Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 174, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIQUENESS OF POSITIVE SOLUTIONS FOR FRACTIONAL q-DIFFERENCE BOUNDARY-VALUE PROBLEMS WITH p-LAPLACIAN OPERATOR

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Abstract. In this article, we study the fractional q-difference boundary-value problems with p-Laplacian operator

$$D_q^\gamma(\phi_p(D_q^\alpha u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3,$$
$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta(D_q u)(\eta),$$

where $0 < \gamma < 1$, $2 < \alpha < 3$, $0 < \beta\eta^{\alpha-2} < 1$, D_{0+}^α is the Riemann-Liouville fractional derivative, $\phi_p(s) = |s|^{p-2}s$, $p > 1$. By using a fixed-point theorem in partially ordered sets, we obtain sufficient conditions for the existence and uniqueness of positive and nondecreasing solutions.

1. Introduction

Recently, an increasing interest in studying the existence of solutions for boundary-value problems of fractional order functional differential equations has been observed [5, 7, 8, 17, 18, 19, 20, 21, 28, 29]. Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, engineering, etc. For an extensive collection of such results, we refer the readers to the monographs by Samko et al [27], Podlubny [25] and Kilbas et al [16].

On the other hand, The q-difference calculus or quantum calculus is an old subject that was first developed by Jackson [13, 14]. It is rich in history and in applications as the reader can confirm in the paper [9].

The origin of the fractional q-difference calculus can be traced back to the works by Al-Salam [3] and Agarwal [1]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, e.g., q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula [4, 26], just to mention some.

Recently, there are few works consider the existence of positive solutions for nonlinear q-fractional boundary value problem (see [10, 11]). As is well-known, the aim of finding positive solutions to boundary value problems is of main importance

2000 *Mathematics Subject Classification.* 39A13, 34B18, 34A08.
Key words and phrases. Fractional q-difference equations; partially ordered sets; fixed-point theorem; positive solution.

Submitted June 15, 2013. Published July 29, 2013.

in various fields of applied mathematics (see the book [2] and references therein). In addition, since q-calculus has a tremendous potential for applications [9], we find it pertinent to investigate such a demand. To the authors' knowledge, no one has studied the existence of positive solutions for nonlinear q-fractional three-point boundary value problem (1.1) and (1.2).

In this article, we study the three-point boundary-value problem

$$D_q^{\gamma}(\phi_p(D_q^{\alpha}u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3, \qquad (1.1) \quad \boxed{\texttt{e1.1}}$$

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = 0, \quad D_{0+}^{\gamma}u(t)|_{t=0} = 0, \qquad (1.2) \quad \boxed{\texttt{e1.2}}$$

where $0 < \beta\eta^{\alpha-2} < 1$, $0 < q < 1$. We will prove the existence and uniqueness of a positive and nondecreasing solution for the boundary value problems (1.1)-(1.2) by using a fixed point theorem in partially ordered sets. Existence of fixed point in partially ordered sets has been considered recently in [6, 12, 22, 23, 24]. This work is motivated by papers [6, 10, 11].

2. Preliminaries

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $\mathbb{N}_0$ is

$$(a-b)^0 = 1, \quad (a-b)^n = \prod_{k=0}^{n-1}(a-bq^k), \quad n \in \mathbb{N},\ a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^{\alpha}\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}\,.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^{\alpha}$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\setminus\{0,-1,-2,\dots\},$$

and satisfies $\Gamma_q(x+1) = [x]\Gamma_q(x)$. The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1}f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)d_qt = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q; i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [15]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}, \tag{2.1}$$ e2.1

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \tag{2.2}$$ e2.2

$$\Big({}_xD_q \int_0^x f(x,t)\,d_qt\Big)(x) = \int_0^x {}_xD_q f(x,t)\,d_qt + f(qx,x). \tag{2.3}$$ e2.3

remark2.1 **Remark 2.1** ([10]). *We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.*

The following definition was considered first in [1].

def2.1 **Definition 2.2.** Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)\,d_qt, \quad \alpha > 0,\ x \in [0,1].$$

def2.2 **Definition 2.3** ([26]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^{\alpha} f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Next, we list some properties that are already known in the literature. Its proof can be found in [1, 26].

lemma2.1 **Lemma 2.4.** *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*

(1) $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x)$,
(2) $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

lemma2.2 **Lemma 2.5** ([10]). *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^{\alpha} D_q^p f)(x) = (D_q^p I_q^{\alpha} f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

