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MAXIMUM NUMBER OF LIMIT CYCLES FOR GENERALIZED LIÉNARD DIFFERENTIAL EQUATIONS

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Abstract. Applying the averaging theory of first and second order to a class of generalized polynomial Liénard differential equations, we improve the known lower bounds for the maximum number of limit cycles that this class can exhibit.

1. Introduction and statement of the main results

One of the main problems in the theory of ordinary differential equations is the study of their limit cycles, their existence, their number and their stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. These last years hundreds of papers studied the limit cycles of planar polynomial differential systems. The Second part of the 16th Hilbert's problem [13] is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. The generalized polynomial Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1.1}$$

was introduced in [17]. Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively. The Liénard equation, which is often taken as the typical example of nonlinear self-excited vibration problem, can be used to model resistor-inductor-capacitor circuits with nonlinear circuit elements. It can also be used to model certain mechanical systems which contain the nonlinear damping coefficients and the restoring force or stiffness. Limit cycles usually arise at a Hopf bifurcation in nonlinear systems with varying parameters. In mechanical systems, the varying parameter is frequently a damping coefficient (see [1, 7]). Lots of papers discussed the possible number of limit cycle of Liénard or generalized mixed Rayleigh-Liénard oscillators. Ding and Leung [7] investigated the generalized mixed Rayleigh-Liénard oscillator with highly nonlinear terms. They consider mainly the number of limit cycle bifurcation diagrams of these systems. For the subclass of polynomial vector fields (1.1) we have a simplified version of Hilbert's problem, see [18, 26].

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate

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singular point, that are so called *small amplitude limit cycles*, see [19, 23]. We denote by $\hat{H}(m,n)$ the maximum number of small amplitude limit cycles for systems of the form (1.1). The values of $\hat{H}(m,n)$ give a lower bound for the maximum number $H(m,n)$ (i.e. the Hilbert number) of limit cycles that the differential equation (1.1) with m and n fixed can have. For more information about the Hilbert's 16th problem and related topics see [14] and [15].

Now we shall describe briefly the main results about the limit cycles on Liénard differential systems as it is described in [21].

- In 1928 Liénard [17] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function , which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (1.1) has a unique limit cycle.
- In 1973 Rychkov [24] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is an odd polynomial of degree five, then equation (1.1) has at most two limit cycles.
- In 1977 Lins, de Melo and Pugh [18] proved that $H(1,1) = 0$ and $H(1,2) = 1$.
- In 1990, 1996, Dumortier, Li and Rousseau in [10] and [8] proved that $H(3,1) = 1$.
- In 1998 Coppel [6] proved that $H(2,1) = 1$.
- In 1997 Dumortier and Chengzhi [9] proved that $H(2,2) = 1$.
- In 2010 Chengzhi Li and Llibre [16] proved that $H(1,3) = 1$.

Blows, Lloyd [3] and Lynch [20, 22] have used inductive arguments in order to prove the following results.

- If g is odd then $\hat{H}(m,n) = [\frac{n}{2}]$.
- If f is even then $\hat{H}(m,n) = n$, whatever g is.
- If f is odd then $\hat{H}(m,2n+1) = [\frac{(m-2)}{2}] + n$.
- If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m,2) = m$.

Christopher and Lynch [5] developed a new algebraic method for determining the Liapunov quantities of system (1.1) and proved the following:

- $\hat{H}(m,2) = [\frac{(2m+1)}{3}]$,
- $\hat{H}(2,n) = [\frac{(2n+1)}{3}]$,
- $\hat{H}(m,3) = 2[\frac{(3m+2)}{8}]$ for all $1 < m \leq 50$,
- $\hat{H}(3,n) = 2[\frac{(3n+2)}{8}]$ for all $1 < n \leq 50$,
- $\hat{H}(4,k) = \hat{H}(k,4)$ for $k = 6,7,8,9$ and $\hat{H}(5,6) = \hat{H}(6,5)$.

