \int_\Omega f w \, dx. \tag{4.12}$$

From (4.9)–(4.12) we infer that

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx \leqslant \int_\Omega f w \, dx.$$

Therefore, for every $w \in C_0^1(\Omega)$,

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx = \int_\Omega f w \, dx.$$

This completes the proof of Theorem 3.4.

We remark that the idea of using the functions $w_h = T_h(u) - w$ in the above proof is taken from [6, Corollary 4.3].

4.4. **Proof of Theorem 3.8.** Suppose that there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that conditions (3.11) and (3.12) are satisfied, and let u be the entropy solution of problem (3.7), (3.8). Let us show that $|\delta u| \in L^1(\Omega)$. In fact, let $i \in \{1, \dots, n\}$. Clearly,

$$|\delta_i u| = (1/\nu_i)^{1/q_i} |\nu_i^{1/q_i} \delta_i u| \quad \text{a.e. in } \Omega. \tag{4.13}$$

Using Young's inequality with the exponents $m_i q_i$ and $m_i q_i/(m_i q_i - 1)$, we obtain

$$(1/\nu_i)^{1/q_i} |\nu_i^{1/q_i} \delta_i u| \leqslant (1/\nu_i)^{m_i} + |\nu_i^{1/q_i} \delta_i u|^{m_i q_i/(m_i q_i - 1)}. \tag{4.14}$$

By condition (3.11), we have $1/\nu_i \in L^{m_i}(\Omega)$ and

$$\frac{m_i q_i}{m_i q_i - 1} < \frac{q_i p_m(\overline{q} - 1)}{p_m(\overline{q} - 1) + \overline{q}}.$$

This along with Proposition 4.3 and (4.13) and (4.14) implies that $|\delta_i u| \in L^1(\Omega)$, $i = 1, \dots, n$. Hence, $|\delta u| \in L^1(\Omega)$. Then, taking into account that conditions (3.9) and (3.10) are satisfied and using Theorem 3.4 and Proposition 3.7, we obtain that u is a W-solution of problem (3.7), (3.8). The proof is complete.

4.5. **An integral identity for the entropy solution.** According to Theorem 3.4, under conditions (3.9) and (3.10) the entropy solution of problem (3.7), (3.8) is a solution in the sense of an integral identity for functions in $C_0^1(\Omega)$. In this subsection, for every function $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ we introduce a function set $\mathcal{M}(u)$ and show that if u is the entropy solution of the problem under consideration, then u satisfies the corresponding integral identity for functions in $\mathcal{M}(u)$. This result, having a self-contained interest, will be used in the proof of Theorem 3.11.

For every function $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$ we set

$$\mathcal{M}(u) = \{w \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega) : a_i(x, \delta u) D_i w \in L^1(\Omega),\ i = 1, \dots, n\}.$$

Clearly, if $u \in \mathring{\mathcal{T}}^{1,q}(\nu, \Omega)$, then the set $\mathcal{M}(u)$ is non-empty.

Proposition 4.4. *Let u be the entropy solution of problem* (3.7), (3.8). *Then for every $w \in \mathcal{M}(u)$,*

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i w \Big\} dx = \int_\Omega f w \, dx.$$

Proof. We fix $w \in \mathcal{M}(u)$ and for every $h \in \mathbb{N}$ we set $w_h = T_h(u) - w$. Then we fix $k \geqslant \|w\|_{L^\infty(\Omega)} + 1$, and let $h \in \mathbb{N}$. Since u is the entropy solution of problem (3.7), (3.8) and $w_h \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, by Definition 3.1, inequality (4.8) holds. Then, arguing as in the proof of Theorem 3.4, for every $h \in \mathbb{N}$ we obtain inequality (4.9). At the same time limit relations (4.10)–(4.12) hold. We only note that now the convergence in (4.10) is justified by the fact that for every $i \in \{1, \dots, n\}$, $a_i(x, \delta u) D_i w \in L^1(\Omega)$, which holds due to the inclusion $w \in \mathcal{M}(u)$. From (4.9)–(4.12) we derive the required result. The proposition is proved. □