The following fixed-point theorems in partially ordered sets are fundamental for the proofs of our main results.

the2.1 **Theorem 2.6** ([12]). *Let $(E, \leq)$ be a partially ordered set and suppose that there exists a metric d in E such that (E, d) is a complete metric space. Assume that E satisfies the condition:*

$$\begin{gathered}\text{if } \{x_n\} \text{ is a nondecreasing sequence in } E \text{ such that } x_n \to x,\\ \text{then } x_n \leq x, \text{ for all } n \in \mathbb{N}.\end{gathered} \tag{2.4}$$
e2.4

Let $T : E \to E$ be nondecreasing mapping such that

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \quad for\ x \geq y,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in E$ with $x_0 \leq T(x_0)$, then T has a fixed point.

If we assume that $(E, \leq)$ satisfies the condition

$$\text{for } x, y \in E \text{ there exists } z \in E \text{ which is comparable to } x \text{ and } y, \tag{2.5}$$
e2.5

then we have the following result.

the2.2 **Theorem 2.7** ([22]). *Adding condition (2.5) to the hypotheses of Theorem 2.6, we obtain uniqueness of the fixed point.*

3. Related lemmas

The basic space used in this paper is $E = C[0, 1]$. Then E is a real Banach space with the norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0, 1], \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad \forall t \in [0, 1].$$

In [22] it is proved that $(C[0, 1], \leq)$ with the classic metric given by

$$d(x, y) = \sup_{0\leq t\leq 1} \{|x(t) - y(t)|\}$$

satisfied condition (2.4) of Theorem 2.6. Moreover, for $x, y \in C[0, 1]$ as the function $\max\{x, y\} \in C[0, 1]$, $(C[0, 1], \leq)$ satisfies condition (2.5).

lemma3.1 **Lemma 3.1.** *If $h \in C[0, 1]$, then the boundary-value problem*

$$(D_q^\alpha u)(t) + h(t) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3, \tag{3.1}$$
e3.1

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = 0 \tag{3.2}$$
e3.2

has a unique solution

$$u(t) = \int_0^1 G(t, qs)h(s)d_qs, \tag{3.3}$$
e3.3

where

$$G(t, s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1 - s)^{(\alpha-2)}t^{\alpha-1} - (t - s)^{(\alpha-1)}, & 0 \leq s \leq t \leq 1,\\ (1 - s)^{(\alpha-2)}t^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.4}$$
e3.4

Proof. In this case $p = 3$. In view of Lemma 2.4 and Lemma 2.5, from (3.1) we see that

$$(I_q^\alpha D_q^3 I_q^{3-\alpha} u)(x) = -I_q^\alpha f(t, u(t))$$

and

$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} - \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)d_qs. \tag{3.5}$$
e3.5

From (3.2), we know that $c_3 = 0$. Differentiating both sides of (3.5), with the help of (2.1) and (2.2), one obtains

$$(D_q u)(t) = [\alpha-1]_q c_1 t^{\alpha-2} + [\alpha-2]_q c_2 t^{\alpha-3} - \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-2)} h(s) d_q s.$$

Using the boundary condition (3.2), we have $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} h(s) d_q s.$$

Therefore, the unique solution of boundary-value problem (3.1), (3.2) is

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} h(s) d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t ((1-qs)^{(\alpha-2)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}) h(s) d_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_t^1 (1-qs)^{(\alpha-2)} t^{\alpha-1} h(s) d_q s \\ &= \int_0^1 G(t,qs) h(s) d_q s. \end{aligned}$$

The proof is complete. $\square$

lemma3.2 **Lemma 3.2.** *If $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, then the boundary-value problem (1.1)-(1.2) is equivalent to the integral equation*

$$u(t) = \int_0^1 G(t,qs) \phi_p^{-1}\left(\frac{1}{\Gamma_q(\gamma)} \int_0^s (s-\tau)^{(\gamma-1)} f(\tau, u(\tau)) d_q \tau \right) d_q s, \tag{3.6}$$ e3.9

where $G(t,s)$ is defined by (3.4).