In 1998, Gasull and Torregrosa [11] obtained upper bounds for $\hat{H}(7,6)$, $\hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$. In 2006, Yu and Han [28] proved that $\hat{H}(m,n) = \hat{H}(n,m)$ for $n = 4$, $m = 10,11,12,13$; $n = 5$, $m = 6,7,8,9$; $n = 6$, $m = 5,6$. In 2009, Llibre, Mereu and Teixeira [21] using the averaging theory studied the maximum number of limit cycles $\tilde{H}(m,n)$ which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential equations of degrees m and n of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \sum_{k\geq 1} \epsilon^k (f_n^k(x)y + g_m^k(x)), \end{aligned} \tag{1.2}$$

where for every k the polynomials $g_m^k(x)$ and $f_n^k(x)$ have degree m and n respectively, and ε is a small parameter. In 2011, Badi and Makhlouf [2] using the averaging theory studied the maximum number of limit cycles $\tilde{H}(m,n)$ which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential equations of degrees m and n as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \sum_{k\geq 1} \epsilon^k (f_n^k(x,y)y + g_m^k(x)), \end{aligned} \tag{1.3}$$

where for every k the polynomial $g_m^k(x)$ has degree m, the polynomial $f_n^k(x,y)$ has degree n on x and y and ε is a small parameter; i.e., the maximal number of *medium amplitude limit cycles* which can bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, perturbed as in (1.3). In fact in [2] the authors computed lower estimations of $\tilde{H}(m,n)$. More precisely they compute the maximum number of limit cycles $\tilde{H}_k(m,n)$ which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory of order k, for $k = 1,2$. Of course $\tilde{H}_k(m,n) \leq \tilde{H}(m,n) \leq H(m,n)$.

In this work using the averaging theory we study the maximum number of limit cycles $\tilde{H}(l,m,n)$ which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential equations of degrees l, m and n of the form

$$\begin{aligned} \dot{x} &= y + \sum_{k\geq 1} \epsilon^k h_l^k(x), \\ \dot{y} &= -x - \sum_{k\geq 1} \epsilon^k (f_n^k(x,y)y + g_m^k(x)), \end{aligned} \tag{1.4}$$

where for every k the polynomials $h_l^k(x)$, $g_m^k(x)$ and $f_n^k(x,y)$ have degree l, m and n respectively and ε is a small parameter, i.e. the maximal number of *medium amplitude limit cycles* which can bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, perturbed as in (1.4).

Let k be a positive integer. We define $E(k)$ as the largest even integer less than or equal to k, and $O(k)$ as the largest odd integer less than or equal to k. The main result that improve the mentioned previous results is the following.

Theorem 1.1. *If for every $k = 1,2$ the polynomials $h_l^k(x)$, $g_m^k(x)$ and $f_n^k(x,y)$ have degree l, m and n respectively, with $l,m,n \geq 1$, then for $|\varepsilon|$ sufficiently small, the maximum number of medium limit cycles of the polynomial Liénard differential systems* (1.4) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory*

(a) *of first order*

$$\tilde{H}_1(l,m,n) = \Big[\frac{\max\{O(l), O(n+1)\} - 1}{2}\Big] = \max\Big\{\Big[\frac{l-1}{2}\Big], \Big[\frac{n}{2}\Big]\Big\}$$

(b) *of second order*

$$\begin{aligned} \tilde{H}_2(l,m,n) = \Big[\Big(&\max\Big\{O(n) + O(m) + 1, O(n) + E(l) + 1, E(m) + E(l), \\ &2O(n) + 2, O(l), O(n+1)\Big\} - 1\Big)/2\Big] \end{aligned}$$

Of course if $H(l,m,n)$ is the Hilbert number for our polynomial Liénard differential systems (1.4), then $\tilde{H}_k(l,m,n) \neq H(l,m,n)$ for $k=1,2$; i.e. the numbers $\tilde{H}_k(l,m,n)$ provide lower bounds for the Hilbert numbers of systems (1.4).

This paper is structured as follows. In section 2 we present a summary of the results on the averaging theory that we we shall need in this paper. In sections 3 and 4 we prove statements (a) and (b) of Theorem 1 respectively.

