Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 175, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF POSITIVE BOUNDED SOLUTIONS FOR NONLINEAR ELLIPTIC SYSTEMS

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Abstract. In this article, we study a class of nonlinear elliptic systems in regular domains of $\mathbb{R}^n (n \geq 3)$ with compact boundary. More precisely, we prove the existence of bounded positive continuous solutions to the system $\Delta u = \lambda f(., u, v)$, $\Delta v = \mu g(., u, v)$, subject to some Dirichlet conditions. Our approach is essentially based on properties of functions in a Kato class $K^\infty(D)$ and the Schauder fixed point theorem.

1. Introduction

The study of elliptic equations has strong motivations. In fact, such equations model many phenomena in biology, ecology, combustion theory [6, 12], chemical reactions, population genetics [13] etc. For instance, many steady state problems arise in the description of physics phenomena such as fluid dynamics [2], wave phenomena, nonlinear field theory [7] etc. As consequence, the study of the existence of positive solutions and their asymtotic behaviour of such problems are of interest. A typical model example of these is the nonlinear eigenvalue problem

$$\Delta u = \lambda f(u) \quad \text{in } D,$$

where λ is a positive parameter. For an extensive review on the existence results of positive solutions of the above problem we refer the reader to the work of Lions [18].

Recently, many researchers extended the study of nonlinear elliptic scalar equations to nonlinear elliptic systems. For some recent results, we give a short account.

Lair and Wood [17] studied the existence of entire nonnegative solutions for the semilinear elliptic system

$$\begin{aligned} \Delta u &= p(|x|)v^r, \\ \Delta v &= q(|x|)u^s, \end{aligned}$$

in $\mathbb{R}^n$, where $r > 0$ and $s > 0$. The authors proved the existence of entire bounded solutions and large ones in the sublinear and superlinear cases, provided that the

2000 *Mathematics Subject Classification.* 34B15, 34B27, 35J66.
Key words and phrases. Green function; Kato class; nonlinear elliptic systems; positive solution; Schauder fixed point theorem.

Submitted August 15, 2012. Published July 29, 2013.

potentials p and q satisfy either

$$\int_0^\infty tp(t)dt < \infty \quad \text{and} \quad \int_0^\infty tq(t)dt < \infty$$

or

$$\int_0^\infty tp(t)dt = \infty \quad \text{and} \quad \int_0^\infty tq(t)dt = \infty.$$

Cirstea and Radulescu [9] studied the semilinear elliptic system

$$\begin{aligned} \Delta u &= p(x)f_1(v), \\ \Delta v &= q(x)f_2(u), \end{aligned}$$

in $\mathbb{R}^n$ ($n \geq 3$), where the functions f_1 and f_2 are nonincreasing on $(0, \infty)$ and p and q are radially symmetric functions in $\mathbb{R}^n$. In particular, the authors established the existence of positive solutions provided that the function $x \to f(cg(x))$ is sublinear at infinity and superliner at 0, for each $c > 0$. Moreover, the authors gave the behavior of solutions, that is, bounded solutions or blow-up ones depending upon some additional conditions related essentially to the potentials p and q. Motivated by this work [9], Ghanmi et al [16] considered the system

$$\begin{gathered} \Delta u = \lambda p(x)f_1(v) \quad \text{in } D, \\ \Delta v = \mu q(x)f_2(u) \quad \text{in } D, \\ u\big|_{\partial D} = a\varphi, \quad v\big|_{\partial D} = b\psi, \\ \lim_{|x|\to+\infty} u(x) = \alpha, \quad \lim_{|x|\to+\infty} v(x) = \beta \quad \text{(if } D \text{ is unbounded)}, \end{gathered} \tag{1.1}$$

where the potentials p and q belong to the Kato class $K^\infty(D)$ defined below (See Definition 1.5), the functions f_1 and f_2 are monotone. Indeed, the authors established two existence results for the problem (1.1) as f_1 and f_2 are nondecreasing or nonincreasing. They used a variant of monotone iteration and the properties of the Green function and potentials belonging to $K^\infty(D)$. We note that the authors extended the results of Toumi and Zeddini [21] and Ahtreya [4] to systems of equations. García-Melián and Rossi [14] considered the elliptic system

$$\begin{aligned} \Delta u &= u^p v^q \quad \text{in } \Omega \\ \Delta v &= u^r v^s \quad \text{in } \Omega \end{aligned}$$

where $p, s > 1, q, r > 0$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, subject to different types of Dirichlet boundary conditions:

(C1) $u = \alpha$, $v = \beta$,
(C2) $u = v = +\infty$ and
(C3) $u = +\infty$, $v = \alpha$ on ∂D, where $\alpha, \beta > 0$.

