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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS TO A FOURTH-ORDER TWO-POINT BOUNDARY-VALUE PROBLEM

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Abstract. In this article, we study the existence and uniqueness of positive solutions for a class of semi-linear, fourth order two point boundary value problems. Some examples are presented to illustrate our main results.

1. Introduction

In this article, we consider the fourth-order two-point boundary-value problem

$$\begin{gathered} u^{(4)}(t) = f(t, u(t), u(t)) + g(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u^{(3)}(1) = -\zeta, \end{gathered} \tag{1.1}$$

where $f : C([0,1] \times \mathbb{R} \times \mathbb{R})$, $g \in C([0,1] \times \mathbb{R})$ are real functions, and $\zeta > 0$.

The existence of solutions for fourth-order boundary-value problems have been studied extensively; see for example [1, 2, 4, 5, 6, 7, 8, 11, 12, 13] and the references therein. Note that in practice, only positive solutions are significant. These results are obtained by the use of the Leray-Schauder method, the topological degree theory, the critical point theory, or the lower and upper solution method. But in the existing literature, there are few papers concerned with the uniqueness of positive solutions. Different from the above works mentioned, in this paper we will use some recent tools on fixed point theory to establish under some hypotheses existence and uniqueness results of positive solutions to (1.1). We present also some examples to illustrate our main results.

2. Preliminaries

Before stating and proving our main theorems, we need some preliminary results.

Lemma 2.1 ([3]). *Problem* (1.1) *has an integral formulation given by*

$$u(t) = \int_0^1 G(t,s)[f(s,u(s),u(s)) + g(s,u(s))]\,ds + \zeta\Phi(t),$$

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where $G : [0,1] \times [0,1] \to \mathbb{R}$ *is the corresponding Green's function*

$$G(t,s) = \frac{1}{6} \begin{cases} s^2(3t-s) & \text{if } s \le t, \\ t^2(3s-t) & \text{if } t \le s, \end{cases}$$

and $\Phi(t) = \frac{t^2}{2} - \frac{t^3}{6}$.

Lemma 2.2 ([10])**.** *For any* $t, s \in [0,1]$*, we have*

$$\frac{1}{3}s^2t^2 \le G(t,s) \le \frac{1}{2}st^2, \quad \frac{1}{3}t^2 \le \Phi(t) \le \frac{1}{2}t^2,$$
$$\frac{1}{2}s^2t \le \frac{\partial G(t,s)}{\partial t} \le st, \quad \frac{1}{2}t \le \Phi'(t) \le 2t.$$

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$; i.e.,

$$x, y \in E, \quad x \preceq y \Leftrightarrow y - x \in P.$$

If $x \preceq y$ and $x \neq y$, then we denote $x \prec y$ or $y \succ x$. By θ_E we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

(1) $x \in P$, $\lambda \ge 0$ implies $\lambda x \in P$;
(2) $-x, x \in P$ implies $x = \theta_E$.

Putting $\operatorname{int}(P) = \{x \in P : x \text{ is an interior point of } P\}$, a cone P is said to be solid if its interior $\operatorname{int}(P)$ is nonempty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta_E \preceq x \preceq y$ implies $\|x\| \le N\|y\|$. in this case, the best constant satisfying this inequality is called the normality constant of P.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that

$$\lambda y \preceq x \preceq \mu y.$$

Clearly, the relation $\sim$ is an equivalence relation. Given $h \succ \theta_E$, we denote by P_h the set

$$P_h = \{x \in E : x \sim h\}.$$

It is easy to see that $P_h \subset P$.

Definition 2.3. An operator $A : E \to E$ is said to be increasing (resp. decreasing) if for all $x, y \in E$, $x \preceq y$ implies $Ax \preceq Ay$ (resp. $Ax \succeq Ay$).

Definition 2.4. An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y; i.e.,

$$(x, y), (u, v) \in P \times P, \quad x \preceq u,\ y \succeq v \Rightarrow A(x, y) \preceq A(u, v).$$

An element $x^* \in P$ is called a fixed point of A if $A(x^*, x^*) = x^*$.

Definition 2.5. An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \succeq tAx, \quad \forall t \in (0,1),\ x \in P.$$

Recently, Zhai and Hao [14] established the following fixed point result.

