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CONTROLLABILITY OF NONLINEAR THIRD-ORDER DISPERSION INCLUSIONS WITH INFINITE DELAY

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Abstract. This article shows the controllability of nonlinear third-order dispersion inclusions with infinite delay. Sufficient conditions are obtained by using a fixed-point theorem for multivalued maps. Particularly, the compactness of the operator semigroups is not assumed in this article.

1. Introduction

In 1895, Korteweg and de Vries considered the following equation as a model for propagation of small amplitude long waves in a uniform channel [16]

$$\eta_t = \frac{3}{2}\sqrt{\frac{g}{l}}\big(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\eta_{\xi\xi}\big)_\xi \tag{1.1}$$

where η is the surface elevation above the equilibrium level l, α is a small constant related to the uniform motion of the liquid, g is the gravitational constant, and $\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$ with surface capillary tension T and density ρ. When posed on the whole real line R or on a periodic domain, (1.1) can always be reduced by certain variable transformations to its standard form

$$x_t + xx_\xi + x_{\xi\xi\xi} = 0$$

where $x \equiv x(\xi, t)$ is a real valued function of two real variables ξ and t and subscript is the corresponding partial derivatives. It is well known that many physical phenomena can be described by the KDV equation. This equation arises in many physical contexts as a model equation incorporating the effects of dispersion, dissipation and nonlinearity. In particular, the equation is now a fundamental model of the weakly nonlinear waves in the weakly dispersive media and has been studied extensively by researchers in various aspects (see [18, 25] and references cited therein).

As one of the fundamental concepts in mathematical control theory, controllability plays an important role in control theory and engineering. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. For the controllability problem, there are different methods for

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various types of nonlinear systems and the details can be found in various papers [6, 15, 24, 26].

Many authors have studied on the controllability problems of third-order dispersion equation. In 1993, Russell and Zhang [22] discussed the controllability and stabilizability of the third-order linear dispersion equation on a periodic domain. They discussed the exponential decay rates with distributed controls of restricted form and for the equation with boundary dissipation. Later on, George, Chalishajar and Nandakumaran [8] discussed the exact controllability of nonlinear third-order dispersion equation. They established the controllability results using two standard types of nonlinearities, namely, Lipschitzian and monotone. Chalishajar [3] studied the exact controllability of nonlinear integro-differential third-order dispersion system by using the Schaefer fixed-point theorem. Recently, Sakthivel, Mahmudov and Ren [27] focused on the approximate controllability for the nonlinear third-order dispersion equation. They discussed the approximate controllability under the assumption that the corresponding linear control system is approximately controllable. More recently, Muthukumar and Rajivganthi [19] studied the approximate controllability of stochastic nonlinear third-order dispersion equation by using fixed-point theory, infinite dimensional semigroup properties, stochastic analysis techniques.

It has been widely argued and accepted [10, 28] that for various reasons, time delay should be taken into consideration in modeling. Obviously, the KDV equation with time delay has more actual significance. Zhao and Xu [30] have studied the existence of solitary waves for KDV equation with time delay. Li and Wang [17] have discussed the controllability of nonlinear third-order dispersion equation with infinite distributed delay.

In recent years the corresponding parts of multivalued analysis were applied to obtain various controllability results for systems governed by semilinear differential and functional differential inclusions in infinite dimensional Banach spaces (refer to [1, 4, 20, 23] and others). The attention of researchers to such systems is caused by the fact that many control processes arising in mathematical physics may be naturally presented in this form (refer to [14]). Specially, it should be point out that Obukhovski and Zecca [20] investigated the controllability problems for a system governed by a semilinear differential inclusion in a Banach space with a noncompact semigroup and as application they considered the controllability for a system governed by a perturbed wave equation.

In this paper, we establish sufficient conditions for the controllability of nonlinear third-order dispersion inclusions with infinite delay by using a fixed-point theorem for multivalued maps combined with a noncompact operator semigroup. To the best of the author's knowledge, the controllability of nonlinear third-order dispersion inclusions has not been studied yet in the literature.