Under several hypotheses on the parameters p, q, r, s, they showed the existence and nonexistence, uniqueness and nonuniqueness of positive solutions. We mention that the proofs in [14] were based on the method of sub and super- solutions and the maximum principle. We remark that numerous works treating nonlinear elliptic systems adopted many techniques employed in the study of scalar equations, namely, the method of sub and super- solutions, variational method, topology degree, fixed point index theory, see [1, 10, 11, 15, 16] for more details and references therein.

In the present article, we consider a $C^{1,1}$-domain D in $\mathbb{R}^n (n \geq 3)$ with compact boundary ∂D. We fix two nontrivial nonnegative continuous functions φ and ψ on

∂D and we will deal with the existence and the asymptotic behaviour of bounded solutions (in the sense of distributions) to the nonlinear elliptic system

$$\begin{gathered}\Delta u=\lambda f(.,u,v)\quad\text{in } D,\\ \Delta v=\mu g(.,u,v)\quad\text{in } D,\\ u\big|_{\partial D}=a\varphi,\quad v\big|_{\partial D}=b\psi,\\ \lim_{|x|\to+\infty}u(x)=\alpha,\quad \lim_{|x|\to+\infty}v(x)=\beta\quad(\text{if } D \text{ is unbounded}),\end{gathered}\tag{1.2}$$

where the nonnegative constants a,b,α and β are such that $a+\alpha>0$, $b+\beta>0$.

For this aim, we will use a fixed point argument to give two existence results for problem (1.2). We are essentially inspired by the work [16].

Hereinafter, we denote by $H_D\varphi$ the bounded continuous solution of the Dirichlet problem

$$\begin{gathered}\Delta u=0\quad\text{in } D,\\ u=\varphi\quad\text{on } \partial D,\\ \lim_{|x|\to+\infty}u(x)=0,\quad\text{if } D \text{ is unbounded},\end{gathered}\tag{1.3}$$

where φ is a nontrivial nonnegative continuous function on ∂D. Morerover, we denote

$$h=1-H_D1 \tag{1.4}$$

and we remark that $h=0$ when D is bounded.

For a nonnegative measurable function f, we denote by Vf the potential function defined in D by

$$Vf(x)=\int_D G_D(x,y)f(y)dy,$$

where G_D is the Green function of the Laplace operator Δ in D with Dirichlet conditions.

Throughout this article, we fix a nontrivial nonnegative continuous function Φ on ∂D and we will use combinations of the following hypotheses

- (H1) f and g are nonnegative measurable functions on $D\times(0,\infty)\times(0,\infty)$ such that for each $x\in D$ the function $(u,v)\mapsto(f(x,u,v),g(x,u,v))$ is continuous on $(0,\infty)\times(0,\infty)$.
- (H2) For all $0<u\le u_1, 0<v\le v_1$ and $x\in D$,
$$f(x,u,v)\le f(x,u_1,v_1),\quad g(x,u,v)\le g(x,u_1,v_1).$$
- (H3) For all $c_1,c_2>0$, the functions $f(.,c_1,c_2)$ and $g(.,c_1,c_2)$ are in $K^\infty(D)$.
- (H4) For $\omega:=aH_D\varphi+\alpha h$ and $\theta:=bH_D\psi+\beta h$, we have
$$\lambda_0=\inf_{x\in D}\frac{\omega(x)}{Vf(.,\omega,\theta)(x)}>0, \tag{1.5}$$
$$\mu_0=\inf_{x\in D}\frac{\theta(x)}{Vg(.,\omega,\theta)(x)}>0. \tag{1.6}$$
- (H5) For all $0\le u\le u_1, 0\le v\le v_1$ and $x\in D$,
$$f(x,u_1,v_1)\le f(x,u,v)\quad\text{and}\quad g(x,u_1,v_1)\le g(x,u,v).$$
- (H6) For $h_0=H_D\Phi$. The functions $x\mapsto\widetilde{p}(x):=\frac{f(x,h_0(x),h_0(x))}{h_0(x)}$ and $x\mapsto\widetilde{q}(x):=\frac{g(x,h_0(x),h_0(x))}{h_0(x)}$ belong to $K^\infty(D)$.