Corollary 4.5. *Let u be the entropy solution of problem* (3.7), (3.8). *Then for every function $w \in \mathring{W}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$ and for every $k > 0$,*

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \delta u) D_i T_k(u - w) \Big\} dx = \int_\Omega f \, T_k(u - w) dx.$$

Proof. Let $w \in \mathring{W}^{1,q}(\nu,\Omega) \cap L^\infty(\Omega)$ and $k > 0$. By Proposition 2.4 and assertion (3.6), we have $T_k(u - w) \in \mathcal{M}(u)$. Then from Proposition 4.4 we deduce the required equality. $\square$

4.6. **Proof of Theorem 3.11.** Suppose that there exists $m \in \mathbb{R}^n$ such that conditions (3.13) and (3.14) are satisfied, and let u be the entropy solution of problem (3.7), (3.8).

Let $i \in \{1,\dots,n\}$. By condition (3.14), we have $p_m(\overline{q}-1) > \overline{q}/(q_i-1)$ and $p_m(\overline{q}-1) > \overline{q}(q_i-1)$. Hence,

$$1 < \frac{q_i p_m(\overline{q}-1)}{p_m(\overline{q}-1)+\overline{q}}, \quad \frac{q_i-1}{q_i} < \frac{p_m(\overline{q}-1)}{p_m(\overline{q}-1)+\overline{q}}\,. \tag{4.15}$$

Since condition (3.13) coincides with condition (3.9), in view of Proposition 4.3 and inequalities (4.15), we have $\nu_i^{1/q_i}\delta_i u \in L^1(\Omega)$ and

$$|\nu_j^{1/q_j}\delta_j u|^{q_j(q_i-1)/q_i} \in L^1(\Omega) \quad \forall j \in \{1,\dots,n\}\,.$$

Therefore, taking into account that, by (3.1),

$$(1/\nu_i)^{1/q_i}|a_i(x,\delta u)| \leqslant (c_1+1)\sum_{j=1}^n |\nu_j^{1/q_j}\delta_j u|^{q_j(q_i-1)/q_i} + g_1^{(q_i-1)/q_i} \quad \text{a.e. in } \Omega,$$

we obtain the inclusion $(1/\nu_i)^{1/q_i} a_i(x,\delta u) \in L^1(\Omega)$.

Thus, $u \in \mathring{\mathcal{T}}^{1,q}(\nu,\Omega)$ and properties (i) and (ii) of Definition 3.10 hold. At the same time, property (ii) of this definition implies that $\mathring{V}^{1,q}(\nu,\Omega) \subset \mathcal{M}(u)$. Then, by Proposition 4.4, property (iii) of Definition 3.10 holds. Hence, u is a weighted weak solution of problem (3.7), (3.8). This completes the proof.

5. Particular cases and examples

First of all we note that Definitions 3.1, 3.3 and 3.6 have the same form with the definitions of the corresponding kinds of solutions studied in [6, 8, 9] in the case of nondegenerate isotropic elliptic second-order equations with L^1-data. It is easy to see that in this case ($q_i = q_1$ and $\nu_i \equiv 1$ for every $i \in \{1,\dots,n\}$) there exist $m,\sigma \in \mathbb{R}^n$, satisfying conditions (3.9) and (3.10), and the existence of $m,\sigma \in \mathbb{R}^n$ with positive coordinates, satisfying conditions (3.11) and (3.12), is equivalent to the requirement $q_1 > 2 - 1/n$. Thus, the results of Section 3 on entropy, T- and W-solutions of problem (3.7), (3.8) generalize the known results concerning solutions of nondegenerate isotropic elliptic second-order equations with L^1-right-hand sides.

In regard to the nondegenerate anisotropic case we state the following proposition.