2. Preliminaries

The purpose of this paper is to study the controllability of the nonlinear third-order dispersion inclusions with infinite delay

$$\frac{\partial x}{\partial t}(\xi,t)+\frac{\partial^3 x}{\partial \xi^3}(\xi,t)\in (Gu)(\xi,t)+F(t,x_t(\xi,\cdot)) \tag{2.1}$$

on the domain $t \in J$, $0 \le \xi \le 2\pi$,with periodic boundary conditions

$$\frac{\partial^k x}{\partial \xi^k}(0,t) = \frac{\partial^k x}{\partial \xi^k}(2\pi,t), \quad k = 0,1,2, \tag{2.2}$$

and initial condition

$$x(\xi,\theta) = x_0(\xi,\theta), \quad -\infty < \theta \le 0,\ 0 \le \xi \le 2\pi, \tag{2.3}$$

where $J = [0,b]$, F is a multivalued continuous function. $x_0 : [0,2\pi]\times(-\infty,0] \to R$ are continuous functions. $x_t(\xi,\theta) = x(\xi,t+\theta)$, $-\infty < \theta \le 0$. u is the control function and the operator G is defined by

$$(Gu)(\xi,t) = g_1(\xi)\big\{u(\xi,t) - \int_0^{2\pi} g_1(s)u(s,t)ds\big\}. \tag{2.4}$$

Then G is a bounded linear operator and $g_1(\xi)$ is a piece-wise continuous nonnegative function on $[0,2\pi]$ such that

$$[g_1] := \int_0^{2\pi} g_1(s)ds = 1.$$

The state $x(\cdot,t)$ takes values in a Banach space $X = L^2(0,2\pi)$ with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. The control function $u(\cdot,t)$ is given in $L^2(J,U)$, a Banach space of all admissible control functions, with $U = L^2(0,2\pi)$ as a Banach space. Define an operator A on X with domain $D = D(A)$ given by

$$D(A) = \big\{x \in H^3(0,2\pi) : \frac{\partial^k x}{\partial \xi^k}(0) = \frac{\partial^k x}{\partial \xi^k}(2\pi);\ k = 0,1,2\big\},$$

such that

$$Ax = -\frac{\partial^3 x}{\partial \xi^3}.$$

It follows from Lemma 8.5.2 and Korteweg-de Vries equation of Pazy [21] that A is the infinitesimal generator of a C_0-group of isometry on X. Let $\{T(t)\}_{t\ge 0}$ be the C_0-group generated by A. Obviously, one can show for all $x \in D(A)$,

$$\langle Ax, x\rangle_{L^2(0,2\pi)} = 0.$$

Also, there exists a constant $M > 0$ such that

$$\sup\{\|T(t)\| : t \in J\} \le M.$$

To study system (2.1)-(2.3), we assume that the histories $x_t : (-\infty,0] \to X$, $x_t(\theta) = x(t+\theta)$ belong to some abstract phase space $\mathcal{B}$, which is defined axiomatically. In this article, we will employ an axiomatic definition of the phase space introduced by Hale and Kato [9] and follow the terminology used in [12]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A1) If $x : (-\infty,\sigma+a) \to X$, $a > 0$, is continuous on $[\sigma,\sigma+a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+a)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$;
  (ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$;
  (iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$.
  Here $H \ge 0$ is a constant, $K, M : [0,+\infty) \to [1,+\infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a]$.

(A3) The space $\mathcal{B}$ is complete.