Remark 1.1. Let $\tau(x) := \delta(x)$ if D is bounded and $\tau(x) := \frac{\delta(x)}{(1+|x|)^{n-1}}$ if D is unbounded. Note that under hypothesis (H5) the condition: "For all $c_1, c_2 > 0$,

$$\frac{f(x, c_1\tau(x), c_2\tau(x))}{\tau(x)} \quad \text{and} \quad \frac{g(x, c_1\tau(x), c_2\tau(x))}{\tau(x)}$$

belong to $K^\infty(D)$" implies (H6). Indeed, from [3, 22], there exists $c > 0$ such that for each $x \in D, h_0(x) \geq c\tau(x)$. Using (H5), we obtain that $\frac{f(x,h_0(x),h_0(x))}{h_0(x)} \leq \frac{f(x,c\tau(x),c\tau(x))}{c\tau(x)} \in K^\infty(D)$. Similarly, we obtain that $\frac{g(x,h_0(x),h_0(x))}{h_0(x)} \in K^\infty(D)$ and so (H6) is satisfied.

Our paper is organized as follows. In Section 2, we give the first existence result concerning problem (1.2). More precisely we prove the following result.

Theorem 1.2. *Assume that* (H1)–(H4) *are satisfied. Then for each* $\lambda \in [0, \lambda_0)$ *and* $\mu \in [0, \mu_0)$*, problem* (1.2) *has a positive continuous bounded solution* (u, v) *satisfying on* D

$$(1 - \frac{\lambda}{\lambda_0})\omega(x) \leq u(x) \leq \omega(x)$$
$$(1 - \frac{\mu}{\mu_0})\theta(x) \leq v(x) \leq \theta(x).$$

As a consequence of Theorem 1.2, we will prove the following result.

Corollary 1.3. *Let* $\xi_1, \xi_2 : (0, +\infty) \to (0, +\infty)$ *be two continuous functions. Assume that* (H1)–(H4) *hold. Then for each* $\lambda \in [0, \lambda_0)$ *and* $\mu \in [0, \mu_0)$*, the problem*

$$\begin{gathered}
\Delta u + \xi_1(u)|\nabla u|^2 = \lambda f(., u, v) \quad \text{in } D, \\
\Delta v + \xi_2(v)|\nabla v|^2 = \mu g(., u, v) \quad \text{in } D, \\
u\big|_{\partial D} = a\varphi, \quad v\big|_{\partial D} = b\psi, \\
\lim_{|x|\to+\infty} u(x) = \alpha, \quad \lim_{|x|\to+\infty} v(x) = \beta \quad (\text{if } D \text{ is unbounded}).
\end{gathered} \tag{1.7}$$

has a positive continuous bounded solution (u, v).

Section 3 is dedicated to the second existence result for system (1.2) for $a = b = 1$ and $\lambda = \mu = 1$. So for a fixed nontrivial nonnegative continuous function Φ on ∂D, we prove the second result of this work.

Theorem 1.4. *Assume* (H1), (H5), (H6) *are satisfied. Then there exists a constant* $c > 1$ *such that if* $\varphi \geq c\Phi$ *and* $\psi \geq c\Phi$ *on* ∂D*, problem* (1.2)*, with* $a = 1$ *and* $b = 1$*, has a positive continuous solution* (u, v)*. Moreover, for each* $x \in D$*,* (u, v) *satisfies*

$$\alpha h(x) + H_D\Phi(x) \leq u(x) \leq \alpha h(x) + H_D\varphi$$
$$\beta h(x) + H_D\Phi(x) \leq v(x) \leq \beta h(x) + H_D\psi.$$

In the remainder of this section we will recall some notation and results needed in the rest of this paper.

$\mathcal{B}(D)$ is the set of Borel measurable functions in D and $\mathcal{C}_0(D)$ is the set of continuous ones vanishing continuously on $\partial D \cup \{\infty\}$. The exponent $+$ means that only the nonnegative functions are considered.

We note that $\mathcal{C}(\overline{D} \cup \{\infty\})$ and $\mathcal{C}(\overline{D} \cup \{\infty\}) \times \mathcal{C}(\overline{D} \cup \{\infty\})$ are two Banach spaces endowed with uniform norm $\|u\|_\infty = \sup_{x\in\overline{D}\cup\{\infty\}} |u(x)|$ and $\|(u, v)\|_\infty = \max(\|u\|_\infty, \|v\|_\infty)$, respectively.

If $f \in L^1_{\rm loc}(D)$ and $Vf \in L^1_{\rm loc}(D)$, then we have $\Delta(Vf) = -f$ in D (in the sense of distributions) see [8].