Lemma 2.6 ([14]). *Let $\beta \in (0,1)$. Let $A : P \times P \to P$ be a mixed monotone operator that satisfies*
$$A(tx, t^{-1}y) \succeq t^{\beta} A(x,y), \quad t \in (0,1),\ x,y \in P. \tag{2.1}$$
Let $B : P \to P$ be an increasing sub-homogeneous operator. Assume that

(i) *there is $h_0 \in P_h$ such that $A(h_0,h_0) \in P_h$ and $Bh_0 \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \succeq \delta_0 Bx$, for all $x,y \in P$.*

Then

(I) *$A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;*
(II) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*
$$rv_0 \preceq u_0 \prec v_0,\ u_0 \preceq A(u_0,v_0) + Bu_0 \preceq A(v_0,u_0) + Bv_0 \preceq v_0;$$
(III) *there exists a unique $x^* \in P_h$ such that $x^* = A(x^*,x^*) + Bx^*$;*
(IV) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*
$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1},\ y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1,2,\dots$$
we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and $F : X \times X \to X$ be a given mapping.

Definition 2.7. We say that $(X, \preceq)$ is directed if for every $(x,y) \in X^2$, there exist $(z,w) \in X^2$ such that $x \preceq z$, $y \preceq z$ and $x \succeq w$, $y \succeq w$.

Definition 2.8. We say that $(X, \preceq, d)$ is regular if the following conditions hold:

(C1) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$, then $x_n \preceq x$ for all n;
(C2) if $\{y_n\}$ is a decreasing sequence in X such that $y_n \to y \in X$, then $y_n \succeq y$ for all n.

Example 2.9. Let $X = C([0,T])$, $T > 0$, be the set of real continuous functions on $[0,T]$. We endow X with the standard metric d given by
$$d(u,v) = \max_{0 \le t \le T} |u(t) - v(t)|, \quad u,v \in X.$$
We define the partial order $\preceq$ on X by
$$u,v \in X, \quad u \preceq v \Leftrightarrow u(t) \le v(t) \text{ for all } t \in [0,T].$$
Let $x,y \in X$. For $z = \max\{x,y\}$, that is, $z(t) = \max\{x(t),y(t)\}$ for all $t \in [0,T]$, and $w = \min\{x,y\}$, that is, $w(t) = \min\{x(t),y(t)\}$ for all $t \in [0,T]$, we have $x \succeq w$ and $y \succeq w$. This implies that $(X, \preceq)$ is directed. Now, let $\{x_n\}$ be a nondecreasing sequence in X such that $d(x_n,x) \to 0$ as $n \to \infty$, for some $x \in X$. Then, for all $t \in [0,T]$, $\{x_n(t)\}$ is a nondecreasing sequence of real numbers converging to $x(t)$. Thus we have $x_n(t) \le x(t)$ for all n; that is, $x_n \preceq x$ for all n. Similarly, if $\{y_n\}$ is a decreasing sequence in X such that $d(y_n,y) \to 0$ as $n \to \infty$, for some $y \in X$, we get that $y_n \succeq y$ for all n. Then we proved that $(X, \preceq, d)$ is regular.

Denote by Σ the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying:

- φ is continuous;
- φ is nondecreasing;
- $\varphi^{-1}(\{0\}) = \{0\}$.

Recently, Harjani, López and Sadarangani [9] established the following fixed point theorem.

Lemma 2.10 (see [9]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that*

$$\psi(d(F(x,y), F(u,v))) \leq \psi(\max\{d(x,u), d(y,v)\}) - \varphi(\max\{d(x,u), d(y,v)\}) \quad (2.2)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, where $\psi, \varphi \in \Sigma$. Suppose also that $(X, \preceq, d)$ is regular, $(X, \preceq)$ is directed, and there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), \; y_0 \succeq F(y_0, x_0).$$

Then F has a unique fixed point $x^* \in X$, that is, there is a unique $x^* \in X$ such that $x^* = F(x^*, x^*)$. Moreover, if $\{x_n\}$ and $\{y_n\}$ are the sequences in X defined by

$$x_{n+1} = F(x_n, y_n), \; y_{n+1} = F(y_n, x_n), \quad n = 0, 1, \dots,$$

then

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(y_n, x^*) = 0. \quad (2.3)$$

Now, we are ready to state and prove our main results.