Example 2.1. The phase space $C_r \times L^p(\rho_1, X)$. Let $r \geq 0, 1 \leq p < \infty$ and let $\rho_1 : (-\infty, -r) \to R$ be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [12]. In other words, this means that ρ_1 is locally integrable and there exists a non-negative, locally bounded function δ on $(-\infty, 0]$ such that $\rho_1(\mu+\nu) \leq \delta(\mu)\rho_1(\nu)$, for all $\mu \leq 0$ and $\nu \in (-\infty, -r)\backslash N_\mu$, where $N_\mu \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho_1, X)$ consists of all classes of functions $\phi : (-\infty, 0] \to X$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho_1\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho_1, X)$ is defined by

$$\|\phi\|_{\mathcal{B}} = \sup\{\|\phi(\nu)\| : -r \leq \nu \leq 0\} + \Big(\int_{-\infty}^{-r} \rho_1(\nu)\|\phi(\nu)\|^p d\nu \Big)^{1/p}.$$

The space $C_r \times L^p(\rho_1, X)$ satisfies axioms (A1), (A2), (A3). Moreover, if $r = 0$ and $p = 2$,the phase space $C_r \times L^p(\rho_1, X)$ is reduced to $\mathcal{B} = C_0 \times L^2(\rho_1, X)$. We can take $H = 1$, $M(t) = \delta(-t)^{1/2}$, and $K(t) = 1 + \big(\int_{-t}^{0} \rho_1(\nu)d\nu \big)^{1/2}$ for $t \geq 0$. We refer the reader to [12] for details.

Next, we introduce definitions, notation and preliminary facts from multivalued analysis which are used throughout this paper.

Let $C(J, X)$ be the Banach space of continuous functions from J to X with the norm $\|x\|_J = \sup\{\|x(t)\| : t \in J\}$. $B(X)$ denotes the Banach space of bounded linear operators from X into itself. A measurable function $x : J \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable (For properties of the Bochner integral see Yosida [29]). $L^1(J, X)$ denotes the Banach space of Bochner integrable functions $x : J \to X$ with norm $\|x\|_{L^1} = \int_0^b \|x(t)\|dt$ for all $x \in L^1(J, X)$.

For a metric space (X, d), we introduce the following symbols:

$$P(X) = \{y \in 2^X, Y \neq \emptyset\}, \quad P_{cl}(X) = \{y \in P(X) : y \text{ is closed}\},$$
$$P_b(X) = \{y \in P(X) : y \text{ is bounded}\}, \quad P_{cp}(X) = \{y \in P(X) : y \text{ is compact}\},$$
$$P_{b,cl}(X) = \{y \in P(X) : y \text{ is bounded and closed}\}.$$

We define $H_d : P(X) \times P(X) \to R_+ \cup \{\infty\}$ by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \quad \sup_{b \in B} d(A, b)\},$$

where

$$d(A, b) = \inf_{a \in A} d(a, b), \ d(a, B) = \inf_{b \in B} d(a, b).$$

Then, $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space.

In what follows, we briefly introduce some facts on multivalued analysis. For more details, one can see [7, 13].

• Γ has a fixed point if there is $x \in X$ such that $x \in \Gamma(x)$. The set of fixed points of the multivalued operator Γ will be denoted by $\mathrm{Fix}\Gamma$.

• A multivalued map $\Gamma : J \to P_{cl}(X)$ is said to be measurable, if for each $x \in X$, the function $Y : J \to \mathbb{R}$, defined by

$$Y(t) = d(x, \Gamma(t)) = \inf\{\|x - z\| : z \in \Gamma(t)\},$$

belongs to $L^1(J,R)$.

Definition 2.2. A multivalued operator $\Gamma : X \to P_{cl}(X)$ is called:

(a) γ-Lipschitz if there exists $\gamma > 0$ such that
$$H_d(\Gamma(x),\Gamma(y)) \leq \gamma d(x,y), \text{ for each } x,y \in X;$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Our main results are based on the following lemma.

Lemma 2.3 ([5]). *Let (X,d) be a complete metric space. If $\Gamma : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix}\Gamma \neq \emptyset$.

By the variation of constant formula, we can write a mild solution of (2.1)-(2.3) as
$$x(\xi,t) = T(t)x(\xi,0) + \int_0^t T(t-s)(Gu)(\xi,s)ds + \int_0^t T(t-s)f(s)(\xi)ds, \tag{2.5}$$
where $f \in S_{F,x} = \{f \in L^1(J,X) : f(t)(\xi) \in F(t,x_t(\xi,\cdot))$, for a.e. $t \in J$, $\xi \in [0,2\pi]\}$.