Definition 1.5 ([5, 19])**.** A Borel measurable function p in D belongs to the class $K^\infty(D)$ if p satisfies

$$\lim_{\alpha\to 0}\Big(\sup_{x\in D}\int_{D\cap B(x,\alpha)}\frac{\rho(y)}{\rho(x)}G_D(x,y)|p(y)|dy\Big)=0, \tag{1.8}$$

and

$$\lim_{M\to+\infty}\Big(\sup_{x\in D}\int_{D\cap(|y|\geq M)}\frac{\rho(y)}{\rho(x)}G_D(x,y)|p(y)|dy\Big)=0 \quad \text{(if D is unbounded)}, \tag{1.9}$$

where $\rho(x)=\min(1,\delta(x))$ and $\delta(x)$ is the Euclidean distance between x and ∂D.

Proposition 1.6. *Let p be a nonnegative function in $K^\infty(D)$, then*

(i) *The function $x \mapsto \frac{\rho(x)}{1+|x|^{n-1}}p(x) \in L^1(D)$.*
(ii) $\alpha_p = \sup_{x,y\in D}\int_D \frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}p(z)dz < \infty$.
(iii) *For any nonnegative superharmonic function h in D we have*

$$\int_D G_D(x,y)h(y)p(y)dy \leq \alpha_p h(x), \forall x\in D. \tag{1.10}$$

(iv) *The potential $Vp \in \mathcal{C}_0(D)$.*
(v) *If h_0 is a positive harmonic function in D, continuous and bounded in $\overline{D}$, then the family of functions*

$$\mathfrak{F}_p = \Big\{\int_D G_D(.,y)h_0(y)v(y)dy : |v|\leq p\Big\}$$

is relatively compact in $\mathcal{C}_0(D)$.

Proof. These properties were proved in [19] for $\mathcal{C}^{1,1}$-bounded domains in $\mathbb{R}^n$ and in [5, 21] for $\mathcal{C}^{1,1}$-unbounded domains with compact boundary. □

2. Proof of Theorem 1.2

In this section, we are concerned with the first existence result for the system (1.2). More precisely, we will give proofs of Theorem 1.2 and Corollary 1.3. Moreover, we will give some examples to illustrate Theorem 1.2.

Proof of Theorem 1.2. We shall use a fixed point argument. Let λ_0, μ_0 be the constants given by (1.5) and (1.6). Let $\lambda \in [0,\lambda_0)$ and $\mu \in [0,\mu_0)$. Recall that $\omega = aH_D\varphi + \alpha h$ and $\theta = bH_D\psi + \beta h$. Consider the non-empty closed convex set Λ given by

$$\Lambda = \Big\{(u,v)\in \mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\}) : (1-\frac{\lambda}{\lambda_0})\omega \leq u\leq \omega,\ (1-\frac{\mu}{\mu_0})\theta\leq v\leq\theta\Big\}.$$

Let T be the integral operator defined on Λ by

$$\begin{aligned} T(u,v) &= (\omega-\lambda\int_D G_D(.,y)f(y,u(y),v(y))dy, \theta-\mu\int_D G_D(.,y)g(y,u(y),v(y))dy) \\ &= (T_1(u,v),T_2(u,v)). \end{aligned}$$

We shall prove that the family $T(\Lambda)$ is relatively compact in $\mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\})$. Let $(u,v)\in\Lambda$. It is obvious to see that $T_1(u,v)\le\omega$ and $T_2(u,v)\le\theta$. Then for each $x\in\overline{D}\cup\{\infty\}$,

$$\|T_1(u,v)\|_\infty\le\|\omega\|_\infty\le\alpha+a\|\varphi\|_\infty:=c_1,$$
$$\|T_2(u,v)\|_\infty\le\|\theta\|_\infty\le\beta+b\|\psi\|_\infty:=c_2.$$

So

$$\|T(u,v)\|_\infty\le\max(c_1,c_2).$$

Hence $T(\Lambda)$ is uniformly bounded.

Next, by hypotheses (H2) and (H3), it follows that for each $(u,v)\in\Lambda$,

$$f(.,u,v)\le f(.,c_1,c_2)=:q_1\in K^\infty(D), \tag{2.1}$$
$$g(.,u,v)\le g(.,c_1,c_2)=:q_2\in K^\infty(D). \tag{2.2}$$

Therefore,

$$\mathcal{A}_1:=\Big\{\int_D G_D(.,y)f(y,u(y),v(y))dy:(u,v)\in\Lambda\Big\}\subseteq\mathfrak{F}_{q_1},$$
$$\mathcal{A}_2:=\Big\{\int_D G_D(.,y)g(y,u(y),v(y))dy:(u,v)\in\Lambda\Big\}\subseteq\mathfrak{F}_{q_2}.$$

Now, by Proposition 1.6 (v), the families $\mathfrak{F}_{q_1}$ and $\mathfrak{F}_{q_2}$ are relatively compact in $\mathcal{C}_0(D)$. Therefore $\mathcal{A}_1$ and $\mathcal{A}_2$ are equicontinuous in $\overline{D}\cup\{\infty\}$. Now, since the functions ω and θ belong to $\mathcal{C}(\overline{D}\cup\{\infty\})$, we deduce that $T_1(\Lambda)$ and $T_2(\Lambda)$ are equicontinuous in $\overline{D}\cup\{\infty\}$. Hence, $T(\Lambda)$ is equicontinuous in $\overline{D}\cup\{\infty\}$. Using Arzela-Ascoli theorem, we obtain that $T(\Lambda)$ is relatively compact in $\mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\})$.