3. Main results

Let $E = C([0,1])$ be the Banach space of continuous functions on $[0,1]$ with the norm

$$\|y\| = \max\{|y(t)| : t \in [0,1]\}. \quad (3.1)$$

Let $P \subset E$ be the cone defined by

$$P = \{y \in C([0,1]) \,|\, y(t) \geq 0, \, t \in [0,1]\}.$$

Our first main result uses the following assumptions:

- (H1) the functions $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous with
$$\int_0^1 s^2 g(s,0)\, ds > 0;$$
- (H2) $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, and $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$;
- (H3) $g(t, \lambda x) \geq \lambda g(t,x)$ for all $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,+\infty)$, and there exists a constant $\beta \in (0,1)$ such that $f(t, \lambda x, \lambda^{-1} y) \geq \lambda^\beta f(t,x,y)$ for all $\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in [0,+\infty)$;
- (H4) there exists a constant $\delta_0 > 0$ such that $f(t,x,y) \geq \delta_0 g(t,x)$ for all $t \in [0,1]$, $x, y \in [0,+\infty)$.

Theorem 3.1. *Assumptions* (H1)–(H4) *hold. Then*

1. *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $r v_0 \preceq u_0 \prec v_0$ and*
$$u_0(t) \leq \int_0^1 G(t,s)[f(s, u_0(s), v_0(s)) + g(s, u_0(s))]\, ds + \zeta\Phi(t), \quad t \in [0,1],$$
$$v_0(t) \geq \int_0^1 G(t,s)[f(s, v_0(s), u_0(s)) + g(s, v_0(s))]\, ds + \zeta\Phi(t), \quad t \in [0,1],$$
where $h(t) = t^2$, $t \in [0,1]$;

(2) *Problem* (1.1) *has a unique positive solution* $x^* \in P_h$;

(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n(t) = \int_0^1 G(t,s)[f(s, x_{n-1}(s), y_{n-1}(s)) + g(s, x_{n-1}(s))]\,ds + \zeta\Phi(t),$$

$$y_n(t) = \int_0^1 G(t,s)[f(s, y_{n-1}(s), x_{n-1}(s)) + g(s, y_{n-1}(s))]\,ds + \zeta\Phi(t),$$

for $n = 1, 2, \dots$, *we have* $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof. Consider the operators $A : P \times P \to E$ and $B : P \to E$ defined by

$$A(u,v)(t) = \int_0^1 G(t,s) f(s, u(s), v(s))\,ds + \zeta\Phi(t),$$

$$(Bu)(t) = \int_0^1 G(t,s) g(s, u(s))\,ds.$$

From Lemma 2.1, u is a solution to (1.1) if and only if $A(u,u) + Bu = u$. From (H1), we show that $A : P \times P \to P$ and $B : P \to P$. Further, it follows from (H2) that A is mixed monotone and B is increasing. On the other hand, for any $\lambda \in (0,1)$, $u, v \in P$, we have from (H3) that

$$\begin{aligned}
A(\lambda u, \lambda^{-1} v)(t) &= \int_0^1 G(t,s) f(s, \lambda u(s), \lambda^{-1} v(s))\,ds + \zeta\Phi(t) \\
&\geq \lambda^\beta \int_0^1 G(t,s) f(s, u(s), v(s))\,ds + \zeta\Phi(t) \\
&\geq \lambda^\beta \Big(\int_0^1 G(t,s) f(s, u(s), v(s))\,ds + \zeta\Phi(t) \Big) \\
&= \lambda^\beta A(u,v)(t).
\end{aligned}$$

Thus we have for all $\lambda \in (0,1)$, $u, v \in P$,

$$A(\lambda u, \lambda^{-1} v) \succeq \lambda^\beta A(u,v).$$

Then condition (2.1) of Lemma 2.6 is satisfied.

From (H3), it follows that for all $\lambda \in (0,1)$, $u \in P$,

$$\begin{aligned}
B(\lambda u)(t) &= \int_0^1 G(t,s) g(s, \lambda u(s))\,ds \\
&\geq \lambda \int_0^1 G(t,s) g(s, u(s))\,ds \\
&= \lambda Bu(t).
\end{aligned}$$

Thus, for all $\lambda \in (0,1)$ and $u \in P$, we have

$$B(\lambda u) \succeq \lambda Bu.$$

Then B is a sub-homogeneous operator.