Proof. By the boundary-value problem (1.1)-(1.2) and Lemma 2.5, we have

$$\phi_p(D_{0+}^{\alpha} u(t)) = ct^{\gamma-1} - \int_0^t \frac{(t-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} f(s,u(s)) d_q s.$$

By $D_{0+}^{\alpha} u(t)|_{t=0} = 0$, there is $c = 0$, and then

$$D_{0+}^{\alpha} u(t) = -\phi_p^{-1}\Big(\int_0^t \frac{(t-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} f(s,u(s)) d_q s \Big).$$

Therefore, boundary-value problem (1.1)-(1.2) is equivalent to the problem

$$\begin{gathered} D_{0+}^{\alpha} u(t) + \phi_p^{-1}\Big(\int_0^t \frac{(t-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} f(s,u(s)) d_q s \Big) = 0, \quad 0 < t < 1,\ 2 < \alpha \le 3, \\ u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = 0. \end{gathered} \tag{3.7}$$ e3.2b

By Lemma 3.1, boundary-value problem (1.1)-(1.2) is equivalent to the integral equation (3.6). The proof is complete. $\square$

lemma3.3 **Lemma 3.3.** *The function G defined by* (3.4) *has the following properties:*

(1) *G is a continuous function and $G(t,qs) \ge 0$;*
(2) *G is strictly increasing in the first variable.*

Proof. The continuity of G is easily checked. On the other hand, let

$$g_1(t,s) = (1-s)^{(\alpha-2)}t^{\alpha-1} - (t-s)^{(\alpha-1)}, \quad 0 \le s \le t \le 1,$$
$$g_2(t,s) = (1-s)^{(\alpha-2)}t^{\alpha-1}, \quad 0 \le t \le s \le 1.$$

It is obvious that $g_2(t,qs) \ge 0$. Now, $g_1(0,qs) = 0$ and, in view of Remark 2.1, for $t \ne 0$

$$\begin{aligned} g_1(t,qs) &= (1-qs)^{(\alpha-2)}t^{\alpha-1} - (1-q\frac{s}{t})^{(\alpha-1)}t^{\alpha-1} \\ &\ge t^{\alpha-1}\Big[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\Big] \ge 0. \end{aligned}$$

Then we conclude that $G(t,qs) \ge 0$ for all $(t,s) \in [0,1]\times[0,1]$. This concludes the proof of Lemma 3.3 (1).

Next, for fixed $s \in [0,1]$, we have

$$\begin{aligned} {}_tD_q g_1(t,qs) &= (1-qs)^{(\alpha-2)}[\alpha-1]_q t^{\alpha-2} - [\alpha-1]_q (t-qs)^{(\alpha-2)} \\ &= (1-qs)^{(\alpha-2)}[\alpha-1]_q t^{\alpha-2} - [\alpha-1]_q (1-q\frac{s}{t})^{(\alpha-2)}t^{\alpha-2} \\ &\ge (1-qs)^{(\alpha-2)}[\alpha-1]_q t^{\alpha-2} - [\alpha-1]_q (1-qs)^{(\alpha-2)}t^{\alpha-2} = 0. \end{aligned}$$

This implies that $g_1(t,qs)$ is an increasing function of t. Obviously, $g_2(t,qs)$ is increasing in t. Therefore $G(t,qs)$ is an increasing function of t for fixed $s \in [0,1]$. The proof is complete. □

4. Main result

For notational convenience, we denote by

$$M = \phi_p^{-1}\left(\frac{1}{\Gamma_q(\gamma)}\right)\sup_{0\le t\le 1}\int_0^1 G(t,qs)d_qs > 0.$$

The main result of this paper is the following.

the3.1 **Theorem 4.1.** *The boundary-value problem* (1.1)-(1.2) *has a unique positive and increasing solution $u(t)$ if the following conditions are satisfied:*

(i) $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous and nondecreasing respect to the second variable;*
(ii) *There exists $0 < \lambda + 1 < M$ such that for $u, v \in [0,+\infty)$ with $u \ge v$ and $t \in [0,1]$*

$$\phi_p(\ln(v+2)) \le f(t,v) \le f(t,u) \le \phi_p(\ln(u+2)(u-v+1)^{\lambda}).$$

Proof. Consider the cone

$$K = \{u \in C[0,1] : u(t) \ge 0\}.$$

As K is a closed set of $C[0,1]$, K is a complete metric space with the distance given by $d(u,v) = \sup_{t\in[0,1]}|u(t)-v(t)|$. Now, we consider the operator T defined by

$$Tu(t) = \int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)}f(\tau,u(\tau))d_q\tau\Big)d_qs.$$

By Lemma 3.3 and condition (i), we have that $T(K) \subset K$.

