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EXPONENTIAL STABILITY OF TRAVELING FRONTS FOR A 2D LATTICE DELAYED DIFFERENTIAL EQUATION WITH GLOBAL INTERACTION

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Abstract. The purpose of this paper is to study traveling wave fronts of a two-dimensional (2D) lattice delayed differential equation with global interaction. Applying the comparison principle combined with the technical weighted-energy method, we prove that any given traveling wave front with large speed is time-asymptotically stable when the initial perturbation around the wave front need decay to zero exponentially as $i\cos\theta + j\sin\theta \to -\infty$, where θ is the direction of propagation, but it can be allowed relatively large in other locations. The result essentially extends the stability of traveling wave fronts for local delayed lattice differential equations obtained by Cheng et al [1] and Yu and Ruan [16].

1. Introduction

The purpose of this paper is to consider the exponential stability of traveling wave fronts for a stage structured population model on a 2D spatial lattice. The population model can be described by the delayed lattice differential equation with global interaction (see Cheng et al [1] and Weng et al [13]):

$$\begin{aligned}\frac{du_{i,j}(t)}{dt} &= D_m[u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{i,j}(t)] - d_m u_{i,j}(t) \\ &\quad + \frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(i-m)\gamma_\alpha(j-n)b\big(u_{m,n}(t-\tau)\big), \quad i,j\in\mathbb{Z},\ t>0,\end{aligned} \tag{1.1}$$

where D_m and d_m represent the diffusion coefficient and the death rate of the matured population, respectively, $d(s)$ and $D(s)$ are the death rate and diffusion rate of the immature population, respectively, at age $s \in (0,\tau)$, $\varpi = e^{\int_0^\tau d(s)ds}$ and $\alpha = \int_0^\tau D(s)ds$ represent the impact of the death rate for immature and the effect of the dispersal rate of immature on the mature population, respectively, and

$$\beta_\alpha(l) = 2e^{-2\alpha}\int_0^\pi \cos(ls)e^{2\alpha\cos s}ds, \quad \gamma_\alpha(l) = 2e^{-2\alpha}\int_0^\pi \cos(ls)e^{2\alpha\cos s}ds, \quad l\in\mathbb{Z}.$$

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The important feature of (1.1) is the reflection of the joint effect of the diffusion dynamics and the nonlocal delayed effect. Under monstable and quasi-monotone assumptions, the authors of [1, 13] established the existence of minimal wave speed $c_* = c_*(\theta)(>0)$, where $\theta \in [0, \frac{\pi}{2}]$ is any fixed direction of propagation, and showed that the minimal wave speed $c_*(\theta)$ coincides with the spreading speed for any fixed direction θ. Moreover, the effects of the maturation period τ and the direction of propagation θ on the spreading speed were considered.

When $D(a) = 0$ for any $0 < a < \tau$ (i.e. the immature population is non-mobile), $\alpha = 0$ and then $\beta_0(0) = \gamma_0(0) = 2\pi$ and $\beta_0(l) = \gamma_0(l) = 0$ for any $l \in \mathbb{Z} \setminus \{0\}$. In this case, (1.1) reduces to the local delayed lattice differential equation

$$\begin{aligned} \frac{du_{i,j}(t)}{dt} &= D_m[u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{i,j}(t)] \\ &\quad - d_m u_{i,j}(t) + \varpi b\big(u_{i,j}(t-\tau)\big). \end{aligned} \tag{1.2}$$

Applying the weighted energy method, Cheng et al [2] proved the asymptotic stability of traveling wave fronts of (1.2). More precisely, they proved that, for the Cauchy problem of (1.2) with initial data

$$u_{i,j}(s) = u^0_{i,j}(s), \quad i, j \in \mathbb{Z},\ s \in [-\tau, 0], \tag{1.3}$$

the traveling wave front $\phi(i\cos\theta + j\sin\theta + ct)$ of (1.2) connecting E^+ and E^- with large speed is time-asymptotically stable, when the initial perturbation around the wave front (i.e. $|u^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs)|$) is sufficiently small in a weighted norm. More recently, the authors of [16] further established the stability of traveling wave fronts of (1.2) for relatively large initial perturbations by using the comparison principle and the weighted-energy method. However, to the best of our knowledge, there has been no results on the stability of traveling wave fronts for the delayed lattice differential equation with global interaction.

The purpose of this paper is to consider the stability of traveling wave fronts of (1.1). More precisely, we shall prove that any given traveling wave front $\phi(i\cos\theta + j\sin\theta + ct)$ with large speed c (i.e. c satisfies (2.6) below) is time-asymptotically stable when the initial perturbation around the wave front (i.e. $|u^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs)|$) need to decay to zero exponentially as $i\cos\theta + j\sin\theta \to -\infty$, where θ is the direction of propagation, but it can be allowed relatively large in other locations (see Theorem 2.3). Here, we use an approach combining the comparison principle and the weighted-energy method, which was developed by [5] to prove the stability of traveling wave fronts of a Nicholson's blowflies equation with diffusion. This approach was further employed by many researchers to prove the stability of traveling wave fronts of various reaction-diffusion equations with local or nonlocal delays; see, e.g., [4, 6, 7, 8, 9, 14, 15].

Although the main idea and methods of the proof for our main theorem are originally encouraged by [4, 6, 7, 8, 9, 14, 15, 16], we mention that difficulties and challenge are existing for our arguments due to the convolution term. For example, in the construction of the weight function, we need to derive some important estimations (see Lemma 2.2). In addition, the proof of the key inequality is more technical (see Lemma 3.3). Similar to [1, 13], we make the following assumptions:

- (A1) $b \in C^2([0, K])$, $b(0) = 0$, $\varpi b(K) = d_m K$, $d_m > \varpi b'(K)$, $\varpi b(u) > d_m u$ for $u \in (0, K)$, where K is a positive constant;
- (A2) $b(u) \le b'(0)u$ and $b'(u) \ge 0$ for $u \in [0, K]$.