Proposition 5.1. *Let $\nu_i \equiv 1$ for all $i \in \{1,\dots,n\}$. Then*

(i) *the existence of $m,\sigma \in \mathbb{R}^n$ satisfying conditions* (3.9) *and* (3.10) *is equivalent to the requirement*

$$q_i < \frac{(n-1)\overline{q}}{n-\overline{q}} \quad \forall i \in \{1,\dots,n\}; \tag{5.1}$$

(ii) *the existence of $m,\sigma \in \mathbb{R}^n$ with positive coordinates satisfying conditions* (3.11) *and* (3.12) *is equivalent to the requirement*

$$\frac{(n-1)\overline{q}}{n(\overline{q}-1)} < q_i < \frac{(n-1)\overline{q}}{n-\overline{q}} \quad \forall i \in \{1,\dots,n\}; \tag{5.2}$$

(iii) *the existence of* $m \in \mathbb{R}^n$ *satisfying conditions* (3.13) *and* (3.14) *is equivalent to requirement* (5.2).

We omit the proof of the proposition because of its simplicity. Observe that requirement (5.2) coincides with the condition imposed on the corresponding exponents in [4, 5] where only the nondegenerate case was considered.

Example 5.2. Let $n \geqslant 3$, $1 < \alpha < n/2$, $\alpha < \beta < n$, and let $q_i = \alpha$ if $i = 1, \dots, n-1$, and $q_n = \beta$. We have

$$\alpha < \frac{\alpha(n-2)}{n-1-\alpha} < n.$$

It is easy to verify that requirement (5.1) is equivalent to the condition

$$\beta < \frac{\alpha(n-2)}{n-1-\alpha}\,, \tag{5.3}$$

and if $n \geqslant 4$ and $\alpha \geqslant 2 - 1/n$, then requirement (5.2) is also equivalent to condition (5.3).

As far as the degenerate isotropic case is concerned, the following proposition holds.

Proposition 5.3. *For every* $i \in \{1, \dots, n\}$, *let* $q_i = q_1$ *and* $\nu_i = \nu_1$. *Then*

(i) *the existence of* $m, \sigma \in \mathbb{R}^n$ *satisfying conditions* (3.9) *and* (3.10) *is equivalent to the existence of* $t, s \in \mathbb{R}$ *such that* $t \geqslant 1/(q_1 - 1)$, $t > n/q_1$, $s > nt/(tq_1 - n)$, $1/\nu_1 \in L^t(\Omega)$ *and* $\nu_1 \in L^s(\Omega)$;

(ii) *the existence of* $m, \sigma \in \mathbb{R}^n$ *with positive coordinates satisfying conditions* (3.11) *and* (3.12) *is equivalent to the existence of* $t, s \in \mathbb{R}$ *such that* $t > n/q_1$, $1/t < q_1 - 2 + 1/n$, $s > nt/(tq_1 - n)$, $1/\nu_1 \in L^t(\Omega)$ *and* $\nu_1 \in L^s(\Omega)$;

(iii) *the existence of* $m \in \mathbb{R}^n$ *satisfying conditions* (3.13) *and* (3.14) *is equivalent to the existence of* $t \in \mathbb{R}$ *such that* $t \geqslant 1/(q_1 - 1)$, $t > n/q_1$, $1/t < q_1(q_1 - 2 + 1/n)$ *and* $1/\nu_1 \in L^t(\Omega)$.

Proof. Let $m, \sigma \in \mathbb{R}^n$, and let conditions (3.9) and (3.10) be satisfied. Setting

$$t = \max\{m_i : i = 1, \dots, n\}, \quad s = \sigma_1, \tag{5.4}$$

by conditions (3.9) and (3.10), we immediately have $t \geqslant 1/(q_1 - 1)$, $1/\nu_1 \in L^t(\Omega)$ and $\nu_1 \in L^s(\Omega)$. Moreover, since $\overline{q} = q_1$ and $q_1/p_m \geqslant 1 - q_1/n + 1/t$, from condition (3.10) we derive that $t > n/q_1$ and $s > nt/(tq_1 - n)$. Conversely, let $t, s \in \mathbb{R}$, and let $t \geqslant 1/(q_1 - 1)$, $t > n/q_1$, $s > nt/(tq_1 - n)$, $1/\nu_1 \in L^t(\Omega)$ and $\nu_1 \in L^s(\Omega)$. Then, taking $m, \sigma \in \mathbb{R}^n$ such that for every $i \in \{1, \dots, n\}$, $m_i = t$ and $\sigma_i = s$, without any difficulties we obtain that conditions (3.9) and (3.10) are satisfied. Thus, assertion (i) is valid.