2. The averaging theory of first and second order

In the proof of our main result we use the averaging theory as it is presented in [4]. Consider the differential system

$$x'(t) = \epsilon F_1(t,x) + \epsilon^2 F_2(t,x) + \epsilon^3 R(t,x,\epsilon), \tag{2.1}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\epsilon_f, \epsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t,.) \in C^1(D)$ for all $t \in \mathbb{R}$, $F_1, F_2, R, D_x F_1$ are locally Lipschitz with respect to x, and R is differentiable with respect to ϵ. We define

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \big[D_z F_1(s,z) y_1(s,z) + F_2(s,z)\big]ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f)\backslash\{0\}$, there exists $a_\epsilon \in V$ such that $F_{10}(a_\epsilon) + \epsilon F_{20}(a_\epsilon) = 0$ and $d_B(F_{10} + \epsilon F_{20}, V, a_\epsilon) \neq 0$.

Then, for $|\epsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(.,\epsilon)$ of the system (2.1) such that $\varphi(0,\epsilon) = a_\epsilon$.

The expression $d_B(F_{10} + \epsilon F_{20}, V, a_\epsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \epsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point a_ϵ is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \epsilon F_{20}$ at a_ϵ is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{20} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of second order*. For more information about the averaging theory see [25, 27].

3. Proof of statement (a) of Theorem 1

We shall need the first order averaging theory to prove statement (a) of Theorem 1. In order to apply the first order averaging method we write system (1.4) with $k=1$, in polar coordinates (r,θ) where $x = rcos(\theta), y = rsin(\theta), r > 0$. In this way system (1.4) is written in the standard form for applying the averaging theory.

If we write $f_n^1(x,y) = \sum_{i+j=0}^n a_{ij}x^iy^j$, $g_m^1(x) = \sum_{i=0}^m b_ix^i$ and $h_l^1(x) = \sum_{i=0}^l c_ix^i$ then system (1.4) becomes

$$\begin{aligned}
\dot{r} &= \epsilon\Big[\sum_{i=0}^{l} c_ir^i\cos^{i+1}(\theta) - r\sin^2(\theta)\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^j(\theta) \\
&\quad - \sin(\theta)\sum_{i=0}^{m} b_ir^icos^i(\theta)\Big] + O(\epsilon^2), \\
\dot{\theta} &= -1 - \frac{\epsilon}{r}\Big[r\cos(\theta)\sin(\theta)\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)sin^j(\theta) \\
&\quad + \cos(\theta)\sum_{i=0}^{m} b_ir^i\cos^i(\theta) + \sin(\theta)\sum_{i=0}^{l} c_ir^i\cos^i(\theta)\Big] + O(\epsilon^2).
\end{aligned} \tag{3.1}$$

Now taking θ as the new independent variable, this system becomes

$$\begin{aligned}
\frac{dr}{d\theta} &= -\epsilon\Big(\sum_{i=0}^{l} c_ir^i\cos^{i+1}(\theta) - r\sin^2(\theta)\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^j(\theta) \\
&\quad - \sin(\theta)\sum_{i=0}^{m} b_ir^i\cos^i(\theta)\Big) + O(\epsilon^2) \\
&= \epsilon F_1(\theta, r) + O(\epsilon^2).
\end{aligned}$$

Using the notation introduced in section 2 we have

$$\begin{aligned}
F_{10}(r) = \frac{-1}{2\pi}\int_0^{2\pi}&\Big(\sum_{i=0}^{l} c_ir^i\cos^{i+1}(\theta) - r\sin^2(\theta)\sum_{i+j=0}^{n} a_{ij}r^{i+j}\cos^i(\theta)\sin^j(\theta) \\
&- \sin(\theta)\sum_{i=0}^{m} b_ir^i\cos^i(\theta)\Big)d\theta.
\end{aligned}$$