Definition 2.4. System (2.1)-(2.3) is said to be exactly controllable on the interval J, if for any given $x_b \in X$ with $[x_b] = 0$, there exists a control $u \in L^2(0,b;L^2(0,2\pi)) = L^2(J,U)$ such that the mild solution $x(.,t)$ of (2.1)-(2.3) satisfies $x(.,b) = x_b$.

For $\theta \leq 0$, $\xi \in [0,2\pi]$ and $\phi \in \mathcal{B}$, we define
$$x(t)(\xi) = x(\xi,t), \quad F(t,\phi)(\xi) = F(t,\phi(\xi,\cdot)), \quad \phi(\theta)(\xi) = \phi(\xi,\theta) = x_0(\xi,\theta). \tag{2.6}$$

Russell and Zhang [22] studied the exact controllability of a corresponding linear system (i.e. with $F \equiv 0$ in (2.1)-(2.3)). In their analysis, they considered controls which conserve the quantity $[x(\cdot,t)]$, which corresponds to the *volume*. The following is their controllability result for the linear system.

Theorem 2.5 ([22]). *Let $b > 0$ be given and let $g_1 \in C^0[0,2\pi]$ be associated with G in (2.4). Given any final state $x_b \in X$ with $[x_b] = 0$, there exists a control $u \in L^2(J,U)$ such that the solution x of*
$$\frac{\partial x}{\partial t}(\xi,t) + \frac{\partial^3 x}{\partial \xi^3}(\xi,t) = (Gu)(\xi,t) \tag{2.7}$$
together with boundary conditions
$$\frac{\partial^k x}{\partial \xi^k}(0,t) = \frac{\partial^k x}{\partial \xi^k}(2\pi,t), \quad k = 0,1,2, \tag{2.8}$$
and initial condition
$$x(\xi,0) = 0 \tag{2.9}$$
satisfies the terminal condition $x(\cdot,b) = x_b$ in X. Moreover, there exist a positive constant C_1 independent of x_b such that
$$\|x\|_{L^2(J,X)} \leq C_1\|x_b\|_X. \tag{2.10}$$

The main purpose of this paper is to obtain sufficient conditions on the perturbed nonlinear term F which will preserve the exact controllability. Usually authors assume the compactness of semigroup while studying the controllability. Here we drop this assumption and prove the controllability result.

3. Controllability

We assume the following conditions hold:

(H1) $F : J \times \mathcal{B} \to P_{cp}(X) : (\cdot, \phi) \to F(\cdot, \phi)$ is measurable for each $\phi \in \mathcal{B}$.

(H2) $H_d(F(t,\phi_1), F(t,\phi_2)) \le l(t)\|\phi_1 - \phi_2\|_{\mathcal{B}}$,for each $t \in J$ and $\phi_1, \phi_2 \in \mathcal{B}$, where $l \in L^1(J, R_+)$ and $d(0, F(t,0)) \le l(t)$,for a.e. $t \in J$.

Denote

$$\Gamma_0^b = \int_0^b T(b-s)GG^*T^*(b-s)ds.$$

Note that the linear system (2.7)-(2.9) is exactly controllable if and only if there exists a $\zeta > 0$ such that

$$\langle \Gamma_0^b x, x\rangle \ge \zeta\|x\|^2, \quad \text{for all } x \in X,$$

Then Γ_0^b is invertible and

$$\|(\Gamma_0^b)^{-1}\| \le \frac{1}{\zeta}.$$

Theorem 3.1. *Assume that conditions* (H1)–(H2) *and* $[x_b] = 0$ *are satisfied. Then the nonlinear third-order dispersion inclusions* (2.1)-(2.3) *is controllable on J provided*

$$(1 + \frac{1}{\zeta}M^2 M_G^2 b)MLK_b < 1, \tag{3.1}$$

where $M_G = \|G\|$, $L = \int_0^b l(s)ds$, $K_b = \sup\{K(t) : t \in J\}$.