Next, let $m, \sigma \in \mathbb{R}^n$, for every $i \in \{1, \dots, n\}$, $m_i > 0$ and $\sigma_i > 0$, and let conditions (3.11) and (3.12) be satisfied. Using these conditions, for $t, s \in \mathbb{R}$ defined by (5.4) we easily establish that $t > n/q_1$, $1/t < q_1 - 2 + 1/n$, $s > nt/(tq_1 - n)$, $1/\nu_1 \in L^t(\Omega)$ and $\nu_1 \in L^s(\Omega)$. Conversely, if we have $t, s \in \mathbb{R}$ with the given properties, then, taking $m, \sigma \in \mathbb{R}^n$ such that for every $i \in \{1, \dots, n\}$, $m_i = t$ and $\sigma_i = s$, we easily get that conditions (3.11) and (3.12) are satisfied. Thus, assertion (ii) is valid.

Finally, let $m \in \mathbb{R}^n$, and let conditions (3.13) and (3.14) be satisfied. Setting $t = \max\{m_i : i = 1, \dots, n\}$, we have

$$1 - \frac{q_1}{n} + \frac{1}{t} \leqslant \frac{q_1}{p_m} \,. \tag{5.5}$$

At the same time, from condition (3.13) we infer that $t \geqslant 1/(q_1 - 1)$ and $1/\nu_1 \in L^t(\Omega)$, and from condition (3.14) we obtain that $q_1/p_m < \min\{(q_1-1)^2, 1\}$. This and (5.5) imply that $t > n/q_1$ and $1/t < q_1(q_1 - 2 + 1/n)$. Conversely, if $t \in \mathbb{R}$, and $t \geqslant 1/(q_1-1)$, $t > n/q_1$, $1/t < q_1(q_1 - 2 + 1/n)$ and $1/\nu_1 \in L^t(\Omega)$, then, taking $m \in \mathbb{R}^n$ such that for every $i \in \{1, \dots, n\}$, $m_i = t$, we easily get that conditions (3.13) and (3.14) are satisfied. Thus, assertion (iii) is valid. This completes the proof of the proposition. □

We remark that the conditions on t, s and ν_1 in assertion (i) of Proposition 5.3 are of the same kind as in [28]. The following two examples concern the degenerate anisotropic case.

Example 5.4. Let $n \geqslant 3$ and $1 < \alpha < n - 1$. We have $\alpha < \alpha(n-2)/(n-1-\alpha)$. Let

$$\alpha \leqslant \beta < \min\big\{\frac{\alpha(n-2)}{n-1-\alpha}, \, n\big\}. \tag{5.6}$$

Since, by (5.6), $\beta(n-1-\alpha) < \alpha(n-2)$, we have $(\beta-\alpha)/(\beta-1) < \alpha/(n-1)$. Let

$$0 < \gamma < n \min\big\{\frac{\alpha}{n-1} - \frac{\beta-\alpha}{\beta-1}, \, \alpha - 1\big\}. \tag{5.7}$$

Since, by (5.7),

$$\frac{\gamma}{n} + \frac{\beta-\alpha}{\beta-1} < \frac{\alpha}{n-1} \,,$$

we have

$$1 - \frac{n-1}{\alpha}\Big(\frac{\gamma}{n} + \frac{\beta-\alpha}{\beta-1}\Big) > 0.$$

Let

$$0 < \tau < n \min\big\{\beta\big[1 - \frac{n-1}{\alpha}\big(\frac{\gamma}{n} + \frac{\beta-\alpha}{\beta-1}\big)\big], \, \beta - 1\big\}. \tag{5.8}$$

Next, assume that $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$. Moreover, let $q_i = \alpha$ and for every $x \in \Omega$, $\nu_i(x) = |x|^\gamma$ if $i = 1, \dots, n-1$, and let $q_n = \beta$ and for every $x \in \Omega$, $\nu_n(x) = |x|^\tau$.