Since

$$\int_0^{2\pi}\cos^{i+1}(\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ is even} \\ \alpha_i \neq 0 & \text{if } i \text{ is odd,} \end{cases}$$

it follows that

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ odd and } j \text{ is odd} \\ \beta_{ij} \neq 0 & \text{if } i \text{ is even and } j \text{ even,} \end{cases}$$

$$\int_0^{2\pi}\sin(\theta)\cos^i(\theta)d\theta = 0$$

for $i = 0, 1, \dots,$ we have

$$\begin{aligned}
F_{10}(r) = \frac{-1}{2\pi}\int_0^{2\pi}&\Big(\sum_{i=1,\, i \text{ odd}}^{l} c_ir^i\cos^{i+1}(\theta) \\
&- \sum_{i+j=0,\, i \text{ even } j \text{ even}}^{n} a_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)\Big)d\theta.
\end{aligned}$$

We define

$$M(l,n)=\begin{cases}\max\{l,n+1\} & \text{if } l \text{ is odd, } n \text{ is even}\\ \max\{l-1,n+1\} & \text{if } l \text{ is even, } n \text{ is even}\\ \max\{l,n\} & \text{if } l \text{ is odd, } n \text{ is odd}\\ \max\{l-1,n\} & \text{if } l \text{ is even, } n \text{ is odd.}\end{cases}$$

Therefore,

$$M(l,n)=\max\{O(l),O(n+1)\}$$

and

$$\Big[\frac{M(l,n)-1}{2}\Big]=\Big[\frac{\max\{O(l),O(n+1)\}-1}{2}\Big]=\max\Big\{\Big[\frac{l-1}{2}\Big],\Big[\frac{n}{2}\Big]\Big\}$$

finally, we have

$$F_{10}(r)=\sum_{k=1,\,k\text{ odd}}^{M(l,n)}\sigma_k r^k,$$

with

$$\sigma_k=\frac{-1}{2\pi}\int_0^{2\pi}\big(c_k\cos^{k+1}(\theta)-a_{(k-1-j)j}\cos^{k-1-j}(\theta)\sin^{j+2}(\theta)\big)\,d\theta,$$

where $k\geq 1$ is an odd integer number and $j\geq 0$ is an even one. Since $F_{10}(r)$ is an odd function, it has at most $[(M(l,n)-1)/2]$ simple positive real roots. From section 2 we obtain that for $|\epsilon|$ sufficiently small, the maximum number of limit cycles of system (1.4) which can bifurcate from the periodic orbits of the linear center $\dot{x}=y$, $\dot{y}=-x$ using the averaging theory of first order is $[(M(l,n)-1)/2]$. Hence statement (a) of Theorem 1 is proved.

4. Proof of statement (b) of Theorem 1

For proving statement (b) of Theorem 1 we shall use the second order averaging theory. In this section we consider the differential systems

$$\begin{gathered}\dot{x}=y+\epsilon h_l^1(x)+\epsilon^2h_l^2(x)+O(\epsilon^3),\\ \dot{y}=-x-\epsilon(f_n^1(x,y)y+g_m^1(x))-\epsilon^2(f_n^2(x,y)y+g_m^2(x))+O(\epsilon^3).\end{gathered}\tag{4.1}$$

where

$$h_l^2(x)=\sum_{i=0}^{l}\hat{c_i}x^i,\quad f_n^2(x,y)=\sum_{i+j=0}^{n}\hat{a_{ij}}x^iy^j,\quad g_m^2(x)=\sum_{i=0}^{m}\hat{b_i}x^i$$

Then system (4.1) in polar coordinates $(r,\theta), r>0$ becomes

$$\begin{aligned}\dot{r}&=\epsilon\frac{xh_l^1(x)-y^2f_n^1(x,y)-yg_m^1(x)}{r}+\epsilon^2\frac{xh_l^2(x)-y^2f_n^2(x,y)-yg_m^2(x)}{r}+O(\epsilon^3),\\ \dot{\theta}&=-1-\epsilon\frac{xyf_n^1(x,y)+xg_m^1(x)+yh_l^1(x)}{r^2}-\epsilon^2\frac{xyf_n^2(x,y)+xg_m^2(x)+yh_l^2(x)}{r^2}\\ &\quad+O(\epsilon^3).\end{aligned}$$