Now, we claim that the operator T maps Λ to itself. Indeed, since $T(\Lambda)$ is equicontinuous on $\overline{D}\cup\{\infty\}$, it follows that for each $(u,v)\in\Lambda, T(u,v)\in\mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\})$. On the other hand, using hypothesis (H2), we conclude that for each $x\in D$,

$$T_1(u,v)(x)\ge\omega(x)-\lambda\int_D G_D(x.,y)f(y,\omega(y),\theta(y))dy.$$

So by (1.5), it follows that

$$T_1(u,v)(x)\ge(1-\frac{\lambda}{\lambda_0})\omega(x). \tag{2.3}$$

Similarly, we have

$$T_2(u,v)(x)\ge(1-\frac{\mu}{\mu_0})\theta(x). \tag{2.4}$$

Then, by (2.3) and (2.4), we deduce by that $T(\Lambda)\subset\Lambda$.

Next, let us prove that T is a continuous mapping in the supremum norm. Let $\{(u_k,v_k)\}_k$ be a sequence in Λ which converges uniformly to a function (u,v) in Λ. Then, for each $x\in D$, we have

$$|T_1(u_k,v_k)(x)-T_1(u,v)(x)|\le\int_D G_D(x,y)|f(y,u_k(y),v_k(y))-f(y,u(y),v(y))|dy.$$

On the other hand, by (H2), we have

$$|f(y,u_k(y),v_k(y))-f(y,u(y),v(y))|\le 2f(y,c_1,c_2)=2q_1(y)\in K^\infty(D).$$

Since, by Proposition 1.6 (iv), the function Vq_1 is bounded, we deduce by (H1) and the dominated convergence theorem that for all $x \in D$,

$$T_1(u_k, v_k)(x) \to T_1(u,v)(x) \quad \text{as } k \to +\infty.$$

Similarly,

$$T_2(u_k, v_k)(x) \to T_2(u,v)(x) \quad \text{as } k \to +\infty.$$

Therefore,

$$T(u_k, v_k)(x) \to T(u,v)(x) \quad \text{as } k \to +\infty.$$

As $T(\Lambda)$ is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\}) \times \mathcal{C}(\overline{D} \cup \{\infty\})$, we conclude that the pointwise convergence implies the uniform convergence; that is,

$$\|T(u_k, v_k) - T(u,v)\|_u \to 0 \quad \text{as } k \to +\infty.$$

Hence T is a compact mapping on from Λ to itself. By the Schauder fixed point theorem, there exists $(u,v) \in \Lambda$ such that $T(u,v) = (u,v)$. That is,

$$u(x) = w(x) - \lambda \int_D G_D(x,y) f(y, u(y), v(y)) dy, \tag{2.5}$$

$$v(x) = \theta(x) - \mu \int_D G_D(x,y) f(y, u(y), v(y)) dy. \tag{2.6}$$

Now, let us prove that (u,v) is a solution of the problem (1.2). Since $q_1, q_2 \in K^\infty(D)$, it follows by Proposition 1.6 (i), that $q_1, q_2 \in L^1_{\rm loc}(D)$. Using (2.1) and (2.2), we deduce that $f(., u, v), g(., u, v) \in L^1_{\rm loc}(D)$ and $Vf(., u, v), Vg(., u, v) \in \mathcal{C}_0(D)$. Thus applying Δ on both sides of (2.5) and (2.6) respectively, we obtain that (u,v) satisfies the elliptic system (in the sense of distributions)

$$\begin{gathered} \Delta u = \lambda f(., u, v) \quad \text{in } D, \\ \Delta v = \mu g(., u, v) \quad \text{in } D. \end{gathered}$$

Moreover, since the functions $Vf(., u, v)$ and $Vg(., u, v)$ are in $\mathcal{C}_0(D)$, we conclude that

$$\begin{gathered} \lim_{x \to z \in \partial D} u(x) = a\varphi(z), \quad \lim_{|x| \to \infty} u(x) = \alpha, \\ \lim_{x \to z \in \partial D} v(x) = b\psi(z), \quad \lim_{|x| \to \infty} v(x) = \beta. \end{gathered}$$