Next, we shall prove that $A(h,h) \in P_h$ and $Bh \in P_h$. Using Lemma 2.2 and (H2), for all $t \in [0,1]$, we have

$$A(h,h)(t) = \int_0^1 G(t,s) f(s, h(s), h(s))\,ds + \zeta\Phi(t)$$

$$
\begin{aligned}
&\geq \int_0^1 G(t,s) f(s,h(s),h(s))\,ds \\
&\geq \frac{t^2}{3}\int_0^1 s^2 f(s,0,1)\,ds \\
&= \Big(\frac{1}{3}\int_0^1 s^2 f(s,0,1)\,ds\Big) h(t).
\end{aligned}
$$

Again, using Lemma 2.2 and (H2), we have

$$
\begin{aligned}
A(h,h)(t) &= \int_0^1 G(t,s) f(s,h(s),h(s))\,ds + \zeta\Phi(t) \\
&\leq \frac{t^2}{2}\int_0^1 s f(s,1,0)\,ds + \zeta\frac{t^2}{2} \\
&= \frac{h(t)}{2}\Big(\int_0^1 s f(s,1,0)\,ds + \zeta\Big).
\end{aligned}
$$

Thus, for all $t \in [0,1]$, we have

$$
\Big(\frac{1}{3}\int_0^1 s^2 f(s,0,1)\,ds\Big) h(t) \leq A(h,h)(t) \leq \frac{1}{2}\Big(\int_0^1 s f(s,1,0)\,ds + \zeta\Big) h(t).
$$

On the other hand, from (H4) and (H1),we have

$$
\int_0^1 s^2 f(s,0,1)\,ds \geq \delta_0 \int_0^1 s^2 g(s,0)\,ds > 0.
$$

Thus we proved that $A(h,h) \in P_h$. Similarly, for all $t \in [0,1]$, we have

$$
\Big(\frac{1}{3}\int_0^1 s^2 g(s,0)\,ds\Big) h(t) \leq Bh(t) \leq \Big(\frac{1}{2}\int_0^1 s g(s,1)\,ds\Big) h(t),
$$

which implies that $Bh \in P_h$.

In the following we show the condition (ii) of Lemma 2.6 is satisfied. Let $u, v \in P$. From (H4), we have

$$
\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,s) f(s,u(s),v(s))\,ds + \zeta\Phi(t) \\
&\geq \int_0^1 G(t,s) f(s,u(s),v(s))\,ds \\
&\geq \delta_0 \int_0^1 G(t,s) g(s,u(s))\,ds \\
&= \delta_0 Bu(t).
\end{aligned}
$$

Hence, for all $u, v \in P$, we have $A(u,v) \succeq \delta_0 Bu$. So the conclusion of Theorem 3.1 follows from Lemma 2.6. $\square$

Example 3.2. Consider the fourth-order two-point boundary-value problem

$$
\begin{gathered}
u^{(4)}(t) = 2\big(t^2 + \sqrt{u(t)}\big) + \frac{1}{\sqrt{u(t)+1}}, \quad 0 < t < 1, \\
u(0) = u'(0) = 0, \\
u''(1) = 0, \quad u^{(3)}(1) = -\zeta,
\end{gathered}
\tag{3.2}
$$

where $\zeta > 0$. Consider the functions $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ defined by

$$f(t,x,y) = t^2 + \sqrt{x} + \frac{1}{\sqrt{y+1}}, \quad g(t,x) = \sqrt{x} + t^2,$$

for all $t \in [0,1]$, $x, y \in [0,+\infty)$. Then (3.2) is equivalent to

$$\begin{gathered} u^{(4)}(t) = f(t,u(t),u(t)) + g(t,u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u^{(3)}(1) = -\zeta. \end{gathered}$$

Clearly, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous functions. Moreover, we have

$$\int_0^1 s^2 g(s,0)\,ds = \int_0^1 s^4\,ds = \frac{1}{5} > 0.$$

Condition (H1) of Theorem 3.1 is satisfied, and Condition (H2) can be checked immediately. Now, let $\lambda \in (0,1)$, $t \in [0,1]$ and $x \geq 0$. We have

$$g(t,\lambda x) = \sqrt{\lambda x} + t^2 = \sqrt{\lambda}\sqrt{x} + t^2 \geq \lambda(\sqrt{x} + t^2) = \lambda g(t,x).$$