Taking θ as the new independent variable, this system becomes

$$\begin{aligned}\frac{dr}{d\theta}&=\epsilon\frac{xh_l^1(x)-y^2f_n^1(x,y)-yg_m^1(x)}{r}-\epsilon^2\Big[\frac{xh_l^2(x)-y^2f_n^2(x,y)-yg_m^2(x)}{r}\\ &\quad-\frac{(xh_l^1(x)-y^2f_n^1(x,y)-yg_m^1(x))(xyf_n^1(x,y)+xg_m^1(x)+yh_l^1(x))}{r^3}\Big]\end{aligned}$$

$$
\begin{aligned}
&- \epsilon^3\Big[\frac{(xh_l^1(x) - y^2 f_n^1(x,y) - yg_m^1(x))(xyf_n^2(x,y) + xg_m^2(x) + yh_l^2(x))}{r^3} \\
&+ \frac{(xh_l^2(x) - y^2 f_n^2(x,y) - yg_m^2(x))(xyf_n^1(x,y) + xg_m^1(x) + yh_l^1(x))}{r^3} \\
&- \frac{(xh_l^1(x) - y^2 f_n^1(x,y) - yg_m^1(x))(xyf_n^1(x,y) + xg_m^1(x) + yh_l^1(x))^2}{r^5}\Big] + O(\epsilon^4) \\
&= \epsilon F_1(\theta,r) + \epsilon^2 F_2(\theta,r) + \epsilon^3 F_3(\theta,r) + O(\epsilon^4),
\end{aligned}
$$

Now we determine the corresponding function

$$
F_{20} = \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{d}{dr}F_1(\theta,r).\int_0^\theta F_1(\phi,r)d\phi + F_2(\theta,r)\Big]d\theta.
$$

For this we put $F_{10} \equiv 0$ which is equivalent to

$$
\begin{gathered}
c_i = 0 \quad \text{for } i \text{ odd, and} \\
a_{ij} = 0 \quad \text{for } i \text{ even and } j \text{ even}
\end{gathered}
$$

First, we have

$$
\begin{aligned}
\frac{d}{dr}F_1(\theta,r) = &- \sum_{i=2,\text{ even}}^{l} ic_i r^{i-1}\cos^{i+1}(\theta) \\
&+ \sum_{i+j=2,\ i \text{ odd or } j \text{ odd}}^{n} (i+j+1)a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+2}(\theta) \\
&+ \sum_{i=1}^{m} ib_i r^{i-1}\cos^i(\theta)\sin(\theta),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\theta F_1(\phi,r)d\phi = &- \sum_{i=0,\, i\text{ even}}^{l} c_i r^i \int_0^\theta \cos^{i+1}(\phi)d\phi \\
&+ \sum_{i+j=1,\ i \text{ odd or } j \text{ odd}}^{n} a_{ij}r^{i+j+1}\int_0^\theta \cos^i(\phi)\sin^{j+2}(\phi)d\phi \\
&+ \sum_{i=0}^{m} b_i r^i \int_0^\theta cos^i(\phi)sin(\phi)d\phi \\
= &- \sum_{i=0,\, i\text{ even}}^{l} c_i r^i A_{i+1}(\theta) + \sum_{i+j=1,\ i \text{ odd or } j \text{ odd}}^{n} a_{ij}r^{i+j+1}A_{i,(j+2)}(\theta) \\
&+ \sum_{i=0}^{m} b_i r^i\Big(\frac{1-cos^{i+1}(\theta)}{i+1}\Big).
\end{aligned}
$$

where

$$
\begin{aligned}
A_i(\theta) &= \int_0^\theta \cos^i(\phi)d\phi \\
&= \sum_{k=1,\, k\text{ odd}}^{i-2} \frac{(i-k)!}{i!}\,\frac{(i-k)^2.(i-(k-2)))^2\dots(i-1)^2}{(i-k)^2}\sin(\theta)\cos^{i-k}(\theta)
\end{aligned}
$$