It is easy to see that for every $i \in \{1, \dots, n\}$, $q_i \in (1, n)$ and $\nu_i \in L^1(\Omega)$. Besides, since in view of (5.7) and (5.8), $\gamma < n(\alpha-1)$ and $\tau < n(\beta-1)$, for every $i \in \{1, \dots, n\}$ we have $(1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega)$.

Taking into account (5.7) and (5.8), we fix a positive number ε_1 such that

$$\varepsilon_1 \leqslant \min\big\{\frac{n(\alpha-1)}{\gamma} - 1, \frac{n(\beta-1)}{\tau} - 1\big\}, \tag{5.9}$$

$$\frac{\varepsilon_1}{n}\big[(n-1)\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\big] < 1 - \frac{\tau}{n\beta} - \frac{n-1}{\alpha}\Big(\frac{\gamma}{n} + \frac{\beta-\alpha}{\beta-1}\Big). \tag{5.10}$$

Now, define $\varepsilon = 1 + \varepsilon_1$, and let $m \in \mathbb{R}^n$ be such that $m_i = \frac{n}{\gamma\varepsilon}$ if $i = 1, \dots, n-1$, and $m_n = \frac{n}{\tau\varepsilon}$.

Using (5.9) and the inequality $\varepsilon > 1$, we establish that condition (3.9) is satisfied. Moreover, using (5.10), we obtain

$$\begin{aligned}\frac{1}{p_m} &= \frac{1}{n}\Big(\sum_{i=1}^{n}\frac{1+m_i}{m_iq_i} - 1\Big)\\ &= \frac{1}{n}\Big\{\frac{n-1}{\alpha} + \frac{1}{\beta} + \frac{(n-1)\gamma}{n\alpha} + \frac{\tau}{n\beta} + \frac{\varepsilon_1}{n}\big[\frac{(n-1)\gamma}{\alpha} + \frac{\tau}{\beta}\big] - 1\Big\}\\ &< \frac{1}{n}\Big\{\frac{1}{\beta} + \frac{(n-1)(\alpha-1)}{\alpha(\beta-1)}\Big\} = \frac{\overline{q}-1}{\overline{q}(\beta-1)}\,.\end{aligned}$$

Hence

$$1 - \frac{(\beta-1)\overline{q}}{p_m(\overline{q}-1)} > 0. \tag{5.11}$$

Then, fixing $\beta_0 > 0$ such that

$$\frac{1}{\beta_0} < 1 - \frac{(\beta-1)\overline{q}}{p_m(\overline{q}-1)}$$

and taking $\sigma \in \mathbb{R}^n$ such that for every $i \in \{1,\dots,n\}$, $\sigma_i = \beta_0$, due to the inequality $\alpha \leqslant \beta$, we establish that condition (3.10) is satisfied.

Next, suppose additionally that $n > 3$ and $\alpha > 2$. Obviously, $\alpha - 1 > 1/(\beta-1)$, and from (5.11) it follows that condition (3.14) is satisfied. Moreover, if additionally we have

$$\begin{gathered}\frac{\gamma}{n} < \alpha - 1 - \frac{1}{\beta-1}\,,\quad \frac{\tau}{n} < \alpha - 1 - \frac{1}{\beta-1}\,,\\ \frac{\gamma}{n}\,\varepsilon_1 < \alpha - 1 - \frac{1}{\beta-1} - \frac{\gamma}{n}\,,\quad \frac{\tau}{n}\,\varepsilon_1 < \alpha - 1 - \frac{1}{\beta-1} - \frac{\tau}{n}\,,\end{gathered}$$

then for every $i \in \{1,\dots,n\}$,

$$\frac{1}{\beta-1} < \alpha - 1 - \frac{1}{m_i}\,,$$

and from (5.11) it follows that condition (3.11) is satisfied.