The rest of this paper is organized as follows. In Section 2, we first introduce some known results on the existence of traveling wave fronts of (1.1), and then present our stability results. The proofs of the main results are given in Section 3.

Notation. Throughout this paper, l^2_w denotes the weighted l^2 space with weight $w(\xi) \in C(\mathbb{R}, \mathbb{R}^+)$ and a fixed $\theta \in [0, \frac{\pi}{2}]$; that is,

$$l^2_w = \Big\{ \varsigma = \{\varsigma_{i,j}\}_{i,j\in\mathbb{Z}}, \varsigma_{i,j} \in \mathbb{R} : \sum_{i,j\in\mathbb{Z}} w(i\cos\theta + j\sin\theta)\varsigma_{i,j}^2 < \infty \Big\}$$

with the norm

$$\|\varsigma\|_{l^2_w} = \Big[\sum_{i,j\in\mathbb{Z}} w(i\cos\theta + j\sin\theta)\varsigma_{i,j}^2 \Big]^{1/2}.$$

In particular, if $w \equiv 1$, we denote l^2_w by l^2.

2. Preliminaries and main results

Throughout this article, a traveling wave solution connecting 0 and K refers to a triplete (ϕ, c, θ), where $\phi = \phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is a function, $c > 0$ and $\theta \in [0, \pi/2]$ are constants, such that $u_{i,j}(t) = \phi(\xi)$, $\xi = i\cos\theta + j\sin\theta + ct$, is a solution of (1.1); that is,

$$\begin{aligned} c\phi'(\xi) = D_m[&\phi(\xi + \cos\theta) + \phi(\xi - \cos\theta) + \phi(\xi + \sin\theta) + \phi(\xi - \sin\theta) - 4\phi(\xi)] \\ &- d_m\phi(\xi) + \frac{\varpi}{4\pi^2} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau)\big) \end{aligned} \tag{2.1}$$

with the boundary conditions

$$\phi(-\infty) = 0, \quad \phi(+\infty) = K. \tag{2.2}$$

The constant θ represents the direction of the wave. We call c the *wave speed* and ϕ the *wave profile*. Moreover, we say ϕ is a *traveling (wave) front* if $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is monotone.

It is clear that the characteristic function for (2.1) with respect to the trivial equilibrium 0 can be represented by

$$\begin{aligned} \Delta(c, \lambda) = c\lambda - D_m\big[&e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4\big] + d_m \\ &- \frac{\varpi}{4\pi^2} b'(0) \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) e^{-\lambda(m\cos\theta + n\sin\theta + c\tau)}. \end{aligned}$$

Properties of $\Delta(c, \lambda)$ and existence of traveling wave fronts of (1.1) were investigated in [1, 13]. For the sake of completeness, we recall them as follows.

Proposition 2.1. *Assume* (A1)–(A2) *hold. Then the following results hold:*

(1) *For each $\theta \in [0, \frac{\pi}{2}]$, there exist $\lambda_* := \lambda_*(\theta) > 0$ and $c_* := c_*(\theta) > 0$ such that*
$$\Delta(c_*, \lambda_*) = 0 \text{ and } \frac{\partial}{\partial\lambda}\Delta(c_*, \lambda)\Big|_{\lambda=\lambda_*} = 0.$$
Furthermore, if $c > c_*(\theta)$, then the equation $\Delta(c, \lambda) = 0$ has two positive real roots $\lambda_1 := \lambda_1(c, \theta)$ and $\lambda_2 := \lambda_2(c, \theta)$ with $\lambda_1 < \lambda_* < \lambda_2$.

(2) *Fix $\theta \in [0, \pi/2]$. Then, for every $c \geq c_*(\theta)$,* (1.1) *has a traveling wave front $\phi(\xi)$ with direction θ and speed c.*

For convenience, we denote

$$L_1 = \max_{u\in[0,K]} b'(u), \quad L_2 = \frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n)\max\big\{1, e^{-m\cos\theta - n\sin\theta}\big\}.$$

Note that if $b''(u) \le 0$ for $u \in [0,K]$, then $L_1 = b'(0)$. Moreover, it is easy to see that $L_2 = 1$ when $\alpha = 0$.

The following result plays an important role for constructing the weight function.

Lemma 2.2. *Assume*

$$d_m > D_m\big(e-1\big) + \frac{1}{2}\varpi b'(K)(1+L_2). \tag{2.3}$$

For any given traveling wave front $\phi(\xi)$ of (1.1) with direction $\theta \in [0,\frac{\pi}{2}]$ and speed $c > c_*(\theta)$ obtained in Proposition 2.1, there exists $\xi_* > 0$ such that for any $\xi \ge \xi_*$,

$$\begin{aligned}
&\frac{\varpi}{4\pi^2} b'\big(\phi(\xi)\big)\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n)\max\{1, e^{-m\cos\theta - n\sin\theta}\}\\
&+ \frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau))\big)\\
&\le \varpi b'(K)(1+L_2) + \bar\epsilon,
\end{aligned}$$

where $\bar\epsilon = d_m - D_m\big(e-1\big) - \frac{1}{2}\varpi b'(K)(1+L_2) > 0$.

