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# MULTIPLE POSITIVE SOLUTIONS FOR DEGENERATE ELLIPTIC EQUATIONS WITH CRITICAL CONE SOBOLEV EXPONENTS ON SINGULAR MANIFOLDS

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Abstract. In this article, we show the existence of multiple positive solutions to a class of degenerate elliptic equations involving critical cone Sobolev exponent and sign-changing weight function on singular manifolds with the help of category theory and the Nehari manifold method.

## 1. Introduction

In this article, we consider the semilinear boundary-value problem

$$\begin{gathered} -\Delta_{\mathbb{B}} u = f_\lambda |u|^{q-2}u + g(x)|u|^{2^*-2}u, \quad x \in \operatorname{int}\mathbb{B}, \\ u = 0, \quad x \in \partial\mathbb{B}, \end{gathered} \tag{1.1}$$

where $1 < q < 2$, $2^* = \frac{2n}{n-2}$ $(n \geq 3)$. Here the domain $\mathbb{B}$ is $[0,1) \times X$ for $X \subseteq \mathbb{R}^{n-1}$ compact, which is regarded as the local model near the conical points on manifolds with conical singularities and $\{0\} \times X \subset \partial\mathbb{B}$. Moreover, the operator $\Delta_{\mathbb{B}}$ in (1.1) is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \dots + \partial_{x_n}^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $x_1 = 0$ (we also call it Fuchsian type Laplacian), and the corresponding gradient operator is denoted by $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$. Near $\partial\mathbb{B}$ we will often use coordinates $(x_1, x') = (x_1, x_2, \dots, x_n)$ for $0 \leq x_1 < 1$, $x \in X$. Our goal is to find the existence of multiple positive solutions for (1.1) in the cone Sobolev space $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$. The definition of such distribution spaces will be given in the next section. Of course, the nonlinear terms in (1.1) need to satisfy the following conditions.

- (H1) the parameter $\lambda > 0$ and $f$, $g : \overline{\mathbb{B}} \to \mathbb{R}$ are continuous and sign-changing functions in $\overline{\mathbb{B}}$. The function $f_\lambda = \lambda f_+ + f_-$ and $f_\pm = \pm\max\{\pm f(x), 0\}$.
- (H2) there exists a non-empty closed set $M = \{x \in \overline{\mathbb{B}}; g(x) = \max_{x\in\overline{\mathbb{B}}} g(x) \equiv 1\}$ and $\rho > n-2$ such that $M \subset \{x \in \operatorname{int}\mathbb{B}; f(x) > 0\}$ and
$$g(z) - g(x) = o(|x-z|_{\mathbb{B}}^\rho) \quad \text{as } x \to z \text{ and uniformly in } z \in M.$$
Here $|\cdot|_{\mathbb{B}}$ means $|x-z|_{\mathbb{B}} = (|\ln\frac{x_1}{z_1}|^2 + |x'-z'|^2)^{1/2}$, where $x = (x_1, x') = (x_1, x_2, \dots, x_n)$ and $z = (z_1, z') = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n_+$.

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**Remark 1.1.** Let $M_r = \{x \in \mathbb{R}^n_+; \mathrm{dist}_{\mathbb{B}}(x, M) < r\}$ for $r > 0$, where $\mathrm{dist}_{\mathbb{B}}(x, M) = \max_{z\in M} |x - z|_{\mathbb{B}}$. Then, by the condition (H2), we may assume that there exist two positive constants $c_0 > 0$ and $r_0 > 0$ such that $f(x)$ and $g(x)$ are positive for all $x \in M_{r_0} \subset \mathbb{B}$ and $g(z) - g(x) = c_0(|x - z|^{\rho}_{\mathbb{B}})$ for all

$$x \in \Omega_{r_0}(z_1, z') := \{(x_1, x') \in \mathbb{R}^n_+; |x - z|_{\mathbb{B}} = (|\ln(\frac{x_1}{z_1})|^2 + |x' - z'|^2)^{1/2} \le r_0\}$$

for all $z \in M$.

The analysis on manifolds with conical singularities and the properties of elliptic, parabolic and hyperbolic equations in this setting have been intensively studied in the previous decades. More specially, in aspects of partial differential equations and pseudo-differential theory of configurations with piecewise smooth geometry, the work of Kondrat'ev (see [9]) has to be mentioned here as the starting point of the analysis of operators on manifolds with conical singularities. The foundations of this analysis have been developed through the fundamental works by Schulze, and subsequently further expended by him and his collaborators, such as Gil, Seiler, Krainer. The main subject of their work is the calculus on manifolds with singularities (see [15] and the references therein). On the other hand, Melrose and his collaborators gave various methods and ideas in the pseudo-differential calculus on manifolds with singularities, cf. Melrose and Mendoza [12]. All these mathematicians investigated deeply the underlying pseudo-differential calculi and the connected functional spaces. While these theories are nowadays well-established, many aspects are still to be interested, for instance, the existence theorem for the corresponding nonlinear elliptic equations on manifolds with singularities.