Let $\lambda \in (0,1)$, $t \in [0,1]$ and $x, y \geq 0$. We have

$$\begin{aligned} f(t,\lambda x,\lambda^{-1}y) &= t^2 + \sqrt{\lambda x} + \frac{1}{\sqrt{\lambda^{-1}y+1}} \\ &= t^2 + \sqrt{\lambda x} + \frac{\sqrt{\lambda}}{\sqrt{y+\lambda}} \\ &\geq \sqrt{\lambda}\Big(t^2 + \sqrt{x} + \frac{1}{\sqrt{y+1}}\Big) \\ &= \lambda^{1/2} f(t,x,y). \end{aligned}$$

Then condition (H3) of Theorem 3.1 is satisfied with $\beta = 1/2$. Let $t \in [0,1]$ and $x, y \in [0,+\infty)$. We have

$$f(t,x,y) = t^2 + \sqrt{x} + \frac{1}{\sqrt{y+1}} \geq t^2 + \sqrt{x} = 1 \cdot g(t,x).$$

Then condition (H4) is also satisfied with $\delta_0 = 1$.

Finally, it follows from Theorem 3.1 that Problem (3.2) has a unique positive solution $x^* \in P_h$, where $h(t) = t^2$, $t \in [0,1]$.

Our second main result uses the following assumptions: is the following.

- (H5) the functions $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous;
- (H6) there exist two positive constants k_f and k_g with $k_f + k_g \in (0,4)$ such that for all $x \geq u$, $y \leq v$, $t \in [0,1]$,
$$0 \leq f(t,x,y) - f(t,u,v) \leq k_f \max\{x-u, v-y\}, \tag{3.3}$$
$$0 \leq g(t,x) - g(t,u) \leq k_g(x-u); \tag{3.4}$$
- (H7) there exist $x_0, y_0 \in P$ such that
$$x_0(t) \leq \int_0^1 G(t,s)[f(s,x_0(s),y_0(s)) + g(s,x_0(s))]\,ds + \zeta\Phi(t), \quad t \in [0,1],$$

$$y_0(t) \geq \int_0^1 G(t,s)[f(s,y_0(s),x_0(s)) + g(s,y_0(s))]\,ds + \zeta\Phi(t), \quad t \in [0,1].$$

Theorem 3.3. *Assume* (H5)–(H7) *ar satisfied. Then*

1. *Problem* (1.1) *has a unique positive solution* $x^* \in P$*;*
2. *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by*
$$x_n(t) = \int_0^1 G(t,s)[f(s,x_{n-1}(s),y_{n-1}(s)) + g(s,x_{n-1}(s))]\,ds + \zeta\Phi(t),$$
$$y_n(t) = \int_0^1 G(t,s)[f(s,y_{n-1}(s),x_{n-1}(s)) + g(s,y_{n-1}(s))]\,ds + \zeta\Phi(t),$$
for $n = 1, 2, \dots$, *converge uniformly to* x^*.

Proof. Define the mapping $F : P \times P \to P$ by

$$F(u,v)(t) = \int_0^1 G(t,s)[f(s,u(s),v(s)) + g(s,u(s))]\,ds + \zeta\Phi(t).$$

Clearly, from (H5), the mapping F is well defined. We endow P with the metric

$$d(u,v) = \|u - v\|, \quad (u,v) \in P \times P,$$

where $\|\cdot\|$ is given by (3.1). We consider the partial order $\preceq$ on P given by

$$u, v \in P, \quad u \preceq v \Longleftrightarrow v - u \in P.$$

In Example 2.9, we proved that $(P, \preceq)$ is directed and $(P, \preceq, d)$ is regular.

From (H6), we show easily that the mapping F has the mixed monotone property (with respect to $\preceq$).