Proof. Define the control function

$$u(\xi,t) = G^*T^*(b-t)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) - \int_0^b T(b-s)f(s)(\xi)ds\big), \tag{3.2}$$

where $f \in S_{F,x}$. Let $Z_b = \{x(\xi,t) \in C((-\infty,b];X) : x_0(\xi,\theta) = \phi(\xi,\theta),\ \phi \in \mathcal{B}\}$. Set $\|\cdot\|_b$ be a seminorm in Z_b defined by

$$\|x(\xi,t)\|_b = \|x_0(\xi,t)\|_{\mathcal{B}} + \sup_{s\in J}\|x(\xi,s)\|, \quad x(\xi,t) \in Z_b.$$

Now, we shall show that, when using the control (3.2), the operators $\Gamma : Z_b \to 2^{Z_b}$ defined by

$$\begin{aligned}(\Gamma x)(\xi,t) = \Big\{&w(\xi,t) \in Z_b : w(\xi,t) = T(t)x(\xi,0) + \int_0^t T(t-s)(Gu)(\xi,s)ds \\ &+ \int_0^t T(t-s)f(s)(\xi)ds,\ t \in J,\ f \in S_{F,x}\Big\}\end{aligned}$$

has a fixed point. This fixed point is then a mild solution of (2.1)-(2.3). Obviously, $x_b \in (\Gamma x)(\cdot, b)$.

Let $\widehat{x}(\xi,t) \in C((-\infty,b],X)$ be the function defined by

$$\widehat{x}(\xi,t) = \begin{cases} x_0(\xi,t), & t \in (-\infty,0], \\ T(t)x(\xi,0), & t \in J. \end{cases}$$

Set $x(\xi,t) = y(\xi,t) + \widehat{x}(\xi,t)$, $t \in (-\infty,b]$. It is easy to see that y satisfies

$$y(\xi,t) = 0, \quad t \in (-\infty,0],$$

$$y(\xi,t) = \int_0^t T(t-s)(Gu)(\xi,s)ds + \int_0^t T(t-s)f(s)(\xi)ds, \quad t \in J,$$

where $f \in S_{F,y} = \{f \in L^1(J,X) : f(t)(\xi) \in F(t, y_t(\xi,\cdot) + \hat{x}_t(\xi,\cdot)),$ for a.e. $t \in J,\ \xi \in [0,2\pi]\}$.

Let $Z_b^0 = \{y(\xi,t) \in Z_b : y(\xi,t) = 0,\ t \in (-\infty,0]\}$. For each $y(\xi,t) \in Z_b^0$, let $\|y(\xi,t)\|_b = \sup_{s\in J}\|y(\xi,s)\|$, thus $(Z_b^0, \|\cdot\|_b)$ is a Banach space. Consider the operator $\Gamma_1 : Z_b^0 \to 2^{Z_b^0}$ defined by

$$\begin{aligned}(\Gamma_1 y)(\xi,t) = \Big\{ v(\xi,t) \in Z_b^0 : v(\xi,t) = \int_0^t T(t-s)(Gu)(\xi,s)ds \\ + \int_0^t T(t-s)f(s)(\xi)ds,\ t \in J,\ f \in S_{F,y}\Big\}.\end{aligned}$$

Next, we shall show that Γ_1 satisfy the hypotheses of Lemma 2.3. The proof will be given in two steps.