Recently, the authors in [3] established the so-called cone Sobolev inequality (see Proposition 2.4) and Poincaré inequality (see Proposition 2.5) for the weighted Sobolev spaces (in Section 2) (see [3] for details). Such kind of inequalities seem to be of fundamental importance to prove the existence of the solutions for such nonlinear problems with totally characteristic degeneracy. In [3], the authors have already obtained the existence theorem for a class of semilinear degenerate equations on manifolds with conical singularities; that is, for the Dirichlet problem

$$\begin{gathered}
-\Delta_{\mathbb{B}} u = |u|^{p-2}u, \quad x \in \operatorname{int}\mathbb{B}, \\
u = 0, \quad x \in \partial\mathbb{B},
\end{gathered}$$

there exists a non-trivial solution $u$ in $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ with $2 < p < 2^* = \frac{2n}{n-2}$. In [4], they proved that the Dirichlet problem

$$\begin{gathered}
-\Delta_{\mathbb{B}} u = \lambda u + |u|^{2^*-2}u, \quad x \in \operatorname{int}\mathbb{B}, \\
u = 0, \qquad x \in \partial\mathbb{B}
\end{gathered} \tag{1.2}$$

admits infinitely many solutions in $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ for $n \ge 7$, where $\lambda > 0$, and $2^* = \frac{2n}{n-2}$. The authors in [2] proved that for any $\lambda \in (0, \lambda_1)$, that (1.2) has a positive solution in $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ for $n \ge 4$, where $\lambda_1$ denotes the first eigenvalue of $-\Delta_{\mathbb{B}}$ with zero Dirichlet condition on $\partial\mathbb{B}$. Also, the existence and multiplicity of solutions of (1.1) may be influenced by the concave and convex nonlinearities is an interesting problem. In this paper, our main result is the following theorem.

**Theorem 1.2.** *For each* $\delta < r_0$, (1.1) *satisfies conditions* (H1) *and* (H2)*, then there exists* $\Lambda_\delta > 0$ *such that for* $\lambda < \Lambda_\delta$, (1.1) *has at least* $\operatorname{cat}_{M_\delta}(M) + 1$ *positive solutions in* $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$.

The notation $\operatorname{cat}_{M_\delta}(M)$ is the Lusternik-Schnirelman category. Now, we introduce the energy functional $J_\lambda$ on $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$:

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{q}\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' - \frac{1}{2^*}\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx', \tag{1.3}$$

Then $J_\lambda(u) \in C^1(\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}), \mathbb{R})$. Thus the semilinear Equation (1.1) is the Euler-Lagrange Equation of variational problem for the energy functional (1.3) and the critical point of $J_\lambda(u)$ in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ is the weak solution of (1.1).

We organize this article as follows: Firstly, we introduce some definitions and results on cone Sobolev spaces in Section 2. Furthermore, we study the decomposition of the Nehari manifold via the combination of concave and convex nonlinearities and get a positive ground-state solution of (1.1) in Section 3. Moreover, we use the idea of category to get multiple positive solutions of (1.1) and give the proof of Theorem 1.2 in Section 4. In this article, positive constants (possibly different) will be denoted by $c$.

## 2. Preliminaries

Here we first introduce the cone Sobolev spaces. Let $X$ be a closed, compact $C^\infty$ manifold of dimension $n-1$, and set $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ which is the local model interpreted as a cone with the base $X$.

A finite dimensional manifold $B$ with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \dots, b_M\} \subset B$ of conical singularities. For the rest of this article, we assume that the manifold $B$ is paracompact and of dimension $n$, and $\mathbb{B}$ the stretched manifold associated with $B$. Then the stretched manifold $\mathbb{B}$ is a $C^\infty$ manifold with compact $C^\infty$ boundary $\partial\mathbb{B} \cong \bigcup_{b\in B_0} X(b)$ such that there is a diffeomophism $B \setminus B_0 \cong \mathbb{B} \setminus \partial\mathbb{B} := \operatorname{int}\mathbb{B}$, the restriction of which to $U_1 \setminus B_0 \cong V_1 \setminus \partial\mathbb{B}$ for an open neighbourhood $U_1 \subset B$ near the points of $B_0$ and a collar neighbourhood $V_1 \subset \mathbb{B}$ with $V_1 \cong \bigcup_{b\in B_0}\{[0,1) \times X(b)\}$. In this article, we consider $\mathbb{B} = [0,1) \times X$, and use the coordinates $(x_1, x') \in \mathbb{B}$.

**Definition 2.1.** For $(x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, we say that $u(x_1, x') \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$ if

$$\|u\|_{L_p} = \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{n-1}} x_1^n |u(x_1, x')|^p \frac{dx_1}{x_1} dx'\Big)^{1/p} < +\infty.$$

The weighted $L_p$-spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$, then $x_1^{-\gamma}u(x_1, x') \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$, and

$$\|u\|_{L_p^\gamma} = \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{n-1}} x_1^n |x_1^{-\gamma}u(x_1, x')|^p \frac{dx_1}{x_1} dx'\Big)^{1/p} < +\infty.$$

Now we can define the weighted Sobolev space for $1 \le p < +\infty$.

**Definition 2.2.** For $m \in \mathbb{N}$, and $\gamma \in \mathbb{R}$, the spaces

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n) := \{u \in \mathcal{D}'(\mathbb{R}_+^n); x_1^{\frac{n}{p}-\gamma}(x_1\partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')\} \tag{2.1}$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^{n-1}$, and $|\alpha| + |\beta| \le m$. In other words, if $u(x_1, x') \in \mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$, then $(x_1\partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$.

It is easy to see that $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$ is a Banach space with norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)} = \sum_{|\alpha|+|\beta|\le m} \Big( \int\int_{\mathbb{R}_+^n} x_1^n |x_1^{-\gamma}(x_1\partial_{x_1})^\alpha \partial_{x'}^\beta u(x_1,x')|^p \frac{dx_1}{x_1} dx' \Big)^{1