Proof. Since $\lim_{\xi\to+\infty} b'\big(\phi(\xi)\big) = b'(K)$, it suffices to show that

$$\lim_{\xi\to+\infty} \frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau))\big) = b'(K). \tag{2.4}$$

Given any $\epsilon > 0$, since

$$\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \beta_\alpha(m) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \beta_\alpha(n) = 1$$

(see [1, Lemma 2.1]), there exists $M, N > 0$ such that

$$\sum_{|m|\ge M} \beta_\alpha(m), \sum_{|n|\ge N} \gamma_\alpha(n) \le \frac{\pi\epsilon}{4L_1}.$$

Noting that $\lim_{\xi\to+\infty} b'\big(\phi(\xi)\big) = b'(K)$, there exists $\xi_* > 0$ such that for any $\xi \ge \xi_* - M - N - c\tau$,

$$|b'\big(\phi(\xi)\big) - b'(K)| < \frac{\epsilon}{4}.$$

Then, for any $\xi \ge \xi_*$, we have

$$\begin{aligned}
&\Big|\frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau)\big) - b'(K)\Big|\\
&= \Big|\frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n)\big[b'\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau)\big) - b'(K)\big]\Big|\\
&\le \frac{1}{4\pi^2}\sum_{|m|\ge M, n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n)\big|b'\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau)\big) - b'(K)\big|\\
&\quad + \frac{1}{4\pi^2}\sum_{|m|\le M, |n|\ge N} \beta_\alpha(m)\gamma_\alpha(n)\big|b'\big(\phi(\xi - m\cos\theta - n\sin\theta - c\tau)\big) - b'(K)\big|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4\pi^2}\sum_{|m|\le M,|n|\le N}\beta_\alpha(m)\gamma_\alpha(n)\big|b'\big(\phi(\xi-m\cos\theta-n\sin\theta-c\tau)\big)-b'(K)\big|\\
&\le 2L_1\frac{1}{2\pi}\sum_{|m|\ge M}\beta_\alpha(m)+2L_1\frac{1}{2\pi}\sum_{|n|\ge N}\gamma_\alpha(n)+\frac{\epsilon}{4}\frac{1}{4\pi^2}\sum_{|m|\le M,|n|\le N}\beta_\alpha(m)\gamma_\alpha(n)<\epsilon.
\end{aligned}
$$

Thus, (2.4) holds. The proof is complete. □

Based on the above lemma, we define the weight function $w(\xi)$ as

$$
w(\xi)=\begin{cases} e^{-(\xi-\xi_*)}, & \text{for } \xi<\xi_*,\\ 1, & \text{for } \xi\ge\xi_*. \end{cases} \tag{2.5}
$$

We can now state our main theorem.

Theorem 2.3. *Assume* (A1)–(A2) *hold and* $b''(u)\le 0$ *for* $u\in[0,K]$. *For any given traveling wave front* $\phi(\xi)$ *of* (1.1) *with direction* $\theta\in[0,\pi/2]$ *and speed* c *obtained in Proposition 2.1, if* (2.3) *holds,*

$$
c>\max\big\{2D_m\big(e-1\big)+b'(0)\varpi(1+L_2)-2d_m,\ c_*(\theta)\big\} \tag{2.6}
$$

and the initial data satisfies $0\le u_{i,j}(s)\le K$ *for* $(i,j,s)\in\mathbb{Z}^2\times[-\tau,0]$, *and*

$$
\{u^0_{i,j}(s)-\phi(i\cos\theta+j\sin\theta+cs)\}_{i,j\in\mathbb{Z}}\in C\big([-\tau,0],l^2_w\big),
$$

then the unique solution $u_{i,j}(t)$ *of the Cauchy problem* (1.1) *and* (1.3) *satisfies* $0\le u_{i,j}(t)\le K$ $(i,j,t)\in\mathbb{Z}^2\times[0,+\infty)$,

$$
\{u_{i,j}(t)-\phi(i\cos\theta+j\sin\theta+ct)\}_{i,j\in\mathbb{Z}}\in C\big([0,+\infty),l^2_w\big),
$$

and there exists positive number μ *such that*

$$
\sup_{i,j\in\mathbb{Z}}\big|u_{i,j}(t)-\phi(i\cos\theta+j\sin\theta+ct)\big|\le C_0e^{-\mu t},\quad t\ge0,
$$

for some constant $C_0>0$.

Remark 2.4. (i) Note that if D_m and $b'(K)$ are relatively small, then the technical assumption (2.3) holds. As mentioned by Mei et al [6], the condition $b'(K)\ll 1$ is natural, see e.g. [6, Remark 1].

(ii) From the condition (2.6) and definitions of the weighted function $w(\xi)$ and the space $C\big([-\tau,0],l^2_w\big)$, we see that the initial perturbation around the wave front must converge to 0 exponentially as $i\cos\theta+j\sin\theta\to-\infty$ in the form

$$
u^0_{i,j}(s)-\phi(i\cos\theta+j\sin\theta+cs)=O(1)e^{-\frac12|i\cos\theta+j\sin\theta|},\ s\in[-\tau,0].
$$

Contrasting to [2], we do not require that the initial perturbation must be sufficiently small in a weighted norm.

(iii) Theorem 2.3 guarantees that any given traveling wave front of (1.1) with large speed is time-asymptotically stable. However, we are unable to prove the stability for any slower waves $c>c_*$, particularly the case of critical waves with $c=c_*$. We leave this for future research.

3. Proof of main results

In this section, we first state the existence of solutions of the Cauchy problem (1.1) and (1.3) and establish the comparison principle. Then we prove our stability results by using the comparison principle together with the weighted energy method. In the sequel, we always assume that all the conditions in Theorem 2.3 hold.

Applying similar methods as in Cheng et al [2, Theorem 2.2], we obtain the following existence result.