Step 1. We show that $(\Gamma_1 y)(\xi,t) \in P_{cl}(Z_b^0)$. Indeed, let $y^{(n)}(\xi,t) \to y^*(\xi,t)$, $\big(v_n(\xi,t)\big)_{n\geq 0} \in (\Gamma_1 y)(\xi,t)$ such that $v_n(\xi,t) \to v_*(\xi,t)$ in Z_b^0. Then $v_*(\xi,t) \in Z_b^0$ and there exists $f_n \in S_{F,y^{(n)}}$ such that, for each $t \in J$,

$$v_n(\xi,t) = \int_0^t T(t-s)(Gu_{y^{(n)}})(\xi,s)ds + \int_0^t T(t-s)f_n(s)(\xi)ds, \quad t \in J,$$

where

$$u_{y^{(n)}}(\xi,t) = G^*T^*(b-t)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) - \int_0^b T(b-s)f_n(s)(\xi)ds\big).$$

Using the fact that F has compact values and (H2) holds, we may pass to a subsequence if necessary to obtain that f_n converges to f_* in $L^1(J,X)$; hence, $f_* \in S_{F,y^*}$. Then, for each $t \in J$,

$$v_n(\xi,t) \to v_*(\xi,t) = \int_0^t T(t-s)(Gu_{y^*})(\xi,s)ds + \int_0^t T(t-s)f_*(s)(\xi)ds, \quad t \in J,$$

where

$$u_{y^*}(\xi,t) = G^*T^*(b-t)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) - \int_0^b T(b-s)f_*(s)(\xi)ds\big).$$

So, $v_*(\xi,t) \in (\Gamma_1 y)(\xi,t)$ and, in particular, $(\Gamma_1 y)(\xi,t) \in P_{cl}(Z_b^0)$.

Step 2. We show that $(\Gamma_1 y)(\xi,t)$ is a contractive multivalued map for each $y(\xi,t) \in Z_b^0$. Let $y(\xi,t), \overline{y}(\xi,t) \in Z_b^0$ and let $v(\xi,t) \in (\Gamma_1 y)(\xi,t)$. Then there exists $f \in S_{F,y}$ such that

$$\begin{aligned}v(\xi,t) &= \int_0^t T(t-s)(Gu)(\xi,s)ds + \int_0^t T(t-s)f(s)(\xi)ds \\ &= \int_0^t T(t-\eta)GG^*T^*(b-\eta)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) \\ &\quad - \int_0^b T(b-s)f(s)(\xi)ds\big)d\eta + \int_0^t T(t-s)f(s)(\xi)ds.\end{aligned}$$

From (H2), it follows that, for each $t \in J$,

$$H_d(F(\phi_1),F(\phi_2)) \leq l(t)\|\phi_1 - \phi_2\|_{\mathcal{B}}, \quad \phi_1,\phi_2 \in \mathcal{B}.$$

Hence, there exists $\omega(t)(\xi) \in F(t, \overline{y}_t(\xi,\cdot) + \widehat{x}_t(\xi,\cdot))$ such that

$$\|f(t) - \omega(t)\| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{B}}.$$

Consider $\Omega : J \to 2^X$,given by

$$\Omega(t) = \{\omega(t) \in X : \|f(t) - \omega(t)\| \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{B}}\}.$$

Since the multivalued operator $W(t) = \Omega(t) \cap F(t, \overline{y}_t + \widehat{x}_t)$ is measurable [2, Proposition III.4], there exists a function $\overline{f}(t)$, which is a measurable selection for W. So, $\overline{f}(t)(\xi) \in F(\overline{y}_t(\xi,\cdot) + \widehat{x}_t(\xi,\cdot))$ and

$$\|f(t) - \overline{f}(t)\| \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{B}}, \quad \text{for each } t \in J.$$

For each $t \in J$, we define

$$\begin{aligned}
\overline{v}(\xi,t) &= \int_0^t T(t-s)(Gu)(\xi,s)ds + \int_0^t T(t-s)\overline{f}(s)(\xi)ds \\
&= \int_0^t T(t-\eta)GG^*T^*(b-\eta)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) \\
&\quad - \int_0^b T(b-s)\overline{f}(s)(\xi)ds\big)d\eta + \int_0^t T(t-s)\overline{f}(s)(\xi)ds.
\end{aligned}$$