Now, let $(x,y), (u,v) \in P \times P$ such that $x \succeq u$ and $y \preceq v$. Again, from (H6), for all $t \in [0,1]$, we have

$$\begin{aligned}
&|F(x,y)(t) - F(u,v)(t)| \\
&= \int_0^1 G(t,s)[(f(s,x(s),y(s)) - f(s,u(s),v(s))) + (g(s,x(s)) - g(s,u(s)))]\,ds \\
&\leq \int_0^1 G(t,s)(k_f + k_g)\max\{x(s) - u(s), v(s) - y(s)\}\,ds \\
&\leq (k_f + k_g)\max\{d(x,u), d(y,v)\}\int_0^1 G(t,s)\,ds \\
&\leq \frac{k_f + k_g}{4}\max\{d(x,u), d(y,v)\}.
\end{aligned}$$

The above inequality follows from Lemma 2.2. Thus, for all $(x,y), (u,v) \in P \times P$ such that $x \succeq u$ and $y \preceq v$, we have

$$d(F(x,y), F(u,v)) \leq \frac{k_f + k_g}{4}\max\{d(x,u), d(y,v)\}.$$

Taking $\psi(r) = r$ and $\varphi(r) = \big(1 - \frac{k_f+k_g}{4}\big)r$ for all $r \geq 0$, since $k_f + k_g \in (0,4)$, we have $\psi, \varphi \in \Sigma$. Moreover, from the above inequality, we have

$$\psi(d(F(x,y), F(u,v))) \leq \psi(\max\{d(x,u), d(y,v)\}) - \varphi(\max\{d(x,u), d(y,v)\}),$$

for all $x \succeq u$ and $y \preceq v$.

On the other hand, from (H7), we have $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Now, the desired result follows immediately from Lemma 2.10. $\square$

We end this article we the following example.

Example 3.4. Consider the fourth-order two-point boundary-value problem

$$\begin{gathered} u^{(4)}(t)=u(t)+\frac{1}{u(t)+1}+q(t), \quad 0<t<1, \\ u(0)=u'(0)=0, \\ u''(1)=0, \quad u^{(3)}(1)=-\zeta, \end{gathered} \tag{3.5}$$

where $\zeta>0$ and $q:[0,1]\to[0,+\infty)$ is a continuous function. Consider the functions $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ and $g:[0,1]\times[0,+\infty)\to[0,+\infty)$ defined by

$$f(t,x,y)=x+\frac{1}{y+1}+q(t), \quad g(t,x)=0,$$

for all $t\in[0,1]$, $x,y\in[0,+\infty)$. Then (3.5) is equivalent to

$$\begin{gathered} u^{(4)}(t)=f(t,u(t),u(t))+g(t,u(t)), \quad 0<t<1, \\ u(0)=u'(0)=0, \\ u''(1)=0, \quad u^{(3)}(1)=-\zeta. \end{gathered}$$

Clearly, f is a continuous function. Let $t\in[0,1]$, $x,y,u,v\in[0,+\infty)$ such that $x\geq u$ and $y\leq v$. We have

$$\begin{aligned} 0\leq f(t,x,y)-f(t,u,v)&=(x-u)+\frac{v-y}{(y+1)(v+1)} \\ &\leq (x-u)+(v-y)\leq 2\max\{x-u,v-y\}. \end{aligned}$$

Then condition (H6) is satisfied with $k_f=2$ and $k_g=0$. Now, we shall prove that condition (H7) is satisfied with

$$x_0\equiv 0, \quad y_0\equiv \zeta+1+\int_0^1 q(s)\,ds.$$

Clearly, we have

$$x_0(t)=0\leq \int_0^1 G(t,s)f(s,0,y_0(s))\,ds+\zeta\Phi(t), \quad \text{for all } t\in[0,1].$$

Now, using Lemma 2.2, for all $t\in[0,1]$, we have

$$\begin{aligned} \int_0^1 G(t,s)f(s,y_0(s),0)\,ds+\zeta\Phi(t) &\leq \frac{1}{2}\Big(\int_0^1 sf(s,y_0(s),0)\,ds+\zeta\Big) \\ &\leq \frac{1}{2}\Big(\int_0^1 f(s,y_0(s),0)\,ds+\zeta\Big) \\ &= \frac{1}{2}\Big(\int_0^1 (y_0+1+q(s))\,ds+\zeta\Big) \\ &= \frac{y_0+1}{2}+\frac{1}{2}\Big(\int_0^1 q(s)\,ds+\zeta\Big)=y_0. \end{aligned}$$

Thus, for all $t\in[0,1]$, we have

$$y_0(t)\geq \int_0^1 G(t,s)f(s,y_0(s),0)\,ds+\zeta\Phi(t).$$

Finally, all the hypotheses of Theorem 3.3 are satisfied. Then, we deduce that Problem (3.5) has a unique positive solution.

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