Lemma 3.1 (Existence). *For any function $u^0(s) = \{u^0_{i,j}(s)\}_{i,j\in\mathbb{Z}} \in C([-\tau,0], l^\infty)$, equation (1.1) has a unique solution $u(t) = \{u_{i,j}(t)\}_{i,j\in\mathbb{Z}} \in C([-\tau,+\infty), l^\infty)$ with $u(s) = u^0(s)$ on $[-\tau,0]$. Furthermore, if*

$$\{u^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs)\}_{i,j\in\mathbb{Z}} \in C\big([-\tau,0], l^2\big),$$

then

$$\{u_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)\}_{i,j\in\mathbb{Z}} \in C\big([0,+\infty), l^2\big).$$

Lemma 3.2 (Comparison Principle). *Let $\{\overline{u}_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ and $\{\underline{u}_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ be the solutions of (1.1) and (1.3) with initial data $\{\overline{u}^0_{i,j}(s)\}_{i,j\in\mathbb{Z}}$ and $\{\underline{u}^0_{i,j}(s)\}_{i,j\in\mathbb{Z}}$, respectively. If*

$$0 \le \underline{u}^0_{i,j}(s) \le \overline{u}^0_{i,j}(s) \le K \quad \text{for } i,j\in\mathbb{Z} \text{ and } s\in[-\tau,0],$$

then

$$0 \le \underline{u}_{i,j}(t) \le \overline{u}_{i,j}(t) \le K \quad \text{for } i,j\in\mathbb{Z} \text{ and } t \ge 0.$$

Proof. Put $w_{i,j}(t) = \overline{u}_{i,j}(t) - \underline{u}_{i,j}(t)$ for $i,j\in\mathbb{Z}$ and $t \ge -\tau$. Direct computation shows that

$$\begin{aligned} w'_{i,j}(t) &= D_m[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] \\ &\quad - d_m w_{i,j}(t) + h_{i,j}(t), \end{aligned}$$

where

$$h_{i,j}(t) = \frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(i-m)\gamma_\alpha(j-n)\big[b\big(\overline{u}_{m,n}(t-\tau)\big) - b\big(\underline{u}_{m,n}(t-\tau)\big)\big].$$

We claim that

$$\begin{aligned} w_{i,j}(t) &= \frac{1}{4\pi^2}e^{-d_m t}\sum_{k,l\in\mathbb{Z}} \beta_{D_m t}(i-k)\gamma_{D_m t}(j-l)w_{k,l}(0) \\ &\quad + \frac{1}{4\pi^2}\sum_{k,l\in\mathbb{Z}}\int_0^t e^{-d_m(t-s)}\beta_{D_m(t-s)}(i-k)\gamma_{D_m(t-s)}(j-l)h_{k,l}(s)ds. \end{aligned}$$

We note that this claim can be proved by using discrete Fourier transformation as in Cheng et al. [2]. For the sake of completeness and reader's convenience, we provide its proof here. Note that the grid function $w_{i,j}(t)$ can be viewed as the discrete spectral of a periodic function $\widehat{w}(t,\lambda)$ by discrete Fourier transformation (see Goldberg, 1965; Titchmarsh, 1962):

$$\widehat{w}(t,\lambda) = \frac{1}{2\pi}\sum_{k,l\in\mathbb{Z}} e^{-\mathbf{i}(k\lambda_1 + l\lambda_2)}w_{k,l}(t),$$

$$w_{k,l}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}(k\lambda_1+l\lambda_2)}\widehat{w}(t,\lambda)d\lambda_1 d\lambda_2,$$

where $\mathbf{i}$ is the imaginary unit and $\lambda = (\lambda_1, \lambda_2)$. Using discrete Fourier transformation, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{w}(t,\lambda) &= D_m[e^{\mathbf{i}\lambda_1} + e^{-\mathbf{i}\lambda_1} + e^{\mathbf{i}\lambda_2} + e^{-\mathbf{i}\lambda_2} - 4]\widehat{w}(t,\lambda) - d_m\widehat{w}(t,\lambda) + \widehat{h}(t,\lambda)\\ &= -\Big[4D_m(\sin^2\frac{\lambda_1}{2} + \sin^2\frac{\lambda_2}{2}) + d_m\Big]\widehat{w}(t,\lambda) + \widehat{h}(t,\lambda).\end{aligned} \tag{3.1}$$

This equation can be solved as:

$$\begin{aligned}\widehat{w}(t,\lambda) &= \widehat{w}(0,\lambda)e^{-4D_m t(\sin^2\frac{\lambda_1}{2}+\sin^2\frac{\lambda_2}{2})}e^{-d_m t}\\ &\quad + \int_0^t \widehat{h}(s,\lambda)e^{-4D_m(t-s)(\sin^2\frac{\lambda_1}{2}+\sin^2\frac{\lambda_2}{2})}e^{-d_m(t-s)}ds.\end{aligned}$$

Note that

$$\widehat{w}(0,\lambda) = \frac{1}{2\pi}\sum_{k,l\in\mathbb{Z}} e^{-\mathbf{i}(k\lambda_1+l\lambda_2)}w_{k,l}(0), \quad \widehat{h}(s,\lambda) = \frac{1}{2\pi}\sum_{k,l\in\mathbb{Z}} e^{-\mathbf{i}(k\lambda_1+l\lambda_2)}h_{k,l}(s).$$