Then, for $t \in J$, we obtain

$$\begin{aligned}
&\|v(\xi,t) - \overline{v}(\xi,t)\| \\
&= \|\big[\int_0^t T(t-\eta)GG^*T^*(b-\eta)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) \\
&\quad - \int_0^b T(b-s)f(s)(\xi)ds\big)d\eta + \int_0^t T(t-s)f(s)(\xi)ds\big] \\
&\quad - \big[\int_0^t T(t-\eta)GG^*T^*(b-\eta)(\Gamma_0^b)^{-1}\big(x_b - T(b)x(\xi,0) \\
&\quad - \int_0^b T(b-s)\overline{f}(s)(\xi)ds\big)d\eta + \int_0^t T(t-s)\overline{f}(s)(\xi)ds\big] \\
&\le \|\int_0^t T(t-\eta)GG^*T^*(b-\eta)(\Gamma_0^b)^{-1}\int_0^b T(b-s)[f(s)(\xi) - \overline{f}(s)(\xi)]dsd\eta\| \\
&\quad + \|\int_0^t T(t-s)[f(s)(\xi) - \overline{f}(s)(\xi)]ds\| \\
&\le (1 + \frac{1}{\zeta}M^2M_G^2b)M\int_0^b l(s)\|y_s - \overline{y}_s\|_{\mathcal{B}}ds \\
&\le (1 + \frac{1}{\zeta}M^2M_G^2b)MLK_b\|y - \overline{y}\|_b
\end{aligned}$$

Then

$$\|v - \overline{v}\|_b \le (1 + \frac{1}{\zeta}M^2M_G^2b)MLK_b\|y - \overline{y}\|_b.$$

By an analogous relation, obtained by interchanging the roles of v and $\overline{v}$, it follows that

$$H_d((\Gamma_1 y)(\xi,t), (\Gamma_1\overline{y})(\xi,t)) \le (1 + \frac{1}{\zeta}M^2M_G^2b)MLK_b\|y - \overline{y}\|_b.$$

In view of (3.1), we conclude that Γ_1 is contractive. As a consequence of Lemma 2.3, we deduce that Γ_1 have a fixed point $y^*(\xi,t) \in Z_b^0$. Let $x(\xi,t) = y^*(\xi,t) + \widehat{x}(\xi,t)$, $t \in (-\infty, b]$. Then x is a fixed point of the operator Γ which is a mild solution of problem (2.1)-(2.3). $\square$

Remark 3.2. We say system (2.1)-(2.3) is approximately controllable on J if for any given $x_b \in X$ and $\epsilon > 0$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(\cdot, t)$ of (2.1)-(2.3) satisfies $\|x(\cdot, b) - x_b\| < \epsilon$. Actually we may also discuss the approximate controllability for system (2.1)-(2.3) under weaker conditions, more precisely, it is possible to formulate and prove sufficient conditions for approximate controllability of nonlinear third-order dispersion inclusions with infinite delay by suitably using techniques similar to those presented in [11, 23, 27]. We will go on to do it as a subsequent work.

Conclusion. We have considered controllability problems of nonlinear third-order dispersion inclusions with infinite delay. By using a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler, sufficient conditions have been given without compactness condition for the semigroup generated by the linear part of the system. In the future research, the controllability of stochastic nonlinear third-order dispersion inclusions may be considered. In addition, it is interesting to investigate the case with both delays and impulsive effects.

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References

[1] N. Abada, M. Benchohra, H. Hammouche; *Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions*, J. Differ. Equat., 246 (2009), 3834-3863.

[2] C. Castaing, M. Valadier; *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, vol. 580, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[3] D. N. Chalishajar; *Controllability of nonlinear integro-differential third order dispersion system*, J. Math. Anal. Appl., 348 (2008), 480-486.

[4] Y. K. Chang, W. T. Li; *Controllability of second-order differential and integrodifferential inclusions in Banach spaces*, J. Optim. Theo. Appl., 129 (2006), 77-87.

[5] H. Covitz, Jr. S. B. Nadler; *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., 8 (1970), 5-11.