This completes the proof. ☐

Proof of Corollary 1.3. Let $i \in \{1,2\}$ and $\rho_i(t) = \int_0^t \exp(\int_0^s \xi_i(r) dr) ds$. Then ρ_i is a $\mathcal{C}^2-$ diffeomorphism from $(0, +\infty)$ to itself. Put $u_1 = \rho_1(u)$ and $v_1 = \rho_2(v)$. Then (u_1, v_1) satisfies

$$\begin{gathered} \Delta u_1 = \lambda \rho_1'(\rho_1^{-1}(u_1)) f(., \rho_1^{-1}(u_1), \rho_2^{-1}(v_1)) \quad \text{in } D, \\ \Delta v_1 = \mu \rho_2'(\rho_2^{-1}(v_1)) g(., \rho_1^{-1}(u_1), \rho_2^{-1}(v_1)) \quad \text{in } D, \\ u_{1/\partial D} = \rho_1(a\varphi), \quad v_{1/\partial D} = \rho_2(b\psi) \\ \lim_{|x| \to +\infty} u_1(x) = \rho_1(\alpha), \quad \lim_{|x| \to +\infty} v_1(x) = \rho_2(\beta) \quad \text{(if } D \text{ is unbounded).} \end{gathered} \tag{2.7}$$

Put

$$\begin{gathered} F(., u_1, v_1) := \rho_1'(\rho_1^{-1}(u_1)) f(., \rho_1^{-1}(u_1), \rho_2^{-1}(v_1)), \\ G(., u_1, v_1) := \rho_2'(\rho_2^{-1}(v_1)) g(., \rho_1^{-1}(u_1), \rho_2^{-1}(v_1)). \end{gathered}$$

Then F and G satisfy (H1)–(H4). Thus by Theorem 1.2, problem (2.7) admits a positive bounded solution (u_1, v_1). So, it is easy to verify that $(\rho_1^{-1}(u_1), \rho_2^{-1}(v_1))$ is a positive bounded solution of the problem $(Q_{a,b})$. This completes the proof. □

Example 2.1. Let D be a $\mathcal{C}^{1,1}$-bounded domain in $\mathbb{R}^n (n \geq 3)$. Let φ and ψ be two nontrivial nonnegative continuous functions on ∂D. Let p, q be two nonnegative functions in $L^k(D), k > \frac{n}{2}$ and suppose that $m_1, m_2 < 1 - \frac{n}{k}$. Let $r_1, r_2, s_1, s_2 > 0$. Then, the system

$$\Delta u = \lambda \frac{p(x)}{(\delta(x))^{m_1}} u^{r_1} v^{s_1} \text{ in } D,$$
$$\Delta v = \mu \frac{q(x)}{(\delta(x))^{m_2}} u^{r_2} v^{s_2} \quad \text{in } D, \quad u\big|_{\partial D} = \varphi,$$
$$v\big|_{\partial D} = \psi.$$

has a positive bounded continuous solution. Indeed, from [20, Proposition 2.3], the functions $p_1(x) := p(x)/(\delta(x))^{m_1}$, and $q_1(x) := q(x)/(\delta(x))^{m_2}$ belong to $K^\infty(D)$ and so (H3) is satisfied. From [20, Proposition 2.7(iii)], there exists a constant $c > 0$ such that we have for each $x \in D$

$$Vp_1(x) \leq c\delta(x).$$

So, for $f(x, u, v) = p_1(x) u^{r_1} v^{s_1}$, we have

$$Vf(., H_D\varphi, H_D\psi)(x) \leq c\|\varphi\|_\infty^{r_1} \|\psi\|_\infty^{s_1} \delta(x).$$

In addition, since the function φ is nontrivial nonnegative on ∂D, then there exists a constant $c_1 > 0$ such that we have on D

$$H_D\varphi(x) \geq c_1\delta(x).$$

Thus,

$$\lambda_0 = \inf_{x \in D} \frac{H_D\varphi(x)}{Vf(., H_D\varphi, H_D\psi)(x)} > \frac{c_1}{c\|\varphi\|_\infty^{r_1} \|\psi\|_\infty^{s_1}} > 0.$$

Similarly, we prove that $\mu_0 > 0$ and so assumption (H4) is satisfied.

Example 2.2. Let $D = \overline{B(0,1)}^c$ be the exterior of the unit ball in $\mathbb{R}^n$ $(n \geq 3)$. Suppose that $\gamma, \sigma > n$. Let $r_1, r_2, s_1, s_2 > 0$. Then, the problem

$$\Delta u = \lambda \frac{1}{|x|^{\sigma-\gamma}(|x|-1)^\gamma} u^{r_1} v^{s_1} \quad \text{in } D,$$
$$\Delta v = \mu \frac{1}{|x|^{\sigma-\gamma}(|x|-1)^\gamma} u^{r_2} v^{s_2} \quad \text{in } D,$$
$$u\big|_{\partial D} = \varphi, \quad v\big|_{\partial D} = \psi,$$
$$\lim_{|x| \to +\infty} u(x) = \alpha, \quad \lim_{|x| \to +\infty} v(x) = \beta.$$

has a positive continuous solution. In fact, from [5] the functions $p(x) := \frac{1}{|x|^\sigma}$ and $q(x) := \frac{1}{|x|^\gamma}$ belong to $K^\infty(D)$. Morerover, from [5, Proposition 3.5], there exists a constant $c > 0$ such that

$$Vp(x) \leq c\frac{|x|-1}{|x|^{n-1}}.$$

So, for $f(x,u,v) := p(x)u^{r_1}v^{s_1}$, $\omega = H_D\varphi + \alpha h$ and $\theta = H_D\psi + \beta h$, there exists a constant $c_1 > 0$ such that

$$Vf(.,\omega,\theta)(x) \le c_1 \frac{|x|-1}{|x|^{n-1}}.$$

On the other hand, from [3, page 258] there exists a constant $c_2 > 0$ such that on D we have

$$\omega(x) \ge c_2 \frac{|x|-1}{|x|^{n-1}}.$$

It follows that $\lambda_0 = \inf_{x\in D} \frac{\omega(x)}{Vf(.,\omega,\theta)(x)} \ge \frac{c_2}{c_1} > 0$. Similarly, we prove that $\mu_0 = \inf_{x\in D} \frac{\theta(x)}{Vg(.,\omega,\theta)(x)} > 0$, for $g(x,u,v) := q(x)u^{r_2}v^{s_2}$. Thus, the assumption (H4) is satisfied.

3. Proof of Theorem 1.4

In this section, we will be interested in (1.2) with $a = b = \lambda = \mu = 1$; that is, we will study the problem

$$\begin{gathered}
\Delta u = f(.,u,v) \quad \text{in } D,\\
\Delta v = g(.,u,v) \quad \text{in } D,\\
u\big|_{\partial D} = \varphi, \quad v\big|_{\partial D} = \psi,\\
\lim_{|x|\to+\infty} u(x) = \alpha, \quad \lim_{|x|\to+\infty} v(x) = \beta \quad (\text{if } D \text{ is unbounded}).
\end{gathered} \tag{3.1}$$

where $\alpha, \beta \ge 0$. So, we recall that Φ is a fixed nontrivial nonnegative continuous function on ∂D and we put $h_0 = H_D\Phi$. First, we give the proof of Theorem 1.4. Then we give an example of application to illustrate Theorem 1.4.

Proof of Theorem 1.4. Let $\alpha_{\widetilde{p}}$ and $\alpha_{\widetilde{q}}$ be the constants defined in Proposition 1.6 (ii) associated to the functions $\widetilde{p}$ and $\widetilde{q}$ given in hypothesis (H6). Put $c = 1+\alpha_{\widetilde{p}}+\alpha_{\widetilde{q}}$ and suppose that

$$\varphi(x) \ge ch_0(x), \quad \psi(x) \ge ch_0(x), \forall x \in \partial D.$$

Then, by the maximum principle, we have

$$H_D\varphi(x) \ge ch_0(x), \quad H_D\psi(x) \ge ch_0(x), \forall x \in D. \tag{3.2}$$

Now, let Γ be the non-empty closed bounded convex set given by

$$\Gamma = \{(w,z) \in \mathcal{C}(\overline{D}\cup\{\infty\}) \times \mathcal{C}(\overline{D}\cup\{\infty\}) : h_0 \le w \le H_D\varphi,\ h_0 \le z \le H_D\psi\}.$$

Consider the operator L defined on Γ by

$$L(w,z) = (L_1(w,z), L_2(w,z)),$$

where

$$L_1(w,z)(x) = H_D\varphi(x) - \int_D G_D(x,y) f(y, w(y)+\alpha h(y), z(y)+\beta h(y))dy,$$
$$L_2(w,z)(x) = H_D\psi(x) - \int_D G_D(x,y) g(y, w(y)+\alpha h(y), z(y)+\beta h(y))dy.$$

We shall prove that the operator L admits a fixed point in Γ. Let $(w,z) \in \Gamma$. Then using (H5) and (H6), it follows that

$$f(., w+\alpha h, z+\beta h)(x) \le f(., h_0, h_0)(x) = h_0(x)\widetilde{p}(x), \tag{3.3}$$

$$g(.,w+\alpha h,z+\beta h)(x)\leq g(.,h_0,h_0)(x)=h_0(x)\widetilde{q}(x). \tag{3.4}$$

Now, using (3.3), (3.4) and (H6), it follows that

$$\mathcal{G}_1:=\Big\{\int_D G_D(.,y)f(y,(w+\alpha h)(y),(z+\beta h)(y))dy:(w,z)\in\Gamma\Big\}\subseteq\mathfrak{F}_{\widetilde{p}},$$
$$\mathcal{G}_2:=\Big\{\int_D G_D(.,y)g(y,(w+\alpha h)(y),(z+\beta h)(y))dy:(w,z)\in\Gamma\Big\}\subseteq\mathfrak{F}_{\widetilde{q}}.$$

By Proposition 1.6 (v), $\mathcal{G}_1$ and $\mathcal{G}_2$ are equicontinuous in $\overline{D}\cup\{\infty\}$. Thus, as in the proof of Theorem 1.2, we conclude that $L(\Gamma)$ is equicontinuous in $\overline{D}\cup\{\infty\}$. Moreover, $L(\Gamma)$ is uniformly bounded.

By Ascoli-Arzela theorem, we conclude that the family $L(\Gamma)$ is relatively compact in $\mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\})$. Next, let us prove that L maps Γ to itself. Let $(w,z)\in\Gamma$, since $L(\Gamma)$ is relatively compact in $\mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\})$, it follows that $L(w,z)\in\mathcal{C}(\overline{D}\cup\{\infty\})\times\mathcal{C}(\overline{D}\cup\{\infty\})$. On the other hand, by Proposition 1.6 (iii) and (3.3), we obtain

$$Vf(.,w+\alpha h,z+\beta h)(x)\leq\alpha_{\widetilde{p}}h_0(x). \tag{3.5}$$

So, by (3.2) and (3.5), we obtain

$$L_1(w,z)(x)\geq(1+\alpha_{\widetilde{q}})h_0(x)\geq h_0(x)>0. \tag{3.6}$$

Similarly, we prove that

$$L_2(w,z)(x)\geq h_0(x)>0. \tag{3.7}$$

Thus, $L(\Gamma)\subset\Gamma$.

Now, we proceed as in the proof of Theorem 1.2 and using hypothesis (H5), we prove the continuity of the operator L in the supremum norm. Thus, we conclude that L is a compact operator mapping from Γ to itself. Hence, the Schauder fixed point theorem ensures the existence of $(w,z)\in\Gamma$ such that

$$w(x)=H_D\varphi(x)-\int_D G_D(x,y)f(y,w(y)+\alpha h(y),z(y)+\beta h(y))dy,$$
$$z(x)=H_D\psi(x)-\int_D G_D(x,y)g(y,w(y)+\alpha h(y),z(y)+\beta h(y))dy.$$

Put $u:=w+\alpha h$ and $v:=z+\alpha h$. It is easy to verify that (u,v) is a positive continuous bounded solution of (3.1). □

Example 3.1. Let D be a $\mathcal{C}^{1,1}-$ domain in $\mathbb{R}^n(n\geq 3)$ with compact boundary. Let h_0 be a positive harmonic bounded function in D and τ be the function defined in Remark 1 and $r_1,r_2,s_1,s_2>0$. Suppose that p and q are two nonnegative functions such that $\widetilde{p}(x):=(\tau(x))^{-(r_1+s_1+1)}p(x)$ and $\widetilde{q}(x):=(\tau(x))^{-(r_2+s_2+1)}q(x)$ belong to $K^\infty(D)$. Then there exists a constant $c>1$ such that if $\varphi\geq ch_0$ and $\psi\geq ch_0$ on ∂D, the system

$$\begin{gathered}\Delta u=p(x)u^{-r_1}v^{-s_1}\quad\text{in } D,\\ \Delta v=q(x)u^{-r_2}v^{-s_2}\quad\text{in } D,\\ u\big|_{\partial D}=\varphi,\quad v\big|_{\partial D}=\psi,\\ \lim_{|x|\to+\infty}u(x)=\alpha,\quad\lim_{|x|\to+\infty}v(x)=\beta.\end{gathered}$$

has a positive bounded continuous solution on D.

Acknowledgements. The author is greatly indebted to Professor Habib Maâgli for many helpful suggestions.

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