Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 180, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF POSITIVE SOLUTIONS FOR KIRCHHOFF TYPE EQUATIONS

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Abstract. In this article, we are interested in the existence of positive solutions for the Kirchhoff type problems

$$
\begin{gathered}
-M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\Delta_p u = \lambda f(u) \quad \text{in } \Omega,\\
u>0 \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $1<p<N$, $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function, λ is a parameter, $f:[0,+\infty)\to\mathbb{R}$ is a C^1 nondecreasing function satisfying $f(0)<0$ (semipositone). Our proof is based on the sub- and super-solutions techniques.

1. Introduction

In this article, we are interested in the existence of positive solutions for Kirchhoff type problems of the form

$$
\begin{gathered}
-M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\Delta_p u = \lambda f(u) \quad \text{in } \Omega,\\
u>0 \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $1<p<N$, $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function, $f:[0,+\infty)\to\mathbb{R}$ is a C^1 nondecreasing function such that $f(0)<0$ (semipositone) and there exist $r>\alpha>0$ such that $f(s)(s-\alpha)\ge 0$.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [4]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$
\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}
$$

presented by Kirchhoff in 1883, see [11]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the

2000 *Mathematics Subject Classification.* 35D05, 35J60.
Key words and phrases. Kirchhoff type problems; semipositone; positive solution; sub-supersolution method.

Submitted April 23, 2013. Published August 7, 2013.

following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [1, 2, 5, 6, 7, 12, 14, 15, 16], in which the authors have used variational method and topological method to get the existence of solutions for (1.1) in the cases when f could satisfy p-superlinear, p-sublinear or p-linear growth condition at infinity. In this paper, motivated by the ideas introduced in [3] and the properties of Kirchhoff type operators in [8, 9, 10], we study problem (1.1) in the semipositone case; i.e., $f(0) < 0$. Using the sub- and supersolutions techniques, we prove the existence of a positive solution for the problem in a range of λ without assuming any condition on f at infinity. To our best knowledge, this is a new research topic for nonlocal problems, see [10].

In order to state precisely our main result we first consider the eigenvalue problem for the p-Laplace operator $-\Delta_p u$:

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (1.3) such that $\phi_1 > 0$ in Ω and $\|\phi_1\|_\infty = 1$. It can be shown that $\frac{\partial \phi_1}{\partial \eta} < 0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m, δ, σ such that

$$\begin{gathered} |\nabla \phi_1|^p - \lambda_1 \phi_1^p \geq m \quad \text{in } \overline{\Omega}_\delta, \\ \phi_1 \geq \sigma \quad \text{in } \Omega \setminus \overline{\Omega}_\delta, \end{gathered} \tag{1.4}$$

where $\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$.

We will also consider the unique solution $e \in C^1(\overline{\Omega})$ of the boundary value problem

$$\begin{gathered} -\Delta_p e = 1 \quad \text{in } \Omega, \\ e = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.5}$$

to discuss our result. It is known that $e > 0$ in Ω and $\frac{\partial e}{\partial \eta} < 0$ on $\partial\Omega$.

For our main result we assume that there exist positive constants M_0, M_∞ and $l_1, l_2 \in (\alpha, r]$ satisfying

- (H1) $M_0 \leq M(t) \leq M_\infty$ for all $t \in \mathbb{R}^+$;
- (H2) $l_2 \geq k l_1$, where $k = k(\Omega) = \frac{p}{p-1} \lambda_1^{\frac{1}{p-1}} \sigma^{\frac{p}{1-p}} \|e\|_{L^\infty(\Omega)}$;
- (H3) $\frac{M_\infty \lambda_1}{f(l_1)} < \frac{m M_0}{|f(0)|}$;
- (H4) $\frac{l_2^{p-1}}{f(l_2)} > \mu \frac{l_1^{p-1}}{f(l_1)}$, where $\mu = \mu(\Omega) = \frac{M_\infty \lambda_1}{M_0 \sigma^p} \left(\frac{p\|e\|_{L^\infty(\Omega)}}{p-1} \right)^{p-1}$.

Our main results reads as follows.

Theorem 1.1. *Under assumptions* (H1)–(H4)*, there exist two positive constants* λ_* *and* λ^* *such that* (1.1) *has a positive solution for all* $\lambda \in (\lambda_*, \lambda^*)$.