Using the inverse discrete Fourier transformation, we obtain

$$\begin{aligned}w_{i,j}(t) &= \frac{1}{2\pi}e^{-d_m t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}(i\lambda_1+j\lambda_2)}\widehat{w}(0,\lambda)e^{-4D_m t(\sin^2\frac{\lambda_1}{2}+\sin^2\frac{\lambda_2}{2})}d\lambda_1 d\lambda_2\\ &\quad + \frac{1}{2\pi}\int_0^t e^{-d_m(t-s)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}(i\lambda_1+j\lambda_2)}\widehat{h}(s,\lambda)\\ &\quad \times e^{-4D_m(t-s)(\sin^2\frac{\lambda_1}{2}+\sin^2\frac{\lambda_2}{2})}d\lambda_1 d\lambda_2 ds\\ &= \frac{1}{4\pi^2}e^{-d_m t}\sum_{k,l\in\mathbb{Z}} w_{k,l}(0)\int_{-\pi}^{\pi} e^{\mathbf{i}(i-k)\lambda_1}e^{-4D_m t\sin^2\frac{\lambda_1}{2}}d\lambda_1\\ &\quad \times \int_{-\pi}^{\pi} e^{\mathbf{i}(j-l)\lambda_2}e^{-4D_m t\sin^2\frac{\lambda_2}{2}}d\lambda_2 + \frac{1}{4\pi^2}\sum_{k,l\in\mathbb{Z}}\int_0^t e^{-d_m(t-s)}h_{k,l}(s)\\ &\quad \times \int_{-\pi}^{\pi} e^{\mathbf{i}(i-k)\lambda_1}e^{-4D_m(t-s)\sin^2\frac{\lambda_1}{2}}d\lambda_1\int_{-\pi}^{\pi} e^{\mathbf{i}(j-l)\lambda_2}e^{-4D_m(t-s)\sin^2\frac{\lambda_2}{2}}d\lambda_2 ds\\ &= \frac{1}{4\pi^2}e^{-d_m t}\sum_{k,l\in\mathbb{Z}}\beta_{D_m t}(i-k)\gamma_{D_m t}(j-l)w_{k,l}(0)\\ &\quad + \frac{1}{4\pi^2}\sum_{k,l\in\mathbb{Z}}\int_0^t e^{-d_m(t-s)}\beta_{D_m(t-s)}(i-k)\gamma_{D_m(t-s)}(j-l)h_{k,l}(s)ds.\end{aligned}$$

Since $b'(u) \ge 0$ for $u \in [0,K]$ and $0 \le \underline{u}^0_{i,j}(s) \le \overline{u}^0_{i,j}(s) \le K$ for $s \in [-\tau, 0]$, we have $w_{i,j}(t) \ge 0$ for $i,j \in \mathbb{Z}$ and $t \in [0,\tau]$. Inductively, we obtain that $w_{i,j}(t) \ge 0$ for $i,j \in \mathbb{Z}$ and $t \ge 0$, i.e. $\underline{u}_{i,j}(t) \le \overline{u}_{i,j}(t)$ for $i,j \in \mathbb{Z}$ and $t > 0$. Similarly, we can show that $\underline{u}_{i,j}(t) \ge 0$ and $\overline{u}_{i,j}(t) \le K$ for $i,j \in \mathbb{Z}$ and $t > 0$. This completes the proof. □

In what follows, we shall prove the stability theorem by means of the comparison principle together with the weighed energy method.

We assume that the initial data $\{u^0_{i,j}(s)\}_{i,j\in\mathbb{Z}}$ of (1.1) satisfying $0 \le u^0_{i,j}(s) \le K$ for $i,j \in \mathbb{Z}$ and $s \in [-\tau, 0]$, and $\{u^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs)\}_{i,j\in\mathbb{Z}} \in$

$C\big([-\tau,0],l_w^2\big)$. Take

$$\varphi^+_{i,j}(s):=\max\big\{u^0_{i,j}(s),\phi(i\cos\theta+j\sin\theta+cs)\big\},$$
$$\varphi^-_{i,j}(s):=\min\big\{u^0_{i,j}(s),\phi(i\cos\theta+j\sin\theta+cs)\big\}$$

for $i,j\in\mathbb{Z}$ and $s\in[-\tau,0]$. Then,

$$\{\varphi^\pm_{i,j}(s)-\phi(i\cos\theta+j\sin\theta+cs)\}_{i,j\in\mathbb{Z}}\in C\big([-\tau,0],l_w^2\big)$$

and

$$0\le\varphi^-_{i,j}(s)\le u^0_{i,j}(s),\quad \phi(i\cos\theta+j\sin\theta+cs)\le\varphi^+_{i,j}(s)\le K.$$

Let $u^\pm(t)=\{u^\pm_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ be the solutions of (1.1) with respect to the initial data $\varphi^\pm(s)=\{\varphi^\pm_{i,j}(s)\}_{i,j\in\mathbb{Z}}$, i.e.

$$\begin{gathered}\frac{du^\pm_{i,j}(t)}{dt}=D_m[u^\pm_{i+1,j}(t)+u^\pm_{i-1,j}(t)+u^\pm_{i,j+1}(t)+u^\pm_{i,j-1}(t)-4u^\pm_{i,j}(t)]-d_mu^\pm_{i,j}(t)\\ +\frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}}\beta_\alpha(i-m)\gamma_\alpha(j-n)b\big(u^\pm_{m,n}(t-\tau)\big),\quad i,j\in\mathbb{Z},\ t>0,\\ u^\pm_{i,j}(s)=\varphi^\pm_{i,j}(s),\quad i,j\in\mathbb{Z},\ s\in[-\tau,0].\end{gathered}\tag{3.2}$$

Applying the comparison principle, we have

$$0\le u^-_{i,j}(t)\le u_{i,j}(t),\quad \phi(i\cos\theta+j\sin\theta+ct)\le u^+_{i,j}(t)\le K\quad\text{for } i,j\in\mathbb{Z},\ t>0.$$

3.1. **Weighted energy estimate.** For convenience, we denote

$$U_{i,j}(t)=u^+_{i,j}(t)-\phi(i\cos\theta+j\sin\theta+ct),\quad \xi_{i,j}(t)=i\cos\theta+j\sin\theta+ct.$$

It is easy to verify that $\{U_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ satisfies

$$\begin{aligned}\frac{dU_{i,j}(t)}{dt}&=D_m[U_{i+1,j}(t)+U_{i-1,j}(t)+U_{i,j+1}(t)+U_{i,j-1}(t)-4U_{i,j}(t)]\\ &\quad+\frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}}\beta_\alpha(m)\gamma_\alpha(n)b'\big(\phi(\xi_{i-m,j-n}(t-\tau))\big)U_{i-m,j-n}(t-\tau)\\ &\quad-d_mU_{i,j}(t)+G_{i,j}(t),\\ U_{i,j}(s)&=\varphi^+_{i,j}(s)-\phi(\xi_{i,j}(s)):=U^0_{i,j}(s),\end{aligned}\tag{3.3}$$

where $i,j\in\mathbb{Z}$, $t>0$, $s\in[-\tau,0]$ and the nonlinear term $G_{i,j}(t)$ is given by

$$\begin{aligned}G_{i,j}(t)&=\frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}}\beta_\alpha(m)\gamma_\alpha(n)\big[b\big(U_{i-m,j-n}(t-\tau)+\phi(\xi_{i-m,j-n}(t-\tau))\big)\\ &\quad-b\big(\phi(\xi_{i-m,j-n}(t-\tau))\big)-b'\big(\phi(\xi_{i-m,j-n}(t-\tau))\big)U_{i-m,j-n}(t-\tau)\big].\end{aligned}$$

To obtain a weighted energy estimate, we need the following key inequality. Take $C_0(\mu)=\min\big\{C_1(\mu),C_2(\mu)\big\}$, where

$$C_1(\mu)=c+2d_m-2D_m(e-1)-L_1\varpi(1+L_2)-2\mu-\varpi L_1L_2(e^{2\mu\tau}-1),$$
$$C_2(\mu)=d_m-D_m(e-1)-\frac{1}{2}\varpi b'(K)(1+L_2)-2\mu-\varpi L_1L_2(e^{2\mu\tau}-1).$$

Define

$$\begin{aligned} B_{i,j}(\mu,t) &= -c\frac{w_\xi(\xi_{i,j}(t))}{w(\xi_{i,j}(t))} + 2(d_m-\mu) - D_m\big[\mathcal{L}(w)(\xi_{i,j}(t)) - 4\big] \\ &\quad - \frac{\varpi}{4\pi^2}e^{2\mu\tau}b'\big(\phi(\xi_{i,j}(t))\big)\sum_{m,n\in\mathbb{Z}}\beta_\alpha(m)\gamma_\alpha(n)\frac{w(\xi_{i+m,j+n}(t+\tau))}{w(\xi_{i,j}(t))} \\ &\quad - \frac{\varpi}{4\pi^2}\sum_{m,n\in\mathbb{Z}}\beta_\alpha(m)\gamma_\alpha(n)b'\big(\phi(\xi_{i-m,j-n}(t-\tau))\big), \end{aligned} \tag{3.4}$$

where

$$\mathcal{L}(w)(\xi_{i,j}(t)) = \frac{w(\xi_{i+1,j}(t))}{w(\xi_{i,j}(t))} + \frac{w(\xi_{i-1,j}(t))}{w(\xi_{i,j}(t))} + \frac{w(\xi_{i,j+1}(t))}{w(\xi_{i,j}(t))} + \frac{w(\xi_{i,j-1}(t))}{w(\xi_{i,j}(t))}.$$

Lemma 3.3 (Key inequality)**.** *Let $w(\xi)$ be the weight function given in* (2.5)*. Then*

$$B_{i,j}(\mu,t) \ge C_0(\mu) > 0,$$

for all $i,j\in\mathbb{Z}$, $t>0$, and $0<\mu<\mu_0 := \min\{\mu_1,\mu_2\}$, where μ_i is the unique solution to the equation $C_i(\mu)=0$, $i=1,2$.

Proof. We distinguish two cases:

Case (i): $\xi_{i,j}(t) < \xi_*$. In this case $w(\xi_{i,j}(t)) = e^{-(\xi_{i,j}(t)-\xi_*)}$. Since $w(\xi)$ is non-increasing in $\mathbb{R}$, we have

$$\begin{aligned} &\frac{w(\xi_{i+m,j+n}(t+\tau))}{w(\xi_{i,j}(t))} \\ &\le \frac{w(\xi_{i+m,j+n}(t))}{w(\xi_{i,j}(t))} \\ &= \begin{cases} e^{(\xi_{i,j}(t)-\xi_*)} \le 1, & \text{if } \xi_{i+m,j+n}(t) \ge \xi_*, \\ e^{(\xi_{i,j}(t)-\xi_{i+m,j+n}(t))} = e^{-m\cos\theta - n\sin\theta}, & \text{if } \xi_{i+m,j+n}(t) < \xi_*. \end{cases} \end{aligned}$$

Hence,

$$\frac{w(\xi_{i+m,j+n}(t+\tau))}{w(\xi_{i,j}(t))} \le \max\{1, e^{-m\cos\theta-n\sin\theta}\} \quad \text{for any } m,n\in\mathbb{Z}.$$

Similarly, it is easy to verify that

$$\begin{aligned} \mathcal{L}(w)(\xi_{i,j}(t)) &\le \frac{w(\xi_{i-1,j}(t))}{w(\xi_{i,j}(t))} + \frac{w(\xi_{i,j-1}(t))}{w(\xi_{i,j}(t))} + 2 \\ &\le \max\{1,e^{\cos\theta}\} + \max\{1,e^{\sin\theta}\} + 2 \le 2(e+1). \end{aligned}$$

Thus, we have

$$\begin{aligned} &B_{i,j}(\mu,t) \\ &\ge c + 2d_m - 2D_m(e-1) - 2\mu - L_1\varpi \\ &\quad - L_1\frac{\varpi}{4\pi^2}e^{2\mu\tau}\sum_{m,n\in\mathbb{Z}}\beta_\alpha(m)\gamma_\alpha(n)\max\{1,e^{-m\cos\theta-n\sin\theta}\} \\ &= c + 2d_m - 2D_m(e-1) - L_1\varpi(1+L_2) - 2\mu - L_1L_2\varpi(e^{2\mu\tau}-1) \\ &= C_1(\mu) > 0 \quad \text{for } 0<\mu<\mu_1, \end{aligned}$$

provided that (2.6) holds.

Case (ii): $\xi_{i,j}(t) \geq \xi_*$. In this case $w(\xi_{i,j}(t)) = 1$. Similarly, we can show that

$$\frac{w(\xi_{i+m,j+n}(t+\tau))}{w(\xi_{i,j}(t))} \leq \max\{1, e^{-m\cos\theta - n\sin\theta}\} \quad \text{for any } m,n \in \mathbb{Z},$$

and

$$\mathcal{L}(w)(\xi_{i,j}(t)) \leq \max\{1, e^{\cos\theta}\} + \max\{1, e^{\sin\theta}\} + 2 \leq 2(e+1).$$

Note that $\xi_{i-m,j-n}(t-\tau) = \xi_{i,j}(t) - m\cos\theta - n\sin\theta - c\tau$. It follows from Lemma 2.2 that

$$\begin{aligned}
B_{i,j}(\mu,t) &\geq 2d_m - 2D_m(e-1) \\
&\quad - \frac{\varpi}{4\pi^2} b'\big(\phi(\xi_{i,j}(t))\big) \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) \max\{1, e^{-m\cos\theta - n\sin\theta}\} \\
&\quad - \frac{\varpi}{4\pi^2} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(t-\tau))\big) - 2\mu \\
&\quad - \frac{\varpi}{4\pi^2}(e^{2\mu\tau}-1) b'\big(\phi(\xi_{i,j}(t))\big) \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) \max\{1, e^{-m\cos\theta - n\sin\theta}\} \\
&\geq 2d_m - 2D_m(e-1) - \varpi b'(K)(1+L_2) - \bar{\epsilon} - 2\mu \\
&\quad - \frac{\varpi}{4\pi^2}(e^{2\mu\tau}-1) L_1 \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) \max\{1, e^{-m\cos\theta - n\sin\theta}\} \\
&= d_m - D_m(e-1) - \frac{1}{2}\varpi b'(K)(1+L_2) - 2\mu - L_1 L_2 \varpi (e^{2\mu\tau}-1) \\
&= C_2(\mu) > 0 \text{ for } 0 < \mu < \mu_2.
\end{aligned}$$

Now, let $0 < \mu < \mu_0 := \min\{\mu_1, \mu_2\}$, then $B_{i,j}(\mu,t) \geq C_0(\mu) > 0$ for all $i,j \in \mathbb{Z}$, $t > 0$. This completes the proof. □

Lemma 3.4 (Weighted energy estimate)**.** *There exists $\mu > 0$ such that*

$$\|U(t)\|_{l^2_w} \leq \Big(\|U^0(0)\|^2_{l^2_w} + C_2 \int_{-\tau}^0 \|U^0(s)\|^2_{l^2_w}\, ds \Big)^{1/2} e^{-\mu t}, \ t \geq 0$$

for some constant $C_2 > 0$.

Proof. Multiplying (3.3) by $e^{2\mu t} w(\xi_{i,j}(t)) U_{i,j}(t)$ for $0 < \mu < \mu_0$, we have

$$\begin{aligned}
&\Big(\frac{1}{2} e^{2\mu t} w U^2_{i,j}(t)\Big)_t + \Big(-\frac{c}{2}\frac{w_\xi}{w} + d_m - \mu\Big) e^{2\mu t} w U^2_{i,j}(t) \\
&- D_m e^{2\mu t} w [U_{i+1,j}(t) + U_{i-1,j}(t) + U_{i,j+1}(t) + U_{i,j-1}(t) - 4U_{i,j}(t)] U_{i,j}(t) \\
&- \frac{\varpi}{4\pi^2} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(t-\tau))\big) U_{i-m,j-n}(t-\tau) e^{2\mu t} w U_{i,j}(t) \\
&= e^{2\mu t} w U_{i,j}(t) G_{i,j}(t),
\end{aligned} \tag{3.5}$$

where $w = w(\xi_{i,j}(t))$. Noting that $2U_{i\pm1,j\pm1}(t)U_{i,j}(t) \leq U^2_{i\pm1,j\pm1}(t) + U^2_{i,j}(t)$, and

$$\begin{aligned}
G_{i,j}(t) = \frac{\varpi}{4\pi^2} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) \frac{1}{2} b''\Big(&\theta_1 U_{i-m,j-n}(t-\tau) \\
&+ \phi(\xi_{i-m,j-n}(t-\tau))\Big) U^2_{i-m,j-n}(t-\tau)\big] \leq 0
\end{aligned}$$

for all $i, j \in \mathbb{Z}$, $t > 0$, where $\theta_1 \in (0, 1)$, substituting these into (3.5), we obtain

$$\begin{aligned} &\big(e^{2\mu t} w U_{i,j}^2(t)\big)_t + \Big(-c\frac{w_\xi}{w} + 2d_m - 2\mu\Big) e^{2\mu t} w U_{i,j}^2(t) \\ &- D_m e^{2\mu t} w [U_{i+1,j}^2(t) + U_{i-1,j}^2(t) + U_{i,j+1}^2(t) + U_{i,j-1}^2(t) - 4U_{i,j}^2(t)] \\ &- \frac{\varpi}{2\pi^2} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(t-\tau))\big) U_{i-m,j-n}(t-\tau) e^{2\mu t} w U_{i,j}(t) \\ &\leq 0. \end{aligned} \tag{3.6}$$

Summing (3.6) about all $i, j \in \mathbb{Z}$ and integrating the inequality over $[0, t]$, we have

$$\begin{aligned} &e^{2\mu t}\|U(t)\|_{l_w^2}^2 - \frac{\varpi}{2\pi^2}\int_0^t \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(s-\tau))\big) \\ &\times U_{i-m,j-n}(s-\tau) e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}(s) ds \\ &+ \int_0^t \sum_{i,j\in\mathbb{Z}} \Big[-c\frac{w_\xi(\xi_{i,j}(t))}{w(\xi_{i,j}(s))} + 2(d_m - \mu) - D_m\big(\mathcal{L}(w)(\xi_{i,j}(s)) - 4\big)\Big] \\ &\times e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s) ds \\ &\leq \|U^0(0)\|_{L_w^2}^2. \end{aligned} \tag{3.7}$$

Using the inequality $2ab \leq a^2 + b^2$ and making changes of variables $s - \tau \to s$, $i - m \to i$, and $j - n \to j$, we obtain

$$\begin{aligned} &\frac{\varpi}{2\pi^2}\int_0^t \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(s-\tau))\big) \\ &\times U_{i-m,j-n}(s-\tau) e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}(s) ds \\ &\leq \frac{\varpi}{4\pi^2}\int_0^t \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(s-\tau))\big) e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s) ds \\ &\quad + \frac{\varpi}{4\pi^2}\int_0^t \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) \\ &\quad \times b'\big(\phi(\xi_{i-m,j-n}(s-\tau))\big) e^{2\mu s} w(\xi_{i,j}(s)) U_{i-m,j-n}^2(s-\tau) ds \\ &\leq \frac{\varpi}{4\pi^2}\int_0^t \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i-m,j-n}(s-\tau))\big) e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s) ds \\ &\quad + \frac{\varpi}{4\pi^2} e^{2\mu\tau}\int_0^t \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i,j}(s))\big) \\ &\quad \times \frac{w(\xi_{i+m,j+n}(s+\tau))}{w(\xi_{i,j}(s))} e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s) ds \\ &\quad + \frac{\varpi}{4\pi^2} e^{2\mu\tau}\int_{-\tau}^0 \sum_{i,j\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i,j}(s))\big) \\ &\quad \times \frac{w(\xi_{i+m,j+n}(s+\tau))}{w(\xi_{i,j}(s))} e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s) ds \end{aligned} \tag{3.8}$$

From the proof of Lemma 3.3, we see that

$$\frac{w(\xi_{i+m,j+n}(t+\tau))}{w(\xi_{i,j}(t))} \le \max\{1, e^{-m\cos\theta - n\sin\theta}\} \quad \text{for any } i,j,m,n \in \mathbb{Z}.$$

Thus, we have

$$\begin{aligned}
&\frac{\varpi}{4\pi^2} e^{2\mu\tau} \int_{-\tau}^{0} \sum_{i,j\in\mathbb{Z}} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) b'\big(\phi(\xi_{i,j}(s))\big) \frac{w(\xi_{i+m,j+n}(s+\tau))}{w(\xi_{i,j}(s))} \\
&\quad \times e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s)ds \\
&\le \frac{\varpi}{4\pi^2} L_1 e^{2\mu\tau} \int_{-\tau}^{0} \sum_{i,j\in\mathbb{Z}} \sum_{m,n\in\mathbb{Z}} \beta_\alpha(m)\gamma_\alpha(n) \max\{1, e^{-m\cos\theta - n\sin\theta}\} \\
&\quad w(\xi_{i,j}(s)) U_{i,j}^2(s)ds \\
&= \varpi L_1 e^{2\mu\tau} L_2 \int_{-\tau}^{0} \|U^0(s)\|_{l_w^2}^2 ds.
\end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we have

$$\begin{aligned}
&e^{2\mu t} \|U(t)\|_{l_w^2}^2 + \int_0^t \sum_{i,j} B_{i,j}(\mu, s) e^{2\mu s} w(\xi_{i,j}(s)) U_{i,j}^2(s)ds \\
&\le \|U^0(0)\|_{l_w^2}^2 + C_2 \int_{-\tau}^{0} \|U^0(s)\|_{l_w^2}^2 ds,
\end{aligned} \tag{3.10}$$

where $C_2 = \varpi L_1 e^{2\mu_0\tau} L_2 > 0$. It then follows from Lemma 3.3 that

$$\|U(t)\|_{l_w^2} \le \Big(\|U^0(0)\|_{l_w^2}^2 + C_2 \int_{-\tau}^{0} \|U^0(s)\|_{l_w^2}^2 ds\Big)^{1/2} e^{-\mu t} \text{ for } t \ge 0.$$

This completes the proof. □

3.2. **Proof of Theorem 2.3.** By Lemma 3.4 and the standard Sobolev's embedding inequality $l^2 \hookrightarrow l^\infty$ and $l_w^2 \hookrightarrow l^2$ for $w(\cdot) \ge 1$ defined as in (2.5), we obtain the convergence of $u_{i,j}^+(t)$, that is there exists a constant $\mu_1^0 > 0$ such that

$$\sup_{i,j\in\mathbb{Z}} |u_{i,j}^+(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le C_2 e^{-\mu_1^0 t}, \quad t \ge 0,$$

for some constant $C_2 > 0$.

Let $V_{i,j}(t) = \phi(i\cos\theta + j\sin\theta + ct) - u_{i,j}^-(t)$. We can similarly prove that $u_{i,j}^-(t)$ converges to $\phi(i\cos\theta + j\sin\theta + ct)$, i.e. there exists a constant $\mu_2^0 > 0$ such that

$$\sup_{i,j\in\mathbb{Z}} |u_{i,j}^-(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le C_3 e^{-\mu_2^0 t}, \quad t \ge 0,$$

for some constant $C_3 > 0$.

Take $\mu^0 = \min\{\mu_1^0, \mu_2^0\}$, Note that $u_{i,j}^-(t) \le u_{i,j}(t) \le u_{i,j}^+(t)$ for $i,j \in \mathbb{Z}, t \ge 0$. Using the Squeeze Theorem, we can easily show that $u_{i,j}(t)$ converges to $\phi(i\cos\theta + j\sin\theta + ct)$; that is,

$$\sup_{i,j\in\mathbb{Z}} |u_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le C_4 e^{-\mu^0 t}, \quad t \ge 0,$$

for some constant $C_4 > 0$. We now complete the proof of Theorem 2.3.

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