We now show that all the conditions of Theorem 2.6 and Theorem 2.7 are satisfied. Firstly, by condition (i), for $u, v \in K$ and $u \geq v$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)} f(\tau,u(\tau))d_q\tau\Big)d_qs \\
&\geq \int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)} f(\tau,v(\tau))d_q\tau\Big)d_qs \\
&= Tv(t).
\end{aligned}$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and by condition (ii) we have

$$\begin{aligned}
d(Tu,Tv) &= \sup_{0\leq t\leq 1} |(Tu)(t)-(Tv)(t)| \\
&= \sup_{0\leq t\leq 1} \big((Tu)(t)-(Tv)(t)\big) \\
&\leq \sup_{0\leq t\leq 1} \Big[\int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)} f(\tau,u(\tau))d_q\tau\Big)d_qs \\
&\quad - \int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)} f(\tau,v(\tau))d_q\tau\Big)d_qs\Big] \\
&\leq \big(\ln(u+2)(u-v+1)^{\lambda} - \ln(v+2)\big) \\
&\quad\times \sup_{0\leq t\leq 1}\int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)} d_q\tau\Big)d_qs \\
&\leq \ln\frac{(u+2)(u-v+1)^{\lambda}}{v+2}\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\Big)\sup_{0\leq t\leq 1}\int_0^1 G(t,qs)d_qs \\
&\leq \big(\lambda+1\big)\ln(u-v+1)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\Big)\sup_{0\leq t\leq 1}\int_0^1 G(t,qs)d_qs.
\end{aligned}$$

Since the function $h(x) = \ln(x+1)$ is nondecreasing, by condition (ii), we have

$$\begin{aligned}
d(Tu,Tv) &\leq (\lambda+1)\ln(\|u-v\|+1)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\Big)\sup_{0\leq t\leq 1}\int_0^1 G(t,qs)d_qs \\
&= (\lambda+1)\ln(\|u-v\|+1)M \\
&\leq \|u-v\| - (\|u-v\| - \ln(\|u-v\|+1)).
\end{aligned}$$

Let $\psi(x) = x - \ln(x+1)$. Obviously $\psi : [0,+\infty) \to [0,+\infty)$ is continuous, nondecreasing, positive in $(0,+\infty)$, $\psi(0) = 0$ and $\lim_{x\to+\infty}\psi(x) = +\infty$. Thus, for $u \geq v$, we have

$$d(Tu,Tv) \leq d(u,v) - \psi(d(u,v)).$$

As $G(t,qs) \geq 0$ and $f \geq 0$, $(T0)(t) = \int_0^1 G(t,qs)f(s,0)d_qs \geq 0$ and by Theorem 2.6 we know that problem (1.1)-(1.2) has at least one nonnegative solution. As $(K, \leq)$ satisfies condition (2.2), thus, Theorem 2.7 implies that uniqueness of the solution. The proof is complete. $\square$

the3.2 **Theorem 4.2.** *If we add the condition $f(t,0) > 0$ for all $t \in [0,1]$ to Theorem 4.1, then the solution $u(t)$ of boundary value problem* (1.1)-(1.2) *obtained from 4.1 is strictly increasing.*

Proof. At first, we take the unique solution $u(t)$ given to us from Theorem 4.1, we will prove that this solution $u(t)$ is strictly increasing function. Next, as $u(0) = \int_0^1 G(0, qs)f(s, u(s))d_qs$ and $G(0, qs) = 0$ we have $u(0) = 0$. Moreover, if we take $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we can consider the following cases.

Case 1: $t_1 = 0$, in this case, $u(t_1) = 0$ and, as $u(t) \geq 0$, suppose that $u(t_2) = 0$. Then

$$\begin{aligned} 0 = u(t_2) = &\int_0^1 G(t_2, qs)f(s, u(s))d_qs \\ &+ \frac{\beta t_2^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs)f(s, u(s))d_qs. \end{aligned}$$

This implies that

$$G(t_2, qs) \cdot f(s, u(s)) = 0, \quad \text{a.e. } (s)$$

and as $G(t_2, s) \neq 0$ a.e.(s) we get $f(s, u(s)) = 0$ a.e. (s). On the other hand, f is nondecreasing respect to the second variable, then we have

$$f(s, 0) \leq f(s, u(s)) = 0, \quad \text{a.e. } (s)$$

which contradicts the condition $f(t, 0) > 0$ for all $t \in [0, 1]$. Thus $u(t_1) = 0 < u(t_2)$.