$$+\frac{(i-1)^2(i-3)^2\dots(2)^2}{i!}\sin(\theta),$$

$$\begin{aligned}
&A_{p,(2n+1)}\\
&=\int_0^\theta \cos^p(\phi)\sin^{2n+1}(\phi)d\phi\\
&=\frac{cos^{p+1}(\theta)}{2n+p+1}\Big\{\sin^{2n}+\sum_{k=1}^{n}\frac{2^k n(n-1)\dots(n-k+1)\sin^{2n-2k}(\theta)}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\Big\},
\end{aligned}$$

$$\begin{aligned}
&A_{p,(2n)}\\
&=\int_0^\theta cos^p(\phi)sin^{2n}(\phi)d\phi\\
&=\frac{-cos^{p+1}(\theta)}{2n+p}\Big\{\sin^{2n-1}+\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\dots(2n-2k+1)\sin^{2n-2k-1}(\theta)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\Big\}\\
&\quad+\frac{(2n-1)!!}{(2n+p).(2n+p-2)\dots(p+2)}\int_0^\theta \cos^p(\theta)d\theta;
\end{aligned}$$

for more details see [12].

From the nine main products of $\frac{d}{dr}F_1(\theta,r)\int_0^\theta F_1(\phi,r)d\phi$, only the following five are not zero when we integrate them between 0 and 2π:

$$\begin{gathered}
\sum_{i=2,\,i\text{ even}}^{l}\ \sum_{k=0,\,k\text{ even}}^{m}\frac{i}{k+1}c_ib_kr^{i+k-1}\cos^{i+k+2}(\theta),\\
-\sum_{i+j=2, i\text{ even and } j\text{ odd}}^{n}\ \sum_{k=0,\,k\text{ even}}^{l}(i+j+1)a_{ij}c_kr^{i+j+k}\cos^i(\theta)\sin^{j+2}(\theta)A_{k+1}(\theta),\\
+\sum_{i+j=2}^{n}\sum_{k+h=1}^{n}(i+j+1)a_{ij}a_{kh}r^{i+j+k+h}\cos^i(\theta)\sin^{j+2}(\theta)A_{i,(j+2)},
\end{gathered}$$

where if i even j is odd, and if i odd j even, and the same for k and h, with $i+k$ odd and $j+h$ is odd too.

$$\begin{aligned}
&+\sum_{i+j=2,\ i\text{ odd and } j\text{ even}}^{n}\ \sum_{k=0,\,k\text{ even}}^{m}(i+j+1)a_{ij}b_kr^{i+j+k}\cos^i(\theta)\\
&\times\sin^{j+2}(\theta)(\frac{1-cos^{k+1}(\theta)}{k+1}),\\
&-\sum_{i=2,\,i\text{ even}}^{m}\ \sum_{k=0,\,k\text{ even}}^{l} ib_ic_kr^{i+k-1}\cos^i(\theta)\sin(\theta)A_{k+1}(\theta).
\end{aligned}$$

Then the last five sums are odd polynomial in the variable r of degree $O(n)+E(l)$, $2O(n)+1$, $O(n)+E(m)$,$E(l)+E(m)-1$, respectively. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}\big[\frac{d}{dr}F_1(\theta,r)\int_0^\theta F_1(\phi,r)d\phi\big]d\theta$$

is an odd polynomial in the variable r and can contribute at most with

$$\Big[\frac{\max\{O(n)+E(l), 2O(n)+1, O(n)+E(m), E(l)+E(m)-1\}-1}{2}\Big]$$

simple positive real roots to the roots of $F_{20}(r)$.