Example 5.5. Let $n \geqslant 3$ and $(2n-3)/(n-1) < \alpha < n-1$. We have $\alpha n > 2(n-1)$ and

$$\max\Big\{\frac{\alpha}{\alpha n - 2(n-1)}\,,\,\alpha\Big\} < \min\Big\{\frac{\alpha(n-2)}{n-1-\alpha}\,,\,n\Big\}.$$

Let

$$\max\Big\{\frac{\alpha}{\alpha n - 2(n-1)}\,,\,\alpha\Big\} < \beta < \min\Big\{\frac{\alpha(n-2)}{n-1-\alpha}\,,\,n\Big\}. \tag{5.12}$$

We set

$$r = n\Big(\frac{n-1}{\alpha} + \frac{1}{\beta}\Big)^{-1}.$$

Since, by (5.12),

$$\frac{\alpha}{\alpha n - 2(n-1)} < \beta < \frac{\alpha(n-2)}{n-1-\alpha}\,,$$

we have

$$\Big(\frac{1}{r} - \frac{1}{n}\Big)\frac{r}{r-1} < \min\Big\{\frac{1}{\beta-1}\,,\,\alpha-1\Big\}. \tag{5.13}$$

Consequently, taking into account that $\alpha < \beta$, we obtain

$$\Big(\frac{1}{r} - \frac{1}{n}\Big)\frac{(\alpha-1)r}{r-1} < 1.$$

We define σ_* by

$$\frac{1}{\sigma_*} = 1 - \Big(\frac{1}{r} - \frac{1}{n}\Big)\frac{(\alpha-1)r}{r-1}$$

and fix γ and τ such that $n/\sigma_* \leqslant \gamma < n$ and $0 < \tau < n$.

Next, assume that $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$. Moreover, let $q_i = \alpha$ and for every $x \in \Omega \setminus \{0\}$, $\nu_i(x) = |x|^{-\gamma}$ if $i = 1, \dots, n-1$, and let $q_n = \beta$ and for every $x \in \Omega \setminus \{0\}$, $\nu_n(x) = |x|^{-\tau}$. It is easy to see that for every $i \in \{1, \dots, n\}$, $q_i \in (1, n)$, $\nu_i \in L^1(\Omega)$ and $(1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega)$. Besides, we have

$$\overline{q} = r. \tag{5.14}$$

Taking into account (5.13), we fix a number r_1 such that

$$\frac{1}{r} - \frac{1}{n} < r_1 < \frac{r-1}{r}\min\Big\{\frac{1}{\beta-1}\,,\,\alpha-1\Big\}, \tag{5.15}$$

and then we fix a number t such that $t \geqslant 1/(\alpha-1)$ and

$$\frac{1}{tr} < \frac{r-1}{r}\min\Big\{\frac{1}{\beta-1}\,,\,\alpha-1\Big\} - r_1. \tag{5.16}$$

Now let $b \in \mathbb{R}^n$ be such that $b_i = t$, $i = 1, \dots, n$. For every $i \in \{1, \dots, n\}$ we have $b_i \geqslant 1/(q_i - 1)$ and $1/\nu_i \in L^{b_i}(\Omega)$. Moreover,

$$\frac{1}{p_b} = \frac{1}{r} - \frac{1}{n} + \frac{1}{tr}\,.$$

This equality along with (5.14)–(5.16) implies that

$$\frac{1}{p_b} < \frac{\overline{q}-1}{\overline{q}}\min\Big\{\frac{1}{\beta-1}\,,\,\alpha-1\Big\}.$$

Hence it follows that for every $i \in \{1, \dots, n\}$,

$$p_b > \frac{\overline{q}}{\overline{q}-1}\max\Big\{\frac{1}{q_i-1}\,,\,q_i-1\Big\}.$$

Thus, we conclude that there exists $m \in \mathbb{R}^n$ such that conditions (3.13) and (3.14) are satisfied. At the same time, since $\gamma\sigma_* \geqslant n$, we have $\nu_1 \notin L^{\sigma_*}(\Omega)$. This and (5.14) imply that there are no $m, \sigma \in \mathbb{R}^n$ such that both conditions (3.9) and (3.10) are satisfied, and there are no $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that both conditions (3.11) and (3.12) are satisfied.

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