2. Preliminaries

We will prove our result by using the method of sub- and supersolutions, we refer the readers to a recent paper [10] on the topic. A function ψ is said to be a subsolution of (1.1) if it is in $W^{1,p}(\Omega)\cap C^0(\overline{\Omega})$ such that $\psi=0$ on $\partial\Omega$ and satisfies

$$M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w\,dx \le \lambda\int_\Omega f(\psi)w\,dx,\quad \forall w\in W, \tag{2.1}$$

where $W:=\{w\in C_0^\infty(\Omega): w\ge 0 \text{ in } \Omega\}$. A function $\phi\in W^{1,p}(\Omega)\cap C^0(\overline{\Omega})$ is said to be a supersolution if $\phi=0$ on $\partial\Omega$ and satisfies

$$M\Big(\int_\Omega |\nabla\phi|^p\,dx\Big)\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w\,dx \ge \lambda\int_\Omega f(\phi)w\,dx,\quad \forall w\in W. \tag{2.2}$$

The following result plays an important role in our arguments. For the readers' convenience, we present its proof in detail.

Lemma 2.1. *Assume that $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function satisfying*

$$M(t)\ge M_0>0 \text{ for all } t\in\mathbb{R}^+.$$

If the functions $u,v\in W_0^{1,p}(\Omega)$ satisfy

$$\begin{aligned} &M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx\\ &\le M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \end{aligned} \tag{2.3}$$

for all $\varphi\in W_0^{1,p}(\Omega)$, $\varphi\ge 0$, then $u\le v$ in Ω.

Proof. Our proof is based on the arguments presented in [8, 9]. Define the functional $\Phi: W_0^{1,p}(\Omega)\to\mathbb{R}$ by the formula

$$\Phi(u):=\frac{1}{p}\widehat{M}\Big(\int_\Omega |\nabla u|^p\,dx\Big),\quad u\in W_0^{1,p}(\Omega).$$

It is obviously that the functional Φ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u\in W_0^{1,p}(\Omega)$ is the functional $\Phi'\in W_0^{-1,p}(\Omega)$, given by

$$\Phi'(u)(\varphi)=M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx,\quad \varphi\in W_0^{1,p}(\Omega).$$

It is obvious that Φ' is continuous and bounded since the function M is continuous. We will show that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$. Indeed, for any $u,v\in W_0^{1,p}(\Omega)$, $u\ne v$, without loss of generality, we may assume that

$$\int_\Omega |\nabla u|^p\,dx \ge \int_\Omega |\nabla v|^p\,dx.$$

(otherwise, changing the role of u and v in the following proof). Therefore, we have

$$M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\ge M\Big(\int_\Omega |\nabla v|^p\,dx\Big) \tag{2.4}$$

since $M(t)$ is a monotone function. Using Cauchy's inequality, we have

$$\nabla u\cdot\nabla v\le |\nabla u||\nabla v|\le \frac{1}{2}(|\nabla u|^2+|\nabla v|^2). \tag{2.5}$$

Using (2.5) we obtain

$$\int_\Omega |\nabla u|^p\,dx-\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx\ge \frac{1}{2}\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2-|\nabla v|^2)\,dx, \tag{2.6}$$

$$\int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,dx \ge \frac12\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2-|\nabla u|^2)\,dx. \tag{2.7}$$

If $|\nabla u|\ge|\nabla v|$, using (2.4)-(2.7), we have

$$\begin{aligned}
I_1 &:= \Phi'(u)(u)-\Phi'(u)(v)-\Phi'(v)(u)+\Phi'(v)(v)\\
&= M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\Big(\int_\Omega |\nabla u|^p\,dx - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx\Big)\\
&\quad - M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\Big(\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,dx - \int_\Omega |\nabla v|^p\,dx\Big)\\
&\ge \frac12 M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2-|\nabla v|^2)\,dx\\
&\quad - \frac12 M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2-|\nabla v|^2)\,dx\\
&= \frac12 M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\int_\Omega (|\nabla u|^{p-2}-|\nabla v|^{p-2})(|\nabla u|^2-|\nabla v|^2)\,dx\\
&\ge \frac{M_0}{2}\int_\Omega (|\nabla u|^{p-2}-|\nabla v|^{p-2})(|\nabla u|^2-|\nabla v|^2)\,dx.
\end{aligned} \tag{2.8}$$

If $|\nabla v|\ge|\nabla u|$, changing the role of u and v in (2.4)-(2.7), we have

$$\begin{aligned}
I_2 &:= \Phi'(v)(v)-\Phi'(v)(u)-\Phi'(u)(v)+\Phi'(u)(u)\\
&= M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\Big(\int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,dx\Big)\\
&\quad - M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\Big(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx - \int_\Omega |\nabla u|^p\,dx\Big)\\
&\ge \frac12 M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2-|\nabla u|^2)\,dx\\
&\quad - \frac12 M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}(|\nabla v|^2-|\nabla u|^2)\,dx\\
&= \frac12 M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\int_\Omega (|\nabla v|^{p-2}-|\nabla u|^{p-2})(|\nabla v|^2-|\nabla u|^2)\,dx\\
&\ge \frac{M_0}{2}\int_\Omega (|\nabla v|^{p-2}-|\nabla u|^{p-2})(|\nabla v|^2-|\nabla u|^2)\,dx.
\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we have

$$\Big(\Phi'(u)-\Phi'(v)\Big)(u-v) = I_1 = I_2 \ge 0, \quad \forall u,v\in W_0^{1,p}(\Omega). \tag{2.10}$$