Now we shall study the contribution of $\frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r)d\theta$ to $F_{20}(r)$. The first part,

$$\frac{xh_l^2(x) - y^2 f_n^2(x,y) - yg_m^2(x)}{r},$$

of $F_2(\theta,r)$, contributes at the roots of $F_{20}(r)$ exactly as the function $F_1(\theta,r)$ contributes to $F_{10}(r)$; i.e. it contributes at most with

$$\Big[\frac{\max\{O(l), O(n+1)\}-1}{2}\Big]$$

simple positive roots to the roots of $F_{20}(r)$. Finally we shall study the contribution of the second part

$$\frac{(xh_l^1(x) - y^2 f_n^1(x,y) - yg_m^1(x))(xyf_n^1(x,y) + xg_m^1(x) + yh_l^1(x))}{r^3}$$

of $F_2(\theta,r)$ to $F_{20}(r)$, which can be written as

$$\begin{aligned}
&\frac{1}{r^2}\Big[\sum_{i=0,\, i \text{ even}}^{l} c_i r^i \cos^{i+1}(\theta) - \sum_{i+j=1,\, i \text{ odd or } j \text{ odd}}^{n} a_{ij} r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)\\
&-\sum_{i=0}^{m} b_i r^i \cos^i(\theta)\sin(\theta)\Big],\\
&\quad\Big[\sum_{i+j=1,\, i \text{ odd or } j \text{ odd}}^{n} a_{ij} r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) + \sum_{i=0}^{m} b_i r^i\cos^{i+1}(\theta)\\
&\quad + \sum_{i=0,\, i \text{ even}}^{l} c_i r^i \cos^i(\theta)\sin(\theta)\Big].
\end{aligned}$$

From the nine products between the different sums, seven ones will not be zero after the integration with respect to θ between 0 and 2π, and two of these seven are equal.

So the terms which will contribute to $F_{20}(r)$ are

$$\begin{aligned}
&\frac{1}{r^2}\Big[\sum_{k=0,\, k \text{ even}}^{l}\ \sum_{i+j=1,\, i \text{ even and } j \text{ odd}}^{n} c_k a_{ij} r^{k+i+j+1}\cos^{k+i+2}(\theta)\sin^{j+1}(\theta)\\
&+\sum_{k=0,\, k \text{ even}}^{l}\ \sum_{i=0,\, i\text{even}}^{m} c_k b_i r^{k+i}\cos^{k+i+2}(\theta)\\
&+\sum_{i+j=1, k+h=1,\ i+k \text{ odd and } j+h \text{ odd}}^{2n} a_{ij}a_{kh} r^{i+j+k+h+2}\cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)\\
&+2\sum_{i+j=1,\, i \text{ odd and } j \text{ even}}^{n}\ \sum_{k=0,\, k \text{ even}}^{m} a_{ij} b_k r^{i+j+k+1}\cos^{i+k+1}(\theta)\sin^{j+2}(\theta)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i+j=1, i \text{ even and } j \text{ odd}}^{n} \sum_{k=0,\, k \text{ even}}^{l} a_{ij} c_k r^{i+j+k+1} \cos^{i+k}(\theta) \sin^{j+3}(\theta) \\
&+\sum_{i=0,\, i \text{ even}}^{m} \sum_{k=0,\, k \text{ even}}^{l} b_i c_k r^{i+k} \cos^{i+k}(\theta) \sin^2(\theta)\Big]
\end{aligned}
$$

So the integral between 0 and 2π with respect to θ of this last expression is an odd polynomial in the variable r of degree $\max\{O(n)+O(m)+1, O(n)+E(l)+1, E(m)+E(l), 2O(n)+2\}$. Consequently the contribution of the second part,

$$
\frac{(xh_l^1(x)-y^2f_n^1(x,y)-yg_m^1(x))(xyf_n^1(x,y)+xg_m^1(x)+yh_l^1(x))}{r^3},
$$

of $F_2(\theta,r)$ to the zeros of $F_{20}(r)$ is at most with

$$
\Big[\frac{\{O(n)+O(m)+1, O(n)+E(l)+1, E(m)+E(l), 2O(n)+2\}-1}{2}\Big]
$$

simple positive real roots.

From the above results, we have that $F_{20}(r)$ has at most

$$
\Big[\frac{\{O(n)+O(m)+1, O(n)+E(l)+1, E(m)+E(l), 2O(n)+2, O(l), O(n+1)\}-1}{2}\Big]
$$

simple positive real roots. So, from the results of section 2 statement (b) of Theorem 1 is proved.

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