Case 2: $0 < t_1$. In this case, let us take $t_2, t_1 \in [0, 1]$ with $t_1 < t_2$, then

$$\begin{aligned} u(t_2) - u(t_1) &= (Tu)(t_2) - (Tu)(t_1) \\ &= \int_0^1 (G(t_2, qs) - G(t_1, qs))f(s, u(s))d_qs \\ &\quad + \frac{\beta(t_2^{\alpha-1} - t_1^{\alpha-1})}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs)f(s, u(s))d_qs. \end{aligned}$$

Taking into account Lemma 3.3 (2) and the fact that $f \geq 0$, we get $u(t_2)-u(t_1) \geq 0$.

Suppose that $u(t_2) = u(t_1)$ then

$$\int_0^1 (G(t_2, qs) - G(t_1, qs))f(s, u(s))d_qs = 0$$

and this implies

$$(G(t_2, qs) - G(t_1, qs))f(s, u(s)) = 0 \quad \text{a.e. } (s).$$

Again, Lemma 3.3 (2) gives us

$$f(s, u(s)) = 0 \quad a.e.(s)$$

and using the same reasoning as above we have that this contradicts condition $f(t, 0) > 0$ for all $t \in [0, 1]$. Thus $u(t_1) = 0 < u(t_2)$. At last, in all cases imply that this solution $u(t)$ is strictly increasing function. The proof is complete. □

5. Examples

Example 5.1. *The fractional boundary-value problem*

$$\begin{gathered} D_{1/2}^{5/2}u(t) + (\frac{1}{10}t^2 + 1)\ln(2 + u(t)) = 0, \quad 0 < t < 1, \\ u(0) = (D_{1/2}u)(0) = 0, \quad (D_{1/2}u)(1) = \frac{1}{2}(D_{1/2}u)(1) \end{gathered} \tag{5.1}$$

e5.1

has a unique and strictly increasing solution.

In this case, $q = 1/2$, $\alpha = 5/2$, $\beta = 1/2$, $\eta = 1$ and $f(t,u) = (\frac{1}{10}t^2+1)\ln(2+u(t))$ for $(t,u) \in [0,1] \times [0,\infty)$. Note that f is a continuous function and $f(t,u) \neq 0$ for $t \in [0,1]$. Moreover, f is nondecreasing respect to the second variable since $\frac{\partial f}{\partial u} = \frac{1}{u+2}(\frac{1}{10}t^2+1) > 0$. On the other hand, for $u \geq v$ and $t \in [0,1]$, we have

$$\begin{aligned}
f(t,u) - f(t,v) &= (\frac{1}{10}t^2+1)\ln(2+u) - (\frac{1}{10}t^2+1)\ln(2+v) \\
&= (\frac{1}{10}t^2+1)\ln\Big(\frac{2+u}{2+v}\Big) \\
&= (\frac{1}{10}t^2+1)\ln\Big(\frac{2+v+u-v}{2+v}\Big) \\
&= (\frac{1}{10}t^2+1)\ln\Big(1+\frac{u-v}{2+v}\Big) \\
&\leq (\frac{1}{10}t^2+1)\ln\big(1+(u-v)\big) \\
&\leq \frac{11}{10}\ln(1+u-v).
\end{aligned}$$

In this case, $\lambda = 11/10$ because

$$\begin{gathered}
M \leq \frac{1-(1-q)^{\alpha-1}}{\Gamma_q(\alpha)} \approx 0.48636, \\
\frac{1}{M+N} \geq 1.41514 > \frac{11}{10} = \lambda.
\end{gathered}$$

Thus Theorem 4.1 implies that boundary value problem (1.1)-(1.2) has a unique solution $u(t)$; i.e.,

$$u(t) = \int_0^1 G(t,qs)\phi_p^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-\tau)^{(\gamma-1)} f(\tau,u(\tau))d_q\tau\Big)d_qs.$$

By Lemma 3.3, we know that G is strictly increasing in the first variable. Therefore, The unique solution $u(t)$ of boundary value problem (5.1) is strictly increasing solution.

Acknowledgments. The authors are very grateful to Professor Ira Herbst for his valuable suggestions and helpful comments which improved the presentation of the original manuscript.

The authors are supported by Foundation for China Postdoctoral Science Foundation, (Grant no. 2012M520665), Youth Foundation for Science and Technology Department of Jilin Province (20130522100JH), Research Foundation during the 12st Five-Year Plan Period of Department of Education of Jilin Province, China (Grant [2013] No. 252), The open project program of Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University (Grant no. 93K172013K03), and Natural Science Foundation of Changchun Normal University.

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