[6] J. P. Dauer, N. I. Mahmudov; *Approximate controllability of semilinear functional equations in Hilbert spaces*, J. Math. Anal. Appl., 273 (2002), 310-327.

[7] K. Deimling; *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[8] R. K. George, D. N. Chalishajar, A. K. Nandakumaran; *Exact controllability of the nonlinear third-order dispersion equation*, J. Math. Anal. Appl., 332 (2007), 1028-1044.

[9] J. K. Hale, J. Kato; *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., 21 (1978), 11-41.

[10] J. K. Hale, S. M. Verduyn Lunel; *Introduction to Functional Differential Equations*, Appl. Math. Sci., vol. 99, Springer, New York, 1993.

[11] B. Hassane, X. L. Fu; *Controllability of neutral functional differential equations with infinite delay*, Acta Math. Scien., 31B(1) (2011), 73-80.

[12] Y. Hino, S. Murakami, T. Naito; *Functional-Differential Equations with Infinite Delay*, in Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.

[13] S. Hu, N. S. Papageorgiou; *Handbook of Multivalued Analysis (Theory)*, Kluwer Academic Publishers, Dordrecht, 1997.

[14] M. Kamenskii, V. Obukhovskii, P. Zecca; *Condensing multivalued maps and semilinear differential inculsions in Banach spaces*, in de Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter, Berlin, 2001.

[15] J. Klamka; *Stochastic controllability of systems with multiple delays in control*, Int. J. Appl. Math. Comput. Sci., 19 (2009), 39-47.

[16] D. J. Korteweg, G. Devres; *On the change of form of long waves advancing in a rectangular canal*, and on a new type of long stationary waves, Philos. Mag., 39 (1895), 422-433.

[17] M. Li, H. Wang; *Controllability of nonlinear third order dispersion equation with distributed delay*, J. Appl. Anal. Comp., 2 (2012), 415-421.

[18] G. Lin, L. Grinberg, G. E. Karniadakis; *Numerical studies of the stochastic Korteweg-de Vries equation*, J. Comput. Physics, 213 (2006), 676-703.

[19] P. Muthukumar, C. Rajivganthi; *Approximate controllability of stochastic nonlinear third-order dispersion equation*, Int. J. Robust. Nonlinnear Control, 2012, DOI: 10.1002/rnc.2908.

[20] V. Obukhovski, P. Zecca; *Controllability for systems governed by semilinear differential inclusions in a Banach space with a noncompact semigroup*, Nonl. Anal., 70 (2009), 3424-3436.

[21] A. Pazy; *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[22] D. L. Russell, B. Y. Zhang; *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control Optim., 31 (1993), 659-676.

[23] K. Rykaczewski; *Approximate controllability of differential inclusions in Hilbert spaces*, Nonl. Anal., 75 (2012), 2701-2712.

[24] R. Sakthivel; *Controllability of nonlinear impulsive ITO type stochastic systems*, Int. J. Appl. Math. Comput. Sci., 19 (2009), 589-595.

[25] R. Sakthivel; *Robust stabilization the Korteweg-de Vries-Burgers equation by boundary control*, Nonlinear Dyn., 58 (2009), 739-744.

[26] R. Sakthivel, N. I. Mahmudov, Juan J. Nieto; *Controllability for a class of fractional-order neutral evolution control systems*, Appl. Math. Comp., 218 (2012), 10334-10340.

[27] R. Sakthivel, N. I. Mahmudov, Y. Ren; *Approximate controllability of the nonlinear third-order dispersion equation*, Appl. Math. Comp., 217 (2011), 8507-8511.

[28] J. H. Wu; *Theory and Applications of Partial Functional Differential Equations*, Appl. Math. Sci., vol. 119, Springer-Verlag, New York, 1996.

[29] K. Yosida; *Functional Analysis*, sixth ed., Springer, Berlin, 1980.

[30] Z. Zhao, Y. Xu; *Solitary waves for Korteweg-de Vries equation with small delay*, J. Math. Anal. Appl., 368 (2010), 43-53.

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