Moreover, if $u\ne v$ and $\Big(\Phi'(u)-\Phi'(v)\Big)(u-v)=0$, then we have

$$\int_\Omega (|\nabla u|^{p-2}-|\nabla v|^{p-2})(|\nabla u|^2-|\nabla v|^2)\,dx = 0,$$

so $|\nabla u|=|\nabla v|$ in Ω. Thus, we deduce that

$$\begin{aligned}
\Big(\Phi'(u)-\Phi'(v)\Big)(u-v) &= \Phi'(u)(u-v)-\Phi'(v)(u-v)\\
&= M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}|\nabla u-\nabla v|^2\,dx = 0;
\end{aligned} \tag{2.11}$$

i.e., $u-v$ is a constant. In view of $u=v=0$ on $\partial\Omega$ we have $u\equiv v$ which is contrary with $u\neq v$. Therefore $\big(\Phi'(u)-\Phi'(v)\big)(u-v)>0$ and Φ' is strictly monotone in $W_0^{1,p}(\Omega)$.

Let u,v be two functions such that (2.3) is satisfied. Taking $\varphi=(u-v)^+$, the positive part of $u-v$, as a test function of (2.3), we have

$$\begin{aligned}(\Phi'(u)-\Phi'(v))(\varphi) &= M\Big(\int_\Omega |\nabla u|^p\,dx\Big)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\ &\quad - M\Big(\int_\Omega |\nabla v|^p\,dx\Big)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \le 0.\end{aligned}\tag{2.12}$$

Relations (2.11) and (2.12) imply that $u\le v$. □

From Lemma 2.1 we obtain the following basic principle of the sub- and super-solutions method.

Theorem 2.2 ([10]). *Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous and increasing function satisfying*

$$M(t)\ge M_0>0 \quad \textit{for all } t\in\mathbb{R}^+.$$

Assume that f satisfies the subcritical growth condition

$$|f(x,t)|\le C(1+|t|^{q-1}),\quad \forall x\in\Omega,\ \forall t\in R,$$

where $1<q<p^*=\frac{Np}{N-p}$, and the function $f(x,t)$ is nondecreasing in $t\in R$. If there exist a subsolution $\underline{u}\in W^{1,p}(\Omega)$ and a supersolution $\overline{u}\in W^{1,p}(\Omega)$ of problem (1.1)*, then* (1.1) *has a minimal solution u_* and a maximal solution u^* in the order interval $[u_*,u^*]$; i.e., $\underline{u}\le u_*\le u^*\le\overline{u}$ and if u is any solution of* (1.1) *such that $\underline{u}\le u\le\overline{u}$, then $u_*\le u\le u^*$.*

In practice problems, it is often known that the subsolution $\underline{u}$ and the supersolution $\overline{u}$ are in $L^\infty(\Omega)$, so the restriction on the growth condition of f is needless. Hence, the following theorem is more suitable for our framework.

Theorem 2.3 ([10]). *Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous and increasing function satisfying*

$$M(t)\ge M_0>0 \quad \textit{for all } t\in\mathbb{R}^+.$$

Assume that $\underline{u},\overline{u}$ are a subsolution and a super-solution of problem (1.1) *such that $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $\underline{u}\le\overline{u}$ in Ω. If $f\in C(\overline{\Omega}\times R,R)$ is nondecreasing in $t\in[\inf_\Omega\underline{u},\sup_\Omega\overline{u}]$ then the conclusion of Theorem 2.2 is valid.*

3. Proof of main result

In this section, we prove Theorem 1.1 by using the sub- and super-solutions method. Our arguments are similar to those presented in [3].

First we construct a positive subsolution of problem (1.1). For this purpose, we let $\psi = l_1 \sigma^{\frac{p}{1-p}} \phi_1^{\frac{p}{p-1}}$. Since $\nabla\psi = \frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\phi_1^{\frac{1}{p-1}}\nabla\phi_1$, we deduce that

$$
\begin{aligned}
& M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega |\psi|^{p-2}\nabla\psi\cdot\nabla w\,dx \\
&= \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega \phi_1|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla w\,dx \\
&= \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega |\nabla\phi_1|^{p-2}\nabla\phi_1\cdot[\nabla(\phi_1 w) - w\nabla\phi_1]\,dx \\
&= \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega |\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla(\phi_1 w)\,dx \\
&\quad - \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega |\nabla\phi_1|^p w\,dx \\
&= \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega \lambda_1|\phi_1|^{p-2}\phi_1(\phi_1 w)\,dx \\
&\quad - \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega |\nabla\phi_1|^p w\,dx \\
&= \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega [\lambda_1\phi_1^p - |\nabla\phi_1|^p] w\,dx.
\end{aligned}
\tag{3.1}
$$

Thus ψ is a subsolution of problem (1.1) if

$$
\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)\int_\Omega [\lambda_1\phi_1^p - |\nabla\phi_1|^p] w\,dx \le \lambda\int_\Omega f(\psi) w\,dx \tag{3.2}
$$

On $\overline{\Omega}_\delta$, we have

$$
|\nabla\phi_1|^p - \lambda_1\phi_1^p \ge m \tag{3.3}
$$

and therefore, by (H1),

$$
\begin{aligned}
&\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \\
&\le -mM_0\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} \le \lambda f(\psi)
\end{aligned}
\tag{3.4}
$$

if

$$
\lambda \le \overline{\lambda} := \frac{mM_0}{|f(0)|}.\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1}. \tag{3.5}
$$

On $\Omega\setminus\Omega_\delta$ we have $\phi_1 \ge \sigma$ and therefore,

$$
\psi = l_1\sigma^{\frac{p}{1-p}}\phi_1^{\frac{p}{p-1}} \ge l_1\sigma^{\frac{p}{1-p}}\sigma^{\frac{p}{p-1}} = l_1. \tag{3.6}
$$

Thus, by (H1),

$$
\begin{aligned}
&\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M\Big(\int_\Omega |\nabla\psi|^p\,dx\Big)[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \\
&\le \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1} M_\infty\lambda_1 \le \lambda f(\psi)
\end{aligned}
\tag{3.7}
$$

if

$$
\lambda \ge \lambda_* := \frac{M_\infty\lambda_1}{f(l_1)}\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1}. \tag{3.8}
$$

By condition (H3), we have $\lambda_* < \overline{\lambda}$. Therefore, ψ is a subsolution of problem (1.1) for all $\lambda_* \le \lambda \le \overline{\lambda}$.

Next, we construct a supersolution of (1.1). Let $\phi = \frac{l_2}{\|e\|_{L^\infty(\Omega)}} e$, in which e is defined by (1.5). Then, by (H1), ϕ is a supersolution of problem (1.1) if

$$\begin{aligned}
&M\Big(\int_\Omega |\nabla\phi|^p\,dx\Big)\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w\,dx \\
&= M\Big(\int_\Omega |\nabla\phi|^p\,dx\Big)\Big(\frac{l_2}{\|e\|_{L^\infty(\Omega)}}\Big)^{p-1}\int_\Omega |\nabla e|^{p-2}\nabla e\cdot\nabla w\,dx \\
&= M\Big(\int_\Omega |\nabla\phi|^p\,dx\Big)\Big(\frac{l_2}{\|e\|_{L^\infty(\Omega)}}\Big)^{p-1}\int_\Omega w\,dx \\
&\ge M_0\Big(\frac{l_2}{\|e\|_{L^\infty(\Omega)}}\Big)^{p-1}\int_\Omega w\,dx \\
&\ge \lambda\int_\Omega f(\phi)w\,dx, \quad \forall w \in W.
\end{aligned} \tag{3.9}$$

But $f(\phi) \le f(l_2)$ and hence ϕ is a supersolution of problem (1.1) if

$$\lambda \le \widehat{\lambda} := \frac{M_0 l_2^{p-1}}{\|e\|_{L^\infty(\Omega)}^{p-1} f(l_2)}. \tag{3.10}$$

By (H4), we have $\widehat{\lambda} > \lambda_*$. We have

$$\begin{aligned}
-\Delta_p\phi &= -\nabla(|\nabla\phi|^{p-2}\nabla\phi) \\
&= -\Big(\frac{l_2}{\|e\|_{L^\infty(\Omega)}}\Big)^{p-1}\nabla(|\nabla e|^{p-2}\nabla e) \\
&= \frac{l_2^{p-1}}{\|e\|_{L^\infty(\Omega)}^{p-1}}
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
-\Delta_p\psi &= -\nabla(|\nabla\psi|^{p-2}\nabla\psi) \\
&= \Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1}[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \\
&\le \lambda_1\Big(\frac{pl_1}{p-1}\sigma^{\frac{p}{1-p}}\Big)^{p-1}.
\end{aligned} \tag{3.12}$$

By condition (H2), using the weak comparison principle for the p-Laplace operator $-\Delta_p u$, we see that $\psi \le \phi$ in Ω.

Set $\lambda^* := \min\{\overline{\lambda}, \widehat{\lambda}\}$. By Theorem 2.3, we conclude that problem (1.1) has a positive solution for any $\lambda \in (\lambda_*, \lambda^*)$.

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