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EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A SEMILINEAR FILTRATION PROBLEM

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Abstract. We first examine the existence and uniqueness of local solutions to the semilinear filtration equation $u_t = \Delta K(u) + \lambda f(u)$, for $\lambda > 0$, with initial data $u_0 \geq 0$ and appropriate boundary conditions. Our main result is the proof of blow-up of solutions for some λ. Moreover, we discuss the existence of solutions for the corresponding steady-state problem. It is found that there exists a critical value λ^* such that for $\lambda > \lambda^*$ the problem has no stationary solution of any kind, while for $\lambda \leq \lambda^*$ there exist classical stationary solutions. Finally, our main result is that the solution for $\lambda > \lambda^*$, blows-up in finite time independently of $u_0 \geq 0$. The functions f, K are positive, increasing and convex and K'/f is integrable at infinity.

1. Introduction

Our purpose in this work is to examine the existence and uniqueness for $\lambda > 0$ and prove the blow-up of local solutions for $\lambda > \lambda^*$, for some λ^*, of the following initial boundary value problem:

$$\left.\begin{aligned} u_t = \Delta K(u) + \lambda f(u), \quad x \in \Omega, \quad t > 0, \\ \mathcal{B}(K(u)) \equiv \frac{\partial K(u)}{\partial n} + \beta(x)K(u) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \end{aligned}\right\} \tag{1.1}$$

where $u = u(x,t)$, n is the outward pointing normal vector field on $\partial\Omega$ and Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary $\partial\Omega$ and having the interior sphere property, [13]. We impose non-negative initial data in order to get non-negative solutions of (1.1). Moreover, taking $f(u) > 0$ for $u \geq 0$ ($f(0) > 0$, the forced case), we avoid degenerating solutions, hence we get classical solutions. To get classical solutions it is enough to have $u_0 \in L^\infty(\Omega)$; for more results and methods concerning semilinear heat and porous medium problems, see [18, 22]. We introduce homogeneous boundary conditions: $\mathcal{B}(\cdot) = 0$. This type of boundary condition is a consequence of Fourier's law for diffusion and conservation of mass, or heat conduction and conservation of energy. The usual type of boundary condition: $\mathcal{B}(u) = \partial u/\partial n + \beta(x)u = 0$, seems to have no physical significance. Instead, one can consider boundary conditions of the form $\partial K(u)/\partial n + \beta(x)u = 0$, which physically

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means zero flux on the boundary. Here $\beta,\ 0\leq\beta=\beta(x)\leq\infty$, is $C^{1+\alpha}(\partial\Omega)$, $\alpha>0$, whenever it is bounded ($\beta\equiv 0$, $\beta\equiv\infty$, $0<\beta<\infty$ means Neumann, Dirichlet and Robin boundary condition respectively). The function $K=K(s)\in C^3([0,\infty))$ satisfies,

$$K(s)>0 \text{ for } s>0 \text{ and } K(0)=0,\ K'(s),\ K''(s)>0, \text{ for } s\geq 0, \tag{1.2}$$

(see also [13, Ch. VI], [21]). Moreover, functions f,K are assumed to satisfy,

$$f(s)>0,\quad f'(s)>0,\quad f''(s)>0,\quad \text{for } s\geq 0, \tag{1.3}$$

$$(a)\quad \int_0^\infty \frac{K'(s)}{f(s)}ds<\infty,\quad \text{which implies}\quad (b)\quad \int_0^\infty \frac{ds}{f(s)}<\infty\,. \tag{1.4}$$

Concerning (1.4)(a), see also below, Subsection 4.1, this is a necessary condition for blow-up of solutions for the equation $z_t=\Delta z+g(z)$ where $z=K(v)$, with $g(z)=f(v)$. This is easily verified for the problem $K'(v)v_t=\Delta K(v)+f(v)$, with v independent of x. In this case, the v problem reduces to $v_t=f(v)/K'(v)$, $v=v(t)$, $t>0$, $v(0)=v_0\geq 0$ and (1.4)(a) implies blow-up of $v=K^{-1}(z)$ and also of z.

Problem (1.1) is a local semilinear filtration problem. If $K(u)=u^q,\ q>1$, then problem (1.1) is the so-called semilinear porous medium problem.

Now, due to $\lambda>0$ and because of the form of functions f and K, as we shall see in Section 3, there exists a critical value of λ , say λ^*, such that for each $\lambda\in(0,\lambda^*)$ the corresponding steady-state problem of (1.1),

$$\Delta K(w)+\lambda f(w)=0,\quad x\in\Omega,\quad \mathcal{B}(K(w))=0,\quad x\in\partial\Omega, \tag{1.5}$$

has at least one (classical) solution $w=w(x)=w(x;\lambda)\in C^2(\Omega)\cap C(\overline{\Omega})$, ($\Delta K(w)=K''(w)|\nabla w|^2+K'(w)\Delta w$).

The response (bifurcation) diagram of (1.5) can be obtained by performing the "pressure transformation",

$$z=K(w)\quad \text{with } g(z)=f(w); \tag{1.6}$$

thus we derive the following problem,

$$\Delta z+\lambda g(z)=0,\quad x\in\Omega,\quad \mathcal{B}(z)=\frac{\partial z}{\partial n}+\beta(x)z=0,\quad x\in\partial\Omega, \tag{1.7}$$

where $g(\sigma)=f(K^{-1}(\sigma))=(f\circ K^{-1})(\sigma)=f(s)$ with $\sigma=K(s)\geq 0$.

From (1.7), on using the pressure transformation (1.6), we can extract many qualitative properties of problem (1.5). Also, several results and methods for the semilinear filtration problem for bounded or unbounded domains can be applied, see [6, 7, 8, 20]. Here, we have to mention that this transformation constrains the function f. This is due to the convexity and the growth requirement on $g(z)$ since $f(w)=f(K^{-1}(z))=g(z)$. We need, in some cases, for $g(z)$ to be increasing and convex, thus a new condition emerges for f. Actually if $g(z),g'(z),g''(z)$ are positive then $f(w),f'(w),K'(w)$ and $(f''(w)K'(w)-f'(w)K''(w))$ are positive while the integrability at infinity implies $\int_0^\infty K'(w)dw/f(w)=\int_{K(0)}^\infty dz/g(z)<\infty$. To avoid such limitations, wherever possible, one can study directly problem (1.5), and substituting these constraints by other conditions on f and K, see Section 3. Thus, without the use of the pressure transformation (1.6), we deduce properties for problem (1.5) from the well known problem (1.7).

For problem (1.7), we know from [1, 9, 10, 19], that if $g,g'>0$ with g superlinear, then there exists a critical value $\lambda^*<\infty$ of the parameter λ such that if $\lambda>\lambda^*$ problem (1.7) does not have any kind of solutions while for $0<\lambda\leq\lambda^*$ it has solutions (unique or multiple solutions), see the $\|z\|$-diagrams ($\|\cdot\| = \sup_\Omega |(\cdot)|$) in Section 3. At the critical value of the parameter $\lambda = \lambda^*$, in the "closed spectrum" case $(0, \lambda^*]$, there exists at least one solution z^* while in the "open spectrum" case $(0, \lambda^*)$, there is no classical solution and there exists only a weak solution, ($\|z(\cdot;\lambda)\| \to \infty$, $z(x;\lambda) \to z^*(x)-$ as $\lambda \to \lambda^*-$ with $z^*(x) = \lambda^* \int_\Omega G(x,y)g(z(y))dy$ where G is the Green's function for $-\Delta$ with appropriate boundary conditions, see [12]).

The response $\|w\|$-diagram near λ^* is equivalent to the $\|z\|$-diagram. This is because $z = K(w)$, $(w = K^{-1}(z))$ and K satisfies (1.2). In what follows, the steady-state problem (1.5) will be studied extensively.

The main purpose, in this work, is to prove that for $\lambda > \lambda^*$ the solution to problem (1.1) becomes infinite in finite time (blows up) for any $u_0(x) \geq 0$. Here, we work on the case that λ^* lies in the spectrum of the stationary problem, similar to the semilinear heat equation, as in [11], also called spectral method; alternatively to [11], one may use concavity arguments, as in [3], or energy methods as in [4]. In addition, it can also be proved that blow-up of solutions occur for sufficiently large initial data and for $\lambda \in (0, \lambda^*)$, [11] and [15, p. 183].

Finally, this work is organized as follows. In Section 2 we briefly discuss the local existence and uniqueness of the time-dependent problem. In Section 3, we examine the steady-state problem, which is the key to our analysis and introduce the corresponding linearized problem (an auxiliary problem that helps us to prove our result). In Section 4, we prove blow-up for large λ by using Kaplan's method; moreover the main result is the blow-up of solutions for $\lambda > \lambda^*$ and for any $u_0 \geq 0$. We end with the Discussion in Section 5.

2. Existence, uniqueness of the time-dependent problem

In this section we study the existence and uniqueness of solutions of problem (1.1) (time-dependent) using (direct) comparison methods.

We give the proof in detail, since, as far as we are aware, it does not appear in the literature. For problem (1.1) the maximum principle holds. Therefore, in order to prove existence and uniqueness we can use comparison techniques (see [2, 14, 19]). Actually, we introduce a system of two iteration schemes which satisfy the problem and on using a proper system of solutions, we get two monotone sequences of solutions. Then, we introduce a weak form of the problem and use the monotone convergence theorem as well as some regularity arguments, we derive that the limits $\underline{u}, \overline{u}$ ($\underline{u} \leq \overline{u}$), of the two mentioned sequences, are classical solutions to the problem. Finally, on using Lipschitz continuity and the maximum principle, we prove that $\underline{u} \geq \overline{u}$. This shows that the system coincides with problem (1.1) and gives local existence and uniqueness.

We begin by proving the existence, therefore we define upper and lower solutions to problem (1.1). Let z, v be such that $z = z(x,t)$, $v = v(x,t) \in C^{2+\alpha,1+\alpha/2}(\Omega_T;\mathbb{R}) \cap C^{\alpha,0}(\overline{\Omega}_T;\mathbb{R})$, $0 < \alpha < 1$, $\Omega_T = \Omega \times (0,T)$. Then z, v are called lower, upper solutions respectively to problem (1.1), if they satisfy,

$$\left.\begin{aligned} S(z) \leq S(u) = 0 \leq S(v), \quad & x \in \Omega,\ 0 < t < T, \\ \mathcal{B}(K(z)) \leq \mathcal{B}(K(u)) = 0 \leq \mathcal{B}(K(v)), \quad & x \in \partial\Omega,\ 0 < t < T, \\ 0 \leq z(x,0) = z_0(x) \leq u_0(x) \leq v(x,0) = v_0(x), \quad & x \in \Omega, \end{aligned}\right\} \tag{2.1}$$

where $S(z) \equiv z_t - \Delta K(z) - \lambda f(z)$. If all the above inequalities are strict, then z, v are called strict lower, upper solutions respectively to (1.1).

Moreover, we can prove that if the inequalities of problem (2.1) are strict, then $z < v$, see [2, 18]. Therefore, we have the following lemma.

Lemma 2.1. *Let z, v be a lower, upper solutions to problem* (1.1)*, then $z \leq u \leq v$, where u is a solution to* (1.1).

Proof. We shall give the proof in two steps:

First step: Let z, v be strict lower, upper solutions to (1.1). We shall prove that $z < u < v$. We give firstly the proof for u, v. For this case, problem (2.1) holds, by substituting ($<$) at the places for ($\leq$). We define $d(x,t) = v(x,t) - u(x,t)$, (see also [2, p. 88], or [18, p. 511, Prop. 52.7]). We assume that the conclusion ($d = v - u > 0$) is false; then, there exists a first time $\bar{t} > 0$ such that $d(\overline{x}, \bar{t}) = 0$ for some $\overline{x} \in \overline{\Omega}$. Assuming now that $\overline{x} \in \partial\Omega$ then $\mathcal{B}(K(v)) = \mathcal{B}(K(u)) = 0$ at $\overline{x}$, (we have used Hopf's boundary lemma and that Ω has the interior sphere property), which contradicts the fact that $\mathcal{B}(K(v)) > 0$. Therefore, we are free to assume that $x \in \Omega$. We also have that $d(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0, \bar{t})$ and $d_t(\overline{x}, \bar{t}) \leq 0$. Moreover, $d(\overline{x}, \bar{t})$ attains its minimum at $x = \overline{x}$, so $\nabla d(\overline{x}, \bar{t}) = \nabla v(\overline{x}, \bar{t}) - \nabla u(\overline{x}, \bar{t}) = 0$ and $\Delta d(\overline{x}, \bar{t}) \geq 0$. Thus at $(\overline{x}, \bar{t})$, $u(\overline{x}, \bar{t}) = v(\overline{x}, \bar{t})$ and

$$
\begin{aligned}
0 \geq d_t(\overline{x}, \bar{t}) &= v_t(\overline{x}, \bar{t}) - u_t(\overline{x}, \bar{t}) \\
&> \Delta K(v(\overline{x}, \bar{t})) - \Delta K(u(\overline{x}, \bar{t})) + \lambda[f(v(\overline{x}, \bar{t})) - f(u(\overline{x}, \bar{t}))] = K'(v)\Delta v \\
&\quad + K''(v)|\nabla v|^2 - K'(u)\Delta u - K''(u)|\nabla u|^2 + \lambda[f(v(\overline{x}, \bar{t})) - f(u(\overline{x}, \bar{t}))] \\
&= K'(u)\Delta d(\overline{x}, \bar{t}) + K''(v)(|\nabla v|^2 - |\nabla u|^2) + \lambda[f(v(\overline{x}, \bar{t})) - f(u(\overline{x}, \bar{t}))] \\
&\geq \lambda[f(v(\overline{x}, \bar{t})) - f(u(\overline{x}, \bar{t}))] = 0.
\end{aligned}
$$

The term with the Laplacian is non-negative, the term with $|\nabla(\cdot)|^2$ is equal to zero and the difference of the nonlinear terms equals zero. Thus, $0 \geq d_t(\overline{x}, \bar{t}) > 0$, which is a contradiction, therefore $v > u$. Similarly, we can prove that $u > z$.

Second step: Let now z, v be a lower, upper solution to (1.1) respectively, we prove that $z \leq u \leq v$. Due to its regularity, f is also Lipschitz continuous, in fact what we need is f to be one-sided Lipschitz continuous in $[z, v]$, that is $f(a+b) - f(b) \leq La$, where L is a positive constant and $0 < a < R$ for some R. We set $v^\varepsilon = v + \varepsilon e^{\sigma t} > v$ for some $\varepsilon, \sigma > 0$ (similarly for z^ε), actually we use $0 < \varepsilon \ll 1$, $\varepsilon e^{\sigma t} < \varepsilon e^{\sigma T} = R$, for some σ and L constants, then (see also [2]):

$$
\begin{aligned}
S(v^\varepsilon) = v_t^\varepsilon - \Delta K(v^\varepsilon) - \lambda f(v^\varepsilon) &= v_t + \varepsilon\sigma e^{\sigma t} - K''(v^\varepsilon)|\nabla v|^2 - K'(v^\varepsilon)\Delta v \\
&- \lambda f(v^\varepsilon) \geq v_t - \Delta K(v) - \lambda f(v) + \varepsilon\sigma e^{\sigma t} - \lambda L\varepsilon e^{\sigma t} \\
&+ (K''(v) - K''(v^\varepsilon))|\nabla(v)|^2 + (K'(v) - K'(v^\varepsilon))\Delta v = S(v) \\
&+ \varepsilon e^{\sigma t}[\sigma - \lambda L - K'''(v)|\nabla v|^2 - K''(v)\Delta v] + O(\varepsilon^2) > S(v) \geq S(u).
\end{aligned}
$$

The last inequality holds since we can take σ large enough, so that the quantity inside the brackets to become non-negative, while R and L are constants; the function v is bounded in $C^{2,1}(\overline{\Omega}_T)$. Then, from the first step $v > u$ and $v^\varepsilon > u$ for $\varepsilon \ll 1$. Now $v^\varepsilon = v + \varepsilon e^{\sigma t} > u$ for every $0 < \varepsilon \ll 1$ and on taking $\varepsilon \to 0$ we get $v \geq u$.

Similar inequality can be proved for the other pair z, u. This completes the proof. $\square$

Next we show that such z, v exist.

Example 2.2. Such z, v exist; $z = Z = 0$ is a lower solution while as an upper solution we choose $v(x) = \widehat{w}(x) = w(x; \widehat{\lambda})$ a steady state (at least for some $\lambda < \lambda^*$), with $\widehat{\lambda} = \lambda + \varepsilon < \lambda^*, \varepsilon > 0$. For $\lambda \geq \lambda^*$ (and also for any $\lambda > 0$), as an upper solution we get $u = V(t)$ satisfying $T - \lambda t = \int_{V(t)}^{\infty} ds/f(s)$, for some T such that $T = \int_{V(0)}^{\infty} ds/f(s) < \int_{u_0}^{\infty} ds/f(s) < \infty$.

Now we define z, v to be lower, upper solutions and an iteration scheme, which starts from $\underline{u}_0(x,t) = Z = 0$, $\overline{u}_0(x,t) = V(t)$, of the form:

$$\underline{I}_n \equiv I(\underline{u}_n) := \underline{u}_{nt} - \Delta(K(\underline{u}_n)) - \lambda f(\underline{u}_{n-1}) = 0, \quad x \in \Omega,\ t > 0, \tag{2.2}$$

$$\overline{I}_n \equiv I(\overline{u}_n) := \overline{u}_{nt} - \Delta(K(\overline{u}_n)) - \lambda f(\overline{u}_{n-1}) = 0, \quad x \in \Omega,\ t > 0, \tag{2.3}$$

$$\mathcal{B}(K(\underline{u}_n)) = \mathcal{B}(K(\overline{u}_n)) = 0, \quad x \in \partial\Omega,\ t > 0 \tag{2.4}$$

$$\underline{u}_n(x,0) = \overline{u}_n(x,0) = u_0(x), \quad x \in \Omega, \tag{2.5}$$

for $n = 1, 2, \dots$.

Next we prove that if we have such an iteration scheme, we can construct two monotone sequences which will converge to the solution of (1.1).

Proposition 2.3. *Let z, v be lower, upper solutions respectively of* (1.1) *and $\underline{u}_n, \overline{u}_n$ satisfy* (2.2), (2.4), (2.5) *and* (2.3), (2.4), (2.5) *respectively, for $n = 1, 2, \dots$ with $\underline{u}_0 = z = Z$ and $\overline{u}_0 = v = V$. Then*

$$\underline{u}_0 < \underline{u}_1 < \dots < \underline{u}_{n-1} < \underline{u}_n < \dots < \overline{u}_n < \overline{u}_{n-1} < \dots < \overline{u}_1 < \overline{u}_0.$$

Proof. We prove this by using induction. First we show that $\underline{u}_{n-1} < \underline{u}_n$ and $\overline{u}_n < \overline{u}_{n-1}$. Thus we have,

$$\underline{I}_1 = \underline{I}(\underline{u}_1) = \underline{u}_{1t} - \Delta K(\underline{u}_1) - \lambda f(\underline{u}_0) = 0 \geq \underline{u}_{0t} - \Delta K(\underline{u}_0) - \lambda f(\underline{u}_0),$$

and on using the maximum principle or Lemma 2.1, for the filtration operator $T(u) = u_t - \Delta K(u)$, we get $\underline{u}_1 \geq \underline{u}_0$. Similarly, $\overline{u}_1 \leq \overline{u}_0$. For the *nth*-step we have:

$$\begin{aligned} \underline{I}_n &= \underline{u}_{nt} - \Delta K(\underline{u}_n) - \lambda f(\underline{u}_{n-1}) = 0 \\ &= \underline{u}_{(n-1)t} - \Delta K(\underline{u}_{n-1}) - \lambda f(\underline{u}_{n-2}). \end{aligned}$$

The above relation gives:

$$[\underline{u}_{nt} - \Delta K(\underline{u}_n)] - [\underline{u}_{(n-1)t} - \Delta K(\underline{u}_{n-1})] = \lambda f(\underline{u}_{n-1}) - \lambda f(\underline{u}_{n-2}) \geq 0,$$

since $\underline{u}_{n-1} \geq \underline{u}_{n-2}$ and $\overline{u}_{n-1} \leq \overline{u}_{n-2}$. Again from maximum principle or Lemma 2.1, we get $\underline{u}_n > \underline{u}_{n-1}$ and $\overline{u}_n < \overline{u}_{n-1}$, $n = 1, 2, \dots$.

Now we prove $\underline{u}_n < \overline{u}_n$, again by induction:

$$\begin{aligned} \underline{I}_n &= \underline{u}_{nt} - \Delta K(\underline{u}_n) - \lambda f(\underline{u}_{n-1}) = 0, \\ \overline{I}_n &= \overline{u}_{nt} - \Delta K(\overline{u}_n) - \lambda f(\overline{u}_{n-1}) = 0, \end{aligned}$$

thus,

$$[\underline{u}_{nt} - \Delta K(\underline{u}_n)] - [\overline{u}_{nt} - \Delta K(\overline{u}_n)] = \lambda f(\underline{u}_{n-1}) - \lambda f(\overline{u}_{n-1}) \leq 0,$$

since $\underline{u}_{n-1} \leq \overline{u}_{n-1}$, which holds for $n = 1, 2, \dots$ (for $n = 1$ see Lemma 2.1). This completes the proof. $\square$

Corollary 2.4. *For the iteration schemes of problems* (2.2)-(2.5) *we have: $\underline{u}_n \nearrow \underline{u}$, $\overline{u}_n \searrow \overline{u}$ pointwise as $n \to \infty$ and hence $\underline{u} \leq \overline{u}$.*

Proof. This is a consequence of the monotonicity of Proposition 2.3 and that the pair (Z,V) is bounded. □

Next we prove that these solutions are indeed classical:

Proposition 2.5. *Functions $\underline{u},\overline{u}$ are classical solutions to the problem*

$$\left.\begin{aligned} S(\underline{u}) = S(\overline{u}) = 0, \quad x\in\Omega,\ t>0,\\ \mathcal{B}(K(\underline{u})) = \mathcal{B}(K(\overline{u})) = 0, \quad x\in\partial\Omega,\ t>0,\\ \underline{u}_0(x,0) = \overline{u}_0(x,0) = u_0(x), \quad x\in\Omega, \end{aligned}\right\} \tag{2.6}$$

with $\underline{u},\overline{u}\in C^{2,1}(\Omega_T)$.

Proof. We shall write down (2.2), (2.4), (2.5) and (2.3), (2.4), (2.5) in a weak form (very weak solutions), see also [13, 22]. More precisely:

$$\begin{aligned} &N(z(x,t))\\ &\equiv \int_\Omega [z(y,s)\eta(y,s)]_0^t dy - \int_0^t\int_\Omega z(y,s)\eta_t(y,s)\,dy\,ds - \int_0^t\int_\Omega K(z)\Delta\eta\ dy\,ds\\ &= \lambda\int_0^t\int_\Omega f(z)\eta\ \ dy\,ds, \end{aligned} \tag{2.7}$$

where $K(z)\in \dot V_2(\Omega_T)\equiv L^\infty((0,T);L^2(\Omega_T))\cap L^2((0,T);L^2_{loc}(\Omega))$ and z, $f(z)\in L^2(\Omega_T)$. The test function $\eta\in W_c^{2,1}(\Omega_T)$, ($\eta$ can also be taken to belong to $C_c^\infty(\Omega_T)$, with compact support), $\eta=\eta(x,t)\geq 0$ with $\Delta\eta<0$, ([13, p. 419], [15]).

The weak version of problem (2.2), (2.4), (2.5) can be written as

$$N(\underline{u}_n) = \lambda\int_0^t\int_\Omega f(\underline{u}_{n-1})\eta\,dy\,ds.$$

Now, passing to the limit as $n\to\infty$, using the monotonicity of $\underline{u}_n,\overline{u}_n$, the monotone convergence theorem (due to the boundedness of z,v, we may also use Lebesgue's dominated convergence theorem) and the fact that $\tau<T$, with T as in Example 2.2 (we only need that $\overline{u}_n$ is uniformly bounded) we get

$$N(\underline{u}) = \lambda\int_0^t\int_\Omega f(\underline{u})\,\eta\,dy\,ds, \quad \text{and similarly,} \quad N(\overline{u}) = \lambda\int_0^t\int_\Omega f(\overline{u})\,\eta\,dy\,ds.$$

Equivalently, in the distributional sense, we have:

$$S(\overline{u}) = S(\underline{u}) = 0, \quad \text{in } \mathcal{D}'(\Omega_T). \tag{2.8}$$

Regularity: In fact, the solutions found above are classical. By using standard regularity theory, (see [13, p. 419]), we see that any bounded (very) weak solution belongs to $C^{\alpha,\alpha/2}(\Omega_T)$ for some $0<\alpha\leq 1$ (Sobolev Embedding Lemma). By bounded (very) weak solutions $\underline{u},\overline{u}$, we mean functions which satisfy (2.8) and $\|\underline{u}\|_\infty,\ \|\overline{u}\|_\infty<\infty$ in Ω_T. Now, by bootstrapping arguments and Schauder type estimates, we get that $\underline{u},\overline{u}\in C^{2+\alpha,1+\alpha/2}(\Omega_T)$. Finally, $\underline{u},\overline{u}\in C^{2,1}(\Omega_T)$. This completes the proof. □

Up to this point, we have proved that $\underline{u}\leq\overline{u}$, next we show that $\underline{u}=\overline{u}$.

Lemma 2.6. *Let f be one-sided Lipschitz continuous: $f(a+b)-f(b)\leq La$ where L positive constant, $0<a<R$ for some R and $\underline{u}$, $\overline{u}\in C^{2,1}(\Omega_T)$. Then $\underline{u}\geq\overline{u}$.*

Proof. Let $\underline{u}^\varepsilon = \underline{u} + \varepsilon e^{\sigma t} > \underline{u}$ for some $\varepsilon > 0$ (similarly for $\overline{u}^\varepsilon$), moreover $0 < \varepsilon \ll 1$, $\varepsilon e^{\sigma t} < \varepsilon e^{\sigma T} = R$ and L constants, then as in Lemma 2.1, we obtain

$$\begin{aligned} S(\underline{u}^\varepsilon) &= \underline{u}^\varepsilon_t - \Delta K(\underline{u}^\varepsilon) - \lambda f(\underline{u}^\varepsilon) \\ &\geq S(\underline{u}) + \varepsilon e^{\sigma t}[\sigma - \lambda L - K'''(\underline{u})|\nabla \underline{u}|^2 - K''(\underline{u})\Delta \underline{u}] + O(\varepsilon^2) \\ &> S(\underline{u}) = S(\overline{u}). \end{aligned}$$

The last inequality holds since σ is taken to be large enough, $\underline{u}$ is bounded in $C^{2,1}(\overline{\Omega}_T)$, $\underline{u} \leq \overline{u}$ and $\underline{u}^\varepsilon \leq \overline{u}$ for $\varepsilon \ll 1$. On using now Lemma 2.1, we get $\underline{u}^\varepsilon = \underline{u} + \varepsilon e^{\sigma t} > \overline{u}$ for any $0 < \varepsilon \ll 1$ and on taking $\varepsilon \to 0$ we derive now that $\underline{u} \geq \overline{u}$. □

Finally, we have the next theorem which gives a result concerning the local existence and uniqueness.

Theorem 2.7. *Problem* (1.1) *has a unique classical solution u with $C^{2,1}(\Omega_T)$ for some $T > 0$.*

Proof. This proof is a consequence of the previous Lemmas and Propositions. For uniqueness see Corollary 2.4 and Lemma 2.6. □

Ending this section, we mention a couple of other works, such as [15, 17] that are related to the local existence and uniqueness of solutions of type (1.1). More precisely, the first work, Levine and Sacks [15], proves that the solution to (1.1) is actually global in time under some extra assumption on f. The second work, Pao [17], concerns the porous medium problem. In this work, the maximum principle is used and a proper iteration scheme of a pair of solutions is constructed, giving local existence and uniqueness.

3. The steady-state and the linearized problem

3.1. **The steady-state problem.** We recall (1.5) that the corresponding steady-state problem of (1.1) is

$$\Delta(K(w(x))) + \lambda f(w(x)) = 0, \quad x \in \Omega, \quad \mathcal{B}(K(w(x))) = 0, \quad x \in \partial\Omega, \tag{3.1}$$

(problem (1.5) and (3.1) are exactly the same). We say that $w = w(x) > 0$ is a classical solution of (3.1), if $z = z(x) = K(w(x))$ is a classical solution ($z \in C^2(\Omega) \cap C^1(\bar{\Omega})$) of

$$\Delta z + \lambda g(z) = 0, \quad x \in \Omega, \quad \mathcal{B}(z) = 0, \quad x \in \partial\Omega, \tag{3.2}$$

where $g(z) = f(K^{-1}(z)) = f(w)$, $z = K(w)$, (again problem (1.7) and (3.2) are exactly the same). Condition (1.2) and especially the monotonicity property of K suggest that both the above steady-state problems are equivalent with respect to the existence and to the multiplicity of solutions (at least close to the supremum of λ, say λ^*, where for $\lambda < \lambda^*$, problem (3.1) has a classical solution, see Figure 1(b)). The equivalence of both problems means that problem (3.1) has a classical solution if and only if problem (3.2) has a classical solution.

On using the pressure transformation (1.6), $z = K(w)$ or $\sigma = K(s)$, we can get many qualitative results, but it constrains the function f through the conditions on $g(\sigma)$; that is, $g(\sigma) = f(s) = f(K^{-1}(\sigma))$ is a convex function with respect to

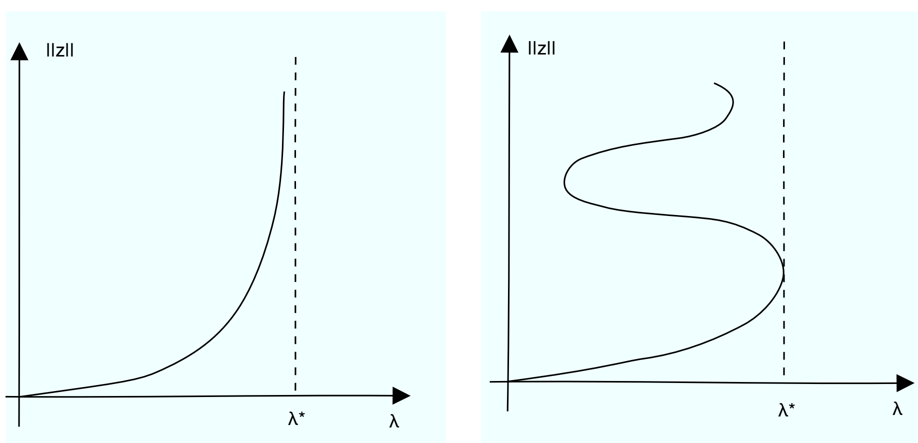
σ; more precisely it is an increasing and convex function. This implies that the following conditions on f and K should hold:

$$g''(\sigma) = \frac{f''(s)K'(s) - f'(s)K''(s)}{(K'(s))^3} = \frac{1}{K'(s)}\Big(\frac{f'(s)}{K'(s)}\Big)' > 0, \quad s \in \mathbb{R}. \tag{3.3}$$

More completely, the functions g, f and K satisfy

$$g(\sigma) = f(s) > 0, \quad g'(\sigma) = \frac{f'(s)}{K'(s)} > 0, \quad g''(\sigma) > 0, \quad \sigma > 0. \tag{3.4}$$

For problem (3.2), we know that if $g, g', g'' > 0$, (g convex) and (1.4) holds, then there exists a critical value $\lambda^* < \infty$ such that if $\lambda > \lambda^*$ problem (3.2) does not have a solution (of any kind) while for $0 < \lambda < \lambda^*$ has at least one, ($\|z\|$-diagrams, Figures 1 (a) and (b)) see [1, 9, 10, 19]. Also a similar diagram holds for $\|w\| = \|K^{-1}(z)\|$. Note especially in Figure 1(b) for λ in the interval $(\lambda^* - \epsilon,\ \lambda^*)$, for some $0 < \epsilon \ll 1$, there exist (at least) two classical solutions. Actually, we are interested in the solutions near $\lambda^* < \infty$ and that $(\lambda^*, \|w^*\|)$ is a bending point of the response diagram of the steady-state problem.



(a) Open spectrum (b) Closed spectrum

FIGURE 1.

Therefore, we also suppose that for any $\lambda \in (\lambda^* - \epsilon, \lambda^*)$, $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ such that the following estimate holds:

$$\|w(x;\lambda\| \le C \quad \text{or equivalently} \quad \|z(x;\lambda)\| = \|K(w(x;\lambda) \le \| \le K(C). \tag{3.5}$$

We note that in the closed spectrum case, at the critical value $\lambda = \lambda^*$ there exists a unique classical solution $z^* = K(w^*)$, while in the open spectrum case a classical solution does not exist but there exists only a weak singular one (see [12, 14]).

Now, if we want to relax f and K from condition (3.3)), we suppose either $N = 1$ or $N \ge 2$ and replace (3.3) by the following condition, see [1],

$$\liminf_{\sigma\to\infty} \frac{g(\sigma)}{\sigma} > c > 0, \quad g(\sigma) \le a + b\,\sigma^{\nu}. \tag{3.6}$$

Relation (3.6) for the functions $g(\sigma) = f(s)$ and $\sigma = K(s)$ gives

$$g(\sigma) = f(s) \le a + b\,K^{\nu}(s), \quad \sigma = K(s), \quad \sigma > 0,\ s \in \mathbb{R}, \tag{3.7}$$

where $\nu < N/(N-1-\delta)$ and $\delta = 0$ for the Dirichlet problem, while $\delta = 1$ for the Neumann and Robin problems, [1, p. 688].

Remarks. (a) The significance of (3.6) is that it controls the growth of g and ensures the existence of the closed spectrum $(0, \lambda^*]$ and that the solution $w^*(x;\lambda^*)$ to (3.1) is classical.

(b) Condition (3.6) or (3.7) does not contradict the conditions of superlinearity; i.e.,

$$\liminf_{\sigma\to\infty} \frac{g(\sigma)}{\sigma} = \liminf_{s\to\infty} \frac{f(s)}{K(s)} > c > 0, \quad \text{or} \quad \liminf_{s\to\infty} \frac{f(s)}{s\,K'(s)} > c > 0, \tag{3.8}$$

for some $c > 0$, which is a consequence of the condition (integrability at ∞):

$$\int_{A=K^{-1}(a)}^{\infty} \frac{ds}{f(s)} < \int_a^{\infty} \frac{d\sigma}{g(\sigma)} = \int_{A=K^{-1}(a)}^{\infty} \frac{dK(s)}{f(s)} = \int_A^{\infty} \frac{K'(s)}{f(s)} ds < \infty, \tag{3.9}$$

for some a or $A \geq 0$. Let us now assume that (3.7) holds, taking into account the positivity, monotonicity of $g(\sigma)$, $f(s)$ and $K(s)$, then on using similar methods as in [10], and a proper successive approximation scheme of the form, $K(w_n(x)) = \int_\Omega G(x,y) f(w_{n-1}(y))dy$, (where $G > 0$ is the Green's function for $-\Delta$ with appropriate boundary conditions and Dini's theorem) we can get the following existence result for $w(x)$:

$$K(w(x)) = \lambda \int_\Omega G(x,y) f(w(y))dy. \tag{3.10}$$

Equation (3.10) is equivalent, provided that (3.5) holds, to the existence of the classical steady-state solution of problem (3.1). The existence of a bounded λ; i.e., $\lambda < \infty$ is given by [9], which is a consequence of superlinearity condition (3.8) that is obtained by (3.9). On taking condition (3.6) or (3.7), we get the closed spectrum diagram for problem (3.1) as in Figure 1(b), by replacing $\|z\|$ with $\|w\|$.

In what follows we consider the closed spectrum case, see Figure 1(b); that is, there exists a unique classical solution $z^* = K(w^*)$ or equivalently $w^* = K^{-1}(z^*)$, at $\lambda = \lambda^*$, for both problems (3.1) and (3.2), and at least one solution (actually two solutions) at each $\lambda \in (\lambda^* - \epsilon,\ \lambda^*)$, $0 < \epsilon \ll 1$. In other words the response diagram (bifurcation) is bending at λ^* and $(\lambda^*, \|w^*\|)$ is the turning point of the response diagram, see Figure 1(b).

3.2. **Linearized problem.** Now we introduce the corresponding linearized problem of (3.1) (or of (1.5)) setting (stability by using perturbations):

$$\begin{gathered} u(x,t) = w(x) + u_1(x,t)\varepsilon + u_2(x,t)\varepsilon^2 + \dots, \quad \text{or} \\ u(x,t) = w(x) + \phi(x)e^{\mu t}\varepsilon + \dots, \quad \text{for } 0 < \varepsilon \ll 1. \end{gathered}$$

Next we substitute in equation (1.1):

$$\begin{aligned} &\varepsilon\phi e^{\mu t}\mu + \dots \\ &= \Delta K(u) + \lambda f(u) = \Delta[K(u) - K(w)] + \Delta K(w) + \lambda f(u) \\ &= \Delta[K'(w)\phi e^{\mu t}\varepsilon + \frac{K''(w)}{2}(u_1\varepsilon + \dots)^2 + \dots] + \lambda(f(u) - f(w)) \\ &= \Delta[K'(w)\phi e^{\mu t}\varepsilon + \frac{K''(w)}{2}(u_1\varepsilon + \dots)^2 + \dots] + \lambda f'(w)\varepsilon\phi e^{\mu t} + \dots \\ &= \varepsilon e^{\mu t}\Delta(K'(w)\phi) + \lambda f'(w)\varepsilon\phi e^{\mu t} + O(\varepsilon^2), \quad x \in \Omega, \quad t > 0, \end{aligned}$$

$$\begin{gathered}\mathcal{B}(K(u)) = \mathcal{B}(K(w)) + e^{\mu t}\mathcal{B}(K'(w)\phi)\varepsilon + O(\varepsilon^2) = 0, \quad x \in \partial\Omega, \quad t > 0,\\ u_0 = u_0(x) = w(x) + \phi(x)\varepsilon + O(\varepsilon^2), \ \ x \in \Omega.\end{gathered}$$

The above procedure is an asymptotic expansion provided that $u_i(x,t)$, $i = 1, 2, \dots$ are uniformly bounded and of order one ($O(1)$), thus on taking terms of the same order with respect to ε, we obtain (3.1) and the problem

$$\begin{gathered}\varepsilon : \quad \phi e^{\mu t}\mu = [\Delta K'(w)\phi]e^{\mu t} + \lambda f'(w)\phi e^{\mu t}, \quad x \in \Omega,\\ \varepsilon : \quad \mathcal{B}(K'(w)\phi) = 0, \quad x \in \partial\Omega.\end{gathered}$$

Thus we get the linearized problem of (3.1) (or of (1.5)):

$$\Delta[K'(w)\phi] + \lambda f'(w)\phi = \mu\phi, \quad x \in \Omega, \quad \mathcal{B}(K'(w)\phi) = 0, \quad x \in \partial\Omega. \tag{3.11}$$

The above problem has a solution for each $\lambda \in (0, \lambda^*]$. This follows from a suitable iteration scheme of the integral representations, [10]:

$$K'(w_n(x))\phi_n(x) = \int_\Omega (\lambda f'(w_{n-1}(y)) - \mu)\phi_{n-1}(y)G(x,y)dy,$$

provided that $f'(0) > \mu/\lambda$. Actually, $w_n \to w > 0$ (due to problem (1.5)), $\phi_n \to \phi > 0$ in Ω, uniformly as $n \to \infty$ and $G > 0$ is the Green's function with appropriate boundary conditions, ($\mathcal{B}(G) = 0$).
Another way of getting such results is by using variational methods, [5, 10], or functional analysis techniques [1]; one can get directly that, if $\lambda < \lambda^*$ then $\phi > 0$. Also, it can be obtained that the response diagram for problem (3.11) is as it appears in Figure 2; as a result the sign and zeros of μ become known. Alternatively, to get the sign and zeros of μ; i.e., Figure 2, we can also use the corresponding linearized problem of (3.2); that is:

$$\Delta\widehat{\phi} + \lambda g'(z)\widehat{\phi} = \widehat{\mu}\widehat{\phi}, \quad x \in \Omega, \quad \mathcal{B}(\widehat{\phi}) = 0, \quad x \in \partial\Omega. \tag{3.12}$$

Thus, it is known that if $\lambda < \lambda^*$ then $\widehat{\phi} > 0$, [1, 5, 10]. Moreover, the response diagram for problem (3.12) is as it appears in Figure 2 (by replacing w with z and μ with $\widehat{\mu}$; the principal eigenpair $(\widehat{\mu}, \widehat{\phi})$ has $\widehat{\phi} > 0$ and the sign of $\widehat{\mu}$ is as in Figure 2). On replacing $z = K(w)$ and $g(z) = f(w)$, then problem (3.2) becomes problem (3.1). Now, on multiplying problem (3.12) by $K'(w)\phi > 0$, problem (3.11) by $\widehat{\phi} > 0$, subtracting these two problems and using Green's identity we obtain

$$\mu = \widehat{\mu}\frac{\int_\Omega \phi\,\widehat{\phi}\,K'(w)\,dx}{\int_\Omega \phi\,\widehat{\phi}\,dx}. \tag{3.13}$$

Hence, the sign and zeros of μ coincide with those of $\widehat{\mu}$, which implies the validity of Figure 2.

In what follows we consider the closed spectrum case, see Figure 1(b), thus we have the following theorem.

Theorem 3.1. *Let f, K satisfy* (1.2)-(1.4)*, and either (i) $N = 1$, or (ii) $N \geq 2$ but now either* (3.3) *and* (3.5) *or* (3.7) *hold, then problem* (3.1)*, has at least one unique classical solution w^* at $\lambda = \lambda^*$ and at least two solutions at each $\lambda \in (\lambda^* - \epsilon,\ \lambda^*)$, $0 < \epsilon \ll 1$. In other words, the response diagram (bifurcation) has at least one turning (bending) point at $(\lambda^*, \|w^*\|)$, see Figure 1(b). Moreover, for every $\lambda \in (0, \lambda^*)$, the minimal positive solution $\underline{w}$ is asymptotically stable and the first eigenvalue $\mu(\lambda)$ of* (3.11) *is negative, while the next bigger classical solution w*

is unstable and the first eigenvalue $\mu(\lambda)$ *of* (3.11) *is positive. The branch* $(\lambda, \|w\|)$ *near* $(\lambda^*, \|w^*\|)$ *forms a continuously differentiable curve and the first eigenvalue of* (3.11) *at* $\lambda = \lambda^*$ *is* $\mu^* = \mu(\lambda^*) = 0$.

Proof. The proof is described briefly below; we use the pressure transformation, the results of problem (3.2) and the papers [1, 5, 9, 10]. Precisely, we can get the response diagram of Figure 2, in which we also note, on using arrows, the stability of steady-state solutions.

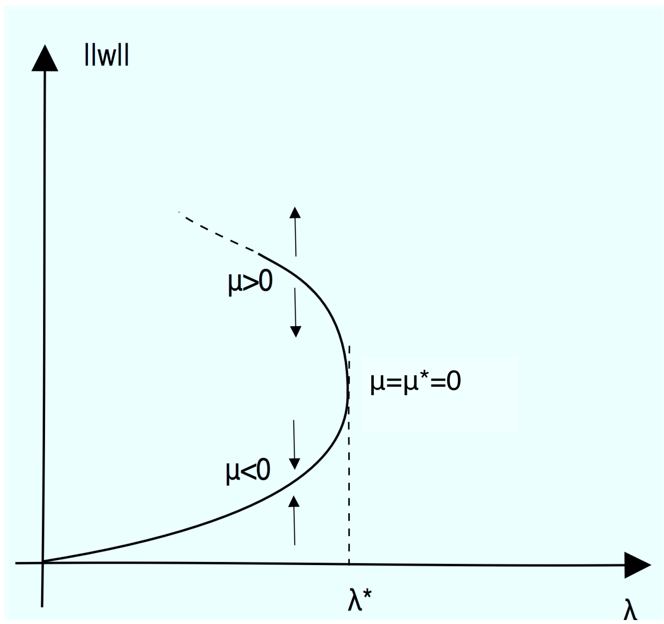


FIGURE 2. Response diagram. Linearized stability.

Actually, at the lower branch we have $\mu < 0$ (the minimal solution $\underline{w}$ is stable, linearized stability) at the upper branch $\mu > 0$ (the maximal solution $\overline{w}$ is unstable). From the continuity of the spectrum [5], for $\lambda > 0$ or at least in a left region of λ^*, we get that $\mu^* = 0$ at $\lambda = \lambda^*$, therefore we have the linearized problem:

$$\left.\begin{aligned} \Delta(K'(w^*)\phi^*) + \lambda^* f'(w^*)\phi^* = 0, \quad w^* = w^*(x), \quad \phi^* = \phi^*(x), \quad x \in \Omega, \\ \mathcal{B}(K'(w^*)\phi^*) = 0, \quad x \in \partial\Omega. \end{aligned}\right\} \tag{3.14}$$

Now, we consider the response diagram at the interval $(\lambda^* - \epsilon, \lambda^*)$ with $0 < \epsilon \ll 1$. This diagram is continuous and concerns classical solutions (this has been proven for problem $\Delta z + \lambda g(z) = 0$, $\mathcal{B}(z) = 0$, with $g(z) = f(K^{-1}(z)) = f(w) > 0$, $g'(z) > 0$, $g''(z) > 0$, see [5]). We have to note that the response diagrams $(\lambda, \|z\|)$ and $(\lambda, \|w\|)$ are similar around the bending point $(\lambda^*, \|w^*\|)$ due to the monotonicity of $K(z)$.

The stability-instability can be obtained by using upper and lower solutions (Sattinger's type arguments [19] and successive approximations). More precisely these results can be obtained as follows. On choosing appropriate initial data $\widehat{u}_0$ and on using comparison methods, we prove that the lower branch $(\lambda, \|\underline{w}\|)$

is asymptotically stable while the upper branch $(\lambda, \|w\|)$ is unstable, Figure 2. Moreover for the linearized problem (3.11) we get that due to the continuity and the monotonicity on each branch of the response diagram we derive: $0 > \mu(\lambda) \nearrow \mu^*-$, when $\lambda \to \lambda^*-$ at the lower branch while $0 < \mu(\lambda) \searrow \mu^*+,\ \lambda \to \lambda^*-$ at the upper branch, therefore $\mu^* = 0$.

The proof of the stability is obtained by taking appropriate initial data $\widehat{u}_0(x) = w(x) + \varepsilon\phi(x)$, $w(x), \phi(x) > 0$, $0 < |\varepsilon| \ll 1$. Particularly,

$$
\begin{aligned}
&\Delta(K(\widehat{u}_0)) + \lambda f(\widehat{u}_0)\\
&= \Delta[K(w+\varepsilon\phi) - K(w)] + \Delta(K(w)) + \lambda f(w) + \lambda f'(w)\phi\,\varepsilon + \frac{\lambda}{2} f''(\xi)\phi^2\varepsilon^2\\
&= [\Delta(K'(w)\phi) + \lambda f'(w)\phi]\varepsilon + \frac{\varepsilon^2}{2}\Delta(K''(\xi)\phi^2) + \frac{\lambda}{2} f''(\xi)\phi^2\varepsilon^2\\
&= \mu\phi\,\varepsilon + O(\varepsilon^2) \equiv R,
\end{aligned}
$$

$$
\operatorname{sgn}(R) = \operatorname{sgn}(\mu\phi\varepsilon) = \operatorname{sgn}(\mu\varepsilon).
$$

From the previous relation we obtain (+), that is the $\operatorname{sgn}(R)$ or $\mu\,\varepsilon > 0$; therefore, $\widehat{u}_0$ is a lower solution to (3.1); (−); that is, the $\operatorname{sgn}(R)$ or $\mu\varepsilon < 0$, therefore $\widehat{u}_0$ is an upper solution to (3.1).

More precisely, on the upper branch at w, (the largest w), we get $\widehat{u}_0 = w + \varepsilon\phi$, $\mu > 0$, $\varepsilon > 0$ and $\widehat{u}_0$ is a lower solution of the steady-state problem; therefore, $u_t(x,t;\widehat{u}_0) = \widehat{u}_t > 0$. Finally, for $0 < \lambda \le \lambda^*$, $\widehat{u} = u(x,t;\widehat{u}_0)$ is increasing with respect to t and unbounded, otherwise, by standard arguments, $u \to \widehat{w}- > w$, which is in contradiction to w being the largest solution, (see also below). Thus, $\|u\| \to \infty$, $t \to T- \le \infty$ and w is unstable from above. Similarly $\mu > 0$, $\varepsilon < 0$ and again on the upper branch at w, then $\widehat{u}_0$ is an upper solution; therefore, $\widehat{u}$ is decreasing with respect to t, hence w is unstable from below. Indeed, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) : \|u_0 - w\| < \delta$ with $\|u(x,t;u_0) - w(x)\| > \varepsilon$ (we interpolate properly $\widehat{u}_0$: $u_0 > \widehat{u}_0 > w$, above w; $u_0 < \widehat{u}_0 < w$, below w).

Similarly, we work on the lower branch at w; say $\underline{w}$, the minimal solution to (1.5), $\mu < 0$, $\varepsilon < 0$, so $\widehat{u} = u(x,t;\widehat{u}_0)$ is increasing with respect to time t and $\underline{w}$ is stable from below. At $\underline{w}$, for $\mu < 0$, $\varepsilon > 0$, so $\widehat{u} = u(x,t;\widehat{u}_0)$ is decreasing with respect to time t and $\underline{w}$ is stable from above.

Thus, $\underline{w}$ is asymptotically stable for any $\lambda \in (\lambda^* - \epsilon, \lambda^*)$, as well as for any $\lambda \in (0, \lambda^*)$. Finally, we get that for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) : \|u_0 - \underline{w}\| < \delta$ with $\|u(x,t;u_0) - \underline{w}(x)\| < \varepsilon$ and $u(x,t;u_0) \to \underline{w}$ pointwise as $t \to \infty$; therefore, $\underline{w}$ is asymptotically stable (we choose appropriate $\widehat{u}_0$; that is below $\underline{w}$, $\underline{w} > u_0 > \widehat{u}_0$, or above $\underline{w}$, $\underline{w} < u_0 < \widehat{u}_0$, and take equicontinuous sequences with respect to t). □

For an alternative way of getting similar results, the sign of μ, see also [5, 10].

4. Blow-up

In this section we give some blow-up results. We recall that (1.2)-(1.4) hold and that $K(0) = 0$, $K'(0), K''(0) > 0$. Firstly we show the unboundness of u; on taking u_0 to be a lower solution to (3.1); i.e., $u_0 = 0$, then u is unbounded and $\|u(\cdot,t;u_0)\| \to \infty$ as $t \to T- \le \infty$ for any $\lambda > \lambda^*$. This is due to the fact that if u was uniformly bounded for $t > 0$ it would converge to w i.e. $u(x,t_n) \to w(x)$ as $t_n \to \infty$. Then, by standard parabolic type arguments (ω-limit set, etc.), [16], see also [17], w will be a stationary solution which is a contradiction for $\lambda > \lambda^*$. The same argument holds, for $0 < \lambda \le \lambda^*$ with $u_0 = w(x) + \varepsilon\phi(x)$, $\phi(x) > 0$, $0 < \varepsilon \ll 1$ where w is the largest stationary solution to (3.1), u is unbounded.

4.1. **Blow-up on using Kaplan's method for $\boldsymbol{\lambda \gg 1}$; $\mathbf{K}, \mathbf{f}$ satisfy** (4.2)**.** Another necessary condition for blow-up of solutions of (1.1) is (1.4)(b) and can be taken from the spatial version of the problem; i.e., $u(x,t) = v(t)$. Now we consider that $u(x,t)$ is uniform with respect to x, so $u(x,t) = v(t)$ and take the spatial derivatives zero. Thus we get the ordinary differential equation

$$\frac{dv}{dt} = \lambda f(v), \quad t > 0, \quad v(0) = \sup_{\Omega} u_0(x),$$

then, due to (1.4)(b), $\lambda t < \int_{v(0)}^{v(t)} ds/f(s) \le \int_{v(0)}^{\infty} ds/f(s) < \infty$. On the other hand, a sufficient blow-up condition of problem (1.1) can be obtained by using Kaplan's method. We set the function $\varphi = \varphi(x)$ to satisfy:

$$\Delta\varphi = -\nu_1\varphi, \quad x \in \Omega, \quad \mathcal{B}(\varphi) = 0, \quad x \in \partial\Omega, \tag{4.1}$$

with $\int_\Omega \varphi dx = 1$ and (ν_1, φ) the first eigenpair of (4.1), with $\nu_1, \varphi(x) > 0$.

At this point we can see the necessity of an extra comparison condition between $K(u)$ and $f(u)$, therefore we additionally assume:

$$\int_\Omega [K(u(x,t)) - f(u(x,t))]\varphi(x)dx \le 0, \quad t > 0, \tag{4.2}$$

where u is the solution to (1.1) and φ satisfies (4.1).

The difference with the next subsection is that here we have blow-up for λ large enough, namely for $\lambda > \nu_1 \ge \lambda^*$, (for $\nu_1 \ge \lambda^*$, see [5, 10],) and that (4.2) is satisfied.

Now we introduce the functional $A(t) = \int_\Omega u(x,t)\varphi(x)dx$, multiply equation (1.1) with a smooth function φ on Ω, integrate over Ω and obtain

$$\begin{aligned} A'(t) &= \int_\Omega u_t(x,t)\varphi(x)dx = \frac{d}{dt}\int_\Omega u(x,t)\varphi(x)dx \\ &= \int_\Omega \varphi\Delta K(u)dx + \lambda\int_\Omega \varphi f(u)dx. \end{aligned} \tag{4.3}$$

Applying Green's identity on (4.3) and using the auxiliary problem (4.1) we obtain

$$A'(t) = -\nu_1 \int_\Omega \varphi K(u)dx + \lambda \int_\Omega \varphi f(u)dx. \tag{4.4}$$

Then (4.2) and (4.4) give

$$A'(t) \ge |\Omega|(\lambda - \nu_1) \fint_\Omega f(u)\varphi dx, \quad \text{where } \fint_\Omega = (1/|\Omega|)\int_\Omega.$$

On using now Jensen's inequality, for $\lambda > \nu_1$, we derive:

$$A'(t) \ge (\lambda - \nu_1) f(A).$$

The above relation implies blow-up of A and hence of u ($A(t) \le C\|u(\cdot,t\|$) for $\lambda > \nu_1$ due to (1.4). Moreover, $\nu_1 \ge \lambda^*$, see [5, 10]; an alternative way to prove the latter is the use of (4.2) by substituting u for w. This is a consequence of the positivity and the monotonicity of K and f; indeed, taking $\widehat{u}_0 = 0$, then $\widehat{u}_0$ is a lower solution to (3.1) and $\widehat{u}_t = u_t(x,t;\widehat{u}_0) > 0$. If now $0 \le \widehat{u}_0 < u_0 < w$, then

$\widehat{u} < u < w$ and taking the limit as $t \to \infty$, we get that $u(x,t) \to w(x;\lambda)-$, $\lambda \le \lambda^*$, and

$$\int_\Omega [K(w(x)) - f(w(x))]\varphi(x)dx \le 0, \quad \varphi > 0. \tag{4.5}$$

Then, if we multiply (3.1) by φ, integrate, and use (4.5), for λ in the spectrum of (3.1), we obtain

$$0 = -\nu_1 \int_\Omega K(w)\varphi dx + \lambda \int_\Omega f(w)\varphi \ge (\lambda - \nu_1)\int_\Omega f(w)\varphi dx,$$

which implies $\nu_1 \ge \lambda$ and then $\nu_1 \ge \lambda^*$; for an alternative proof without the requirement (4.5) see [5, 10].

4.2. **Blow-up for $\lambda > \lambda^*$ and any non-negative initial data.** In this paragraph, we prove our main result, which is the blow-up of solutions of (1.1) when $\lambda > \lambda^*$ and for any $u_0 \ge 0$. We mainly follow the method that was first applied by Lacey in [11], also called the spectral method. Thus we have the following theorem.

Theorem 4.1. *Let f, K satisfy* (1.2)-(1.4) *and either* (3.3) *or* (3.7)*, then the solution to* (1.1) *blows up in finite time $t^* < \infty$ for any $\lambda > \lambda^*$ and any non-negative initial data.*

Proof. We consider problem (3.14); i.e., the linearized problem of (3.1), for $\lambda = \lambda^*$, with first eigenpair $(\mu^*, \phi^*) = (0, \phi^*)$, $\phi^* > 0$ in Ω, see Theorem 3.1:

$$\Delta(K'(w^*)\phi^*) + \lambda^* f'(w^*)\phi^* = 0, \quad x \in \Omega, \quad \mathcal{B}(K'(w^*)\phi^*) = 0, \quad x \in \partial\Omega. \tag{4.6}$$

We multiply problem (1.1) by $K'(w^*)\phi^*$, and integrate over Ω,

$$\begin{aligned}\int_\Omega u_t K'(w^*)\phi^* dx &= \int_\Omega K'(w^*)\phi^*(\Delta K(u))dx + \lambda \int_\Omega K'(w^*)f(u)\phi^* dx\\ &= \int_\Omega K'(w^*)\phi^*[\Delta(K(u) - K(w^*))\\ &\quad + \Delta K(w^*)]dx + \lambda \int_\Omega K'(w^*)f(u)\phi^* dx;\end{aligned}$$

then we use the Green's identity and from (4.6), we derive

$$\begin{aligned}\int_\Omega u_t K'(w^*)\phi^* dx = -\lambda^* \int_\Omega \phi^* f'(w^*)[K(u) - K(w^*]dx + \lambda \int_\Omega K'(w^*)f(u)\phi^* dx\\ - \lambda^* \int_\Omega K'(w^*)f(w^*)\phi^* dx.\end{aligned}$$

We add and subtract $\lambda^* \int_\Omega f(u)K'(w^*)\phi^* dx$ and define $\alpha(t) = \int_\Omega K'(w^*)\phi^* u dx$, thus we obtain

$$\begin{aligned}\alpha'(t) &= \int_\Omega K'(w^*)\phi^* u_t dx\\ &= (\lambda - \lambda^*)\int_\Omega K'(w^*)\phi^* f(u)dx - \lambda^* \int_\Omega \phi^* f'(w^*)[K(u) - K(w^*]dx\\ &\quad + \lambda^* \int_\Omega \phi^* K'(w^*)[f(u) - f(w^*)]dx\\ &\ge (\lambda - \lambda^*)I_B + \lambda^* \int_\Omega [K'(w^*)(f(u) - f(w^*)) - f'(w^*)(K(u) - K(w^*))]\phi^* dx,\end{aligned}$$

where $I_B = \inf_t \int_\Omega K'(w^*)\phi^* f(u)dx = \int_\Omega K'(w^*)\phi^* \inf_t f(u)dx > 0$.

We set now $u = u(x,t) = w^* + v = w^*(x) + v(x,t)$, and use for simplicity, in some parts of calculations s as a general variable, instead of v; i.e., $u = w^* + s$. Next we shall construct a function $h(s)$ such that $h(0) = 0$, $h(s) > 0$ for $s \in \mathbb{R}^*$, h convex and $h(s) = \Lambda[f(s) - f(0) - sf'(0)] \le \inf_{x\in\Omega}[f(w^*+s) - f(w^*) - sf'(w^*)]$ for $s \ge u_B - M$, where $u_B = \inf_\Omega u_0(x) \ge 0$, $\max_{x\in\Omega} w^*(x) = M < \infty$, $0 < \Lambda \le 1/2$ and $\int_b^\infty ds/h(s) < \infty$ for every $b \ge 0$, see [11, p. 1355]. Therefore,

$$
\begin{aligned}
F = F(s;w^*) &\equiv [f(w^*+s) - f(w^*)]K'(w^*) - [K(w^*+s) - K(w^*)]f'(w^*) \\
&= \begin{cases} F_1(s;w^*), & \text{for } 0 \le w^* + s \le w^* \text{ or } s \le 0, \\ F_2(s;w^*), & \text{for } w^* + s > w^* \text{ or } s > 0, \end{cases}
\end{aligned}
$$

with s a general variable and $0 \le m = \min_{x\in\Omega} w^*(x) \le w^*(x) \le \max_{x\in\Omega} w^*(x) = M < \infty$.

Interval I:. For $s \le 0$, so that $u_B - M \le s \le 0$, that is $w^* + s \le w^*$, and extending the domain of f and K to be defined also for negative values, we obtain

$$
\begin{aligned}
F = F_1(s;w^*) &= [f(w^*+s) - f(w^*)]K'(w^*) - [K(w^*+s) - K(w^*)]f'(w^*) \\
&= F_1(0;w^*) + F_1'(0;w^*)s + F_1''(\eta;w^*)s^2/2 = F_1''(\eta;w^*)s^2/2 \\
&= [f''(w^*+\eta)K'(w^*) - K''(w^*+\eta)f'(w^*)]s^2/2, \quad \text{for } s < \eta < 0.
\end{aligned} \tag{4.7}
$$

In the previous expression, F_1 has been expanded with respect to s as a Taylor series about 0, $F_1'(0;w^*) = [\frac{d}{ds}F_1(s;w^*)]_{s=0}$, etc. and $F_1(0;w^*) = F_1'(0;w^*) = 0$. From relation (3.3), $\theta = w^* + \eta \le w^*$ with $u_B - M \le \eta \le 0$, thus we have

$$
\frac{f''(\theta)}{K''(\theta)} > \frac{f'(\theta)}{K'(\theta)} \quad \text{and} \quad \frac{f''(w^*)}{K''(w^*)} > \frac{f'(w^*)}{K'(w^*)} > \frac{f'(\theta)}{K'(\theta)}, \quad \theta \in \mathbb{R}. \tag{4.8}
$$

If now $\frac{f''(\theta)}{K''(\theta)} = \frac{f''(w^*+\eta)}{K''(w^*+\eta)} \le \frac{f'(w^*)}{K'(w^*)}$, for any $\eta \le 0$, then on taking $\eta \to 0-$ we obtain $\frac{f''(w^*)}{K''(w^*)} \le \frac{f'(w^*)}{K'(w^*)}$, contradicting (4.8); therefore

$$
\frac{f''(\theta)}{K''(\theta)} = \frac{f''(w^*+\eta)}{K''(w^*+\eta)} > \frac{f'(w^*)}{K'(w^*)}, \quad s \le \eta \le 0. \tag{4.9}
$$

Hence, relations (4.7) and (4.9) (replacing w^* with l_0 and η with η_0) imply that

$$
\begin{aligned}
F = F_1(s;w^*) &\ge \inf_\eta [f''(w^*+\eta)K'(w^*) - K''(w^*+\eta)f'(w^*)]s^2/2 \\
&\ge \min_{x\in\overline{\Omega}} [f''(w^*+\eta_0)K'(w^*) - K''(w^*+\eta_0)f'(w^*)]s^2/2 \\
&= [f''(l_0+\eta_0)K'(l_0) - K''(l_0+\eta_0)f'(l_0)]s^2/2 = \Lambda_1 s^2,
\end{aligned}
$$

where $l_0 = w^*(x_0)$, for some $x_0 \in \overline{\Omega}$ and some $\eta_0 \in [s, 0]$; moreover $\Lambda_1 > 0$ due to (4.8).

So there is a $\Lambda_2 > 0$ small enough such that

$$
F = F_1(s;w^*) \ge \Lambda_1 s^2 \ge \Lambda_2[f(s) - f(0) - sf'(0)], \quad u_B - M < s \le 0. \tag{4.10}
$$

Interval II: For $s > 0$, but now $s < S$, for some S (see below (4.14)), and $w^* < w^* + s$, we have

$$
\begin{aligned}
F = F_2(s;w^*) &= [f(w^*+s) - f(w^*)]K'(w^*) - [K(w^*+s) - K(w^*)]f'(w^*) \\
&= F_2(0;w^*) + F_2'(0;w^*)s + F_2''(\eta;w^*)s^2/2 = F_2''(\eta;w^*)s^2/2 \\
&= [f''(w^*+\eta)K'(w^*) - K''(w^*+\eta)f'(w^*)]s^2/2, \quad \text{for } 0 < \eta < s \le S,
\end{aligned}
$$

Relation (3.3) now gives:

$$\frac{f''(\theta)}{K''(\theta)} > \frac{f'(\theta)}{K'(\theta)} > \frac{f'(w^*)}{K'(w^*)}, \quad \theta = w^* + \eta > 0, \quad \text{with } 0 \le \eta \le S. \tag{4.11}$$

Hence, from (4.11), we have

$$\begin{aligned}
F_2(s; w^*) &= [f''(w^* + \eta)K'(w^*) - K''(w^* + \eta)f'(w^*)]s^2/2 \\
&\ge \inf_{\eta}[f''(w^* + \eta)K'(w^*) - K''(w^* + \eta)f'(w^*)]s^2/2 \\
&= [f''(w^* + \eta_1)K'(w^*) - K''(w^* + \eta_1)f'(w^*)]s^2/2 \\
&\ge \min_{x\in\overline{\Omega}}[f''(w^* + \eta_1)K'(w^*) - K''(w^* + \eta_1)f'(w^*)]s^2/2 \\
&= [f''(l_1 + \eta_1)K'(l_1) - K''(l_1 + \eta_1)f'(l_1)]s^2/2 \\
&= \Lambda_3 s^2 > 0, \quad l_1 = w^*(x_1),\ x_1 \in \overline{\Omega},
\end{aligned}$$

for some x_1; moreover $\Lambda_3 > 0$ due to (4.11). So there is a $\Lambda_4 > 0$ small enough such that

$$F = F_2(s; w^*) \ge \Lambda_3 s^2 \ge \Lambda_4[f(s) - f(0) - sf'(0)], \quad 0 < s < S. \tag{4.12}$$

Interval III: For $s > 0$, actually for $s > S$ for some $S > 0$ and $w^* < w^* + s$ we get that

$$\begin{aligned}
F(s; w^*) &= F_2(s; w^*) = A(s; w^*) - B(s; w^*) \\
&= [f(w^* + s) - f(w^*)]K'(w^*) - [K(w^* + s) - K(w^*)]f'(w^*) \\
&\ge K'(w^*)[f(w^* + s) - f(w^*) - \Lambda_5 sK'(w^* + s)]
\end{aligned} \tag{4.13}$$

where $\Lambda_5 = f'(M)/K'(m)$, $(K'(m) > 0,\ m \ge 0)$. From the growth condition (1.4)(a), we have

$$\frac{f(s)}{K'(s)} > c_1 s + c_2, \quad s > S = S(c_1, c_2); \tag{4.14}$$

for any $c_1, c_2 > 0$, there exists $S = S(c_1, c_2) \ge 0$ which is the largest root of the equation $f(s)/K'(s) = c_1 s + c_2$. Relation (4.14) implies

$$\frac{f(w^* + s)}{K'(w^* + s)} > c_1(w^* + s) + c_2 \quad \text{for } s > S, \tag{4.15}$$

where $w^* + s \ge s > S$. From relation (4.15) we get that

$$\begin{aligned}
f(w^* + s) &> c_1(w^* + s)K'(w^* + s) + c_2K'(w^* + s) \\
&\ge c_1 sK'(w^* + s) + c_2K'(m) \\
&= 2[\Lambda_5 sK'(w^* + s) + f(M)],
\end{aligned} \tag{4.16}$$

on taking $c_1 = 2\Lambda_5 = 2f'(M)/K'(m)$ and $c_2 = 2f(M)/K'(m)$. From (4.13), (4.16) we derive:

$$\begin{aligned}
F_2(s; w^*) &\geq K'(w^*)[f(w^* + s) - f(M) - \Lambda_5 s K'(w^* + s)] \\
&\geq K'(w^*)[f(w^* + s) - \frac{1}{2} f(w^* + s)] \\
&\geq K'(w^*)\frac{1}{2} f(w^* + s) \\
&\geq K'(w^*)\frac{1}{2} f(s) \geq K'(m)\frac{1}{2} f(s) \\
&> \Lambda_6[f(s) - f(0) - s f'(0)] > 0,
\end{aligned} \tag{4.17}$$

where $0 < \Lambda_6 < K'(m)/2$.

Remarks. (a) We remind the reader that parameter λ^* is fixed and $w^* = w^*(x) = w^*(x; \lambda^*) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a fixed unique solution to (3.1) at $\lambda = \lambda^*$. This implies that $\inf_x w^*(x) = m \geq 0$, (depending on the boundary conditions) and $\sup_x w^*(x) = M$. Therefore m and M are fixed and $\Lambda_5 = \Lambda_5(m, M)$.

(b) Before going further, it is worth noting that from interval III, we shall get the blow-up of solutions, while from intervals I, II together with III we shall get an upper bound for the blow-up time t^*. More precisely, intervals I, II are used for finding a better estimate of the upper bound of t^*; while in these intervals u is uniformly bounded with respect to x, with $t < T$ for some $T > 0$, i.e. $u = u(x, t) < M + S$. It is remarkable that condition (3.3) is used only on the intervals I, II and contributes to better estimation of the upper bound of blow-up time.

Next, on using (4.10), (4.12) and (4.17) we take $h = h(s)$ such that

$$h(s) = \Lambda[f(s) - f(0) - s f'(0)] > 0, \tag{4.18}$$

where $\Lambda = \min\{\Lambda_2, \Lambda_4, \Lambda_6\}$ depending upon K'; moreover h satisfies

$$h(s) > 0, \quad s \in \mathbb{R}^*, \quad h'(s) > 0, \quad h(0) = h'(0) = 0, \quad h''(s) > 0, \quad s \in \mathbb{R}, \tag{4.19}$$

and its minimum is $(0, h(0)) = (0, 0)$.

Thus, for $v = u - w^*$, $A(t) = \int_\Omega K'(w^*)\phi^* v dx = \alpha(t) - \int_\Omega K'(w^*)\phi^* w^* dx$, with $a(t) = \int_\Omega K'(w^*)\phi^* u dx$, normalizing $\int_\Omega K'(w^*)\phi^* dx = 1$, using (4.19) and Jensen's inequality, then for $\lambda > \lambda^*$, we have

$$\begin{aligned}
&\alpha'(t) \\
&= A'(t) = \int_\Omega K'(w^*)\phi^* v_t dx \\
&\geq (\lambda - \lambda^*) I_B + \lambda^* \int_\Omega \{K'(w^*)[(f(u) - f(w^*)] - f'(w^*)[K(u) - K(w^*))]\phi^*\} dx \\
&\geq (\lambda - \lambda^*) I_B + \lambda^* \int_\Omega K'(w^*)\Lambda[f(v) - f(0) - v f'(0)]\phi^* dx \\
&\geq (\lambda - \lambda^*) I_B + \lambda^* \int_\Omega K'(w^*) h(v)\phi^* dx \\
&\geq \lambda^* \int_\Omega K'(w^*) h(v)\phi^* dx \geq \lambda^* h(A).
\end{aligned} \tag{4.20}$$

Thus, $A'(t) \geq \lambda^* h(A)$, which implies, due to (1.4) and (4.19), blow-up of $A(t)$ and since $A(t) \leq \|v(\cdot, t)\|$, blow-up of v at t^*_v and hence of u at $t^* < \infty$ where $t^* \leq t^*_v$, since $u = w^* + v > v$ in Ω and w^* is bounded. This completes the proof of the theorem. $\square$

Upper bound for the blow-up time: Now following [11], we give an upper bound for the blow-up time: Again from (3.3) and (4.20), we obtain

$$\begin{aligned} A'(t) &= \int_\Omega K'(w^*)\phi^* v_t dx \\ &\geq (\lambda - \lambda^*)I_B + \lambda^* \int_\Omega K'(w^*)h(v)\phi^* dx \\ &\geq (\lambda - \lambda^*)I_B + \lambda^* h(A). \end{aligned} \tag{4.21}$$

If

$$A_0 = \int_\Omega K'(w^*)v_0\phi^* dx = \int_\Omega K'(w^*)(u_0 - w^*)\phi^* dx$$

and $A_1 < \min\{0, A_0\}$, we choose A_2 such that $0 \leq A_2 \leq -A_1$. Then, from (4.21) and noting that $H(s) \equiv [(\lambda - \lambda^*)I_B + \lambda^* h(s)]^{-1} < (\lambda^* h(s))^{-1}$, h is defined by (4.18), we obtain

$$\begin{aligned} 0 < t &= \int_{A_1}^{A(t)} H(s)ds < t_v^* = \int_{A_1}^{\infty} H(s)ds \\ &< \int_{A_1}^{-A_2} H(s)ds + \int_{-A_2}^{A_2} H(s)ds + \int_{A_2}^{\infty} H(s)ds \\ &\leq \frac{1}{\lambda^*}\int_{A_1}^{-A_2} \frac{ds}{h(s)} + \int_{-\infty}^{\infty} [(\lambda - \lambda^*)I_B + C_1 s^2]^{-1} ds + \frac{1}{\lambda^*}\int_{A_2}^{\infty} \frac{ds}{h(s)} \\ &\leq C_2 + \pi\,[(\lambda - \lambda^*)I_B C_1]^{-1/2} \equiv T_B, \end{aligned}$$

where $C_1 = \frac{1}{2}\lambda^* \inf_{|s|<A_2} h''(s) > 0$ and

$$C_2 = \frac{1}{\lambda^*}[\int_{A_1}^{-A_2} (h(s))^{-1} ds + \int_{A_2}^{\infty} (h(s))^{-1} ds].$$

For the integral in the middle, we use (4.19), the fact that the maximum of $H(s)$ is taken at $s = 0$ and that $\int_{|s|>A_2} H(s)ds \ll \int_{|s|<A_2} H(s)ds$, as well as $\int_{-A_2}^{A_2} H(s)ds \lesssim \int_{-\infty}^{\infty} H(s)ds$. Moreover,

$$\begin{aligned} \int_{-A_2}^{A_2} [(\lambda - \lambda^*)I_B + C_1 s^2]^{-1} ds &\lesssim \int_{-\infty}^{\infty} [(\lambda - \lambda^*)I_B + C_1 s^2]^{-1} ds \\ &= \pi[(\lambda - \lambda^*)I_B C_1]^{-1/2}, \end{aligned}$$

for $0 < (\lambda - \lambda^*) \ll 1$. Finally, the solution u blows up in finite time $t^* \leq t_v^* \leq T_B$, where T_B is an upper bound of t^*.

Discussion

In this work, our essential outcome is the proof of the blow-up of solutions under some particular conditions. Beforehand, we discuss the local existence and uniqueness of solutions of (1.1) using comparison methods. We examine the stationary solutions of (3.1), following mainly [1, 10] and choose the case of a response (bifurcation) diagram with at least one turning point. For a turning point $(\lambda^*, \|w^*\|)$ (limit of minimal solution $w(x;\lambda)$), we consider that the critical value λ^* lies in the spectrum of (3.1) and that a solution w^* exists; actually the existence of a bounded w^* is the key of this work. If λ^* does not lie in the spectrum, i.e. there is no classical bounded solution w^*, then the proof of blow-up of u, in general, is an open question, see also [11]. Due to the fact that $f(0) > 0$, and $f(s), f'(s), f''(s) > 0$, $s \in \mathbb{R}$, the u-solutions are classical and do not degenerate. Our main result is that for $K(s) > 0$, $s > 0$, $K'(s), K''(s) > 0$, $s \geq 0$, $K(0) = 0$, and for any non-negative initial data, the u-solutions blow up in finite time for any $\lambda > \lambda^*$. For the proof of this result, we use spectral properties of the stationary problem and of corresponding linearized problem. Following similar ideas as in [11] (spectral method), for the semilinear heat equation, we construct a proper function and through this function we prove blow-up for a functional, from which we obtain the blow-up of u. The requirement that λ^* lies in the spectrum (this has the form $(0, \lambda^*]$) of (1.1) may not be necessary, [2, 3]; for the semilinear heat problem, this requirement has been replaced by a concavity assumption, see [3]. Moreover, we also give an upper bound estimate for the blow-up time.

Some other blow-up results, especially when $K(0) = K'(0) = K''(0) = 0$, or for the degenerate problems, as well as blow-up with respect to the initial data, will be presented in a forthcoming work.

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References

[1] H. Amann; *Fixed point equations and nonlinear eigenvalue problems in ordered Banach Spaces,* SIAM Rev. **18**, (1976), 620–709.

[2] J. W. Bebernes, D. Eberly; *Mathematical Problems from Combustion Theory*, Springer–Verlag, New York, (1989).

[3] H. Bellout; *A criterion for blow-up of solutions to semilinear heat equations,* SIAM J. Math. Anal. **18**, (1987), 722–727.

[4] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa; *Blow-up for $u_t = \Delta u + \lambda g(u)$ revisited,* Adv. Diff. Eq, (1996), 73–90.

[5] M. G. Crandall, P. H. Rabinowitz; *Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems,* Arch. Rational Mech. Anal. **58**, (1975), no. 3, 207–218.

[6] M. Fila; *Boundedness of Global Solutions of Nonlinear Diffusion Equations*, J. Diff. Eqs, **98**, (1992), 226–240.

[7] V. A. Galaktionov, J. L. Vazquez; *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions,* Comm. Pure Appl. Math. **50**, (1997), 1-67.

[8] V. A. Galaktionov, J. L. Vazquez; *Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional quasilinear heat equations*, Arch. Rational Mech. Anal. **129**, (1995), 225–244.

[9] J. Kazdan, F. Warner; *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **43**, (1983), 1350–1366.

[10] H. B. Keller, D. S. Cohen; *Some positone problems suggested by nonlinear heat generation,* J. Math. Mech. **16**, (1967), 1361–1376.

[11] A. A. Lacey; *Mathematical analysis of thermal runaway for spartially inhomogeneous reactions*, SIAM J. Appl. Math. **43**, (1983), 1350–1366.

[12] A. A. Lacey, D. E. Tzanetis; *Global existence and convergence to a singular steady state for a semilinear heat equation,* Proc. Royal Soc. Edinb. **105A**, (1987), 289-305.

[13] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva; *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monog. Vol. **23**, Amer. Math. Soc., Providence, R.I., (1968).
[14] E. A. Latos, D. E. Tzanetis; *Existence and blow-up of solutions for a non-local filtration and Porous Medium problem*, Proc. Edinb. Math. Soc. (2) **53**, (2010), no. 1, 195–209.
[15] H. Levine, P. Sacks; *Some existence and nonexistence theorems for solutions of degenerate parabolic equations,* J. Diff. Eqns. **52**, (1984), 135–161.
[16] W.-M. Ni, P. E. Sacks, J. Tavantzis; *On the asymptotic behavior of solutions of certain quasilinear parabolic equations,* J. Differential Equations **54** (1984), no. 1, 97-120.
[17] C. V. Pao; *Singular reaction diffusion equations of porous medium type,* Nonlinear Analysis TMA, **71**, (2009), 2033-2052.
[18] P. Quittner, P. Souplet; *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, Birkhäuser Advanced Texts, (2007).
[19] D. H. Sattinger; *Monotone methods in nonlinear elliptic and parabolic boundary value problems,* Indiana Univ. Math. J. **21**, (1972), 979–1000.
[20] Y. Seki, N. Umeda, R. Suzuki; *Blow-up directions for quasilinear parabolic equations*, Proc. Royal Soc. Edin. **138A**, 379–405, (2008).
[21] R. Suzuki; *Boundeness of global solutions of the one dimensional quasilinear degenerate parabolic equation,* J. Math. Soc. Japan **50**, No. 1, (1998), 119–138.
[22] J. L. Vazquez; *The Porous Medium equation: Mathematical theory*, Oxford Mathematical Monographs, Madrid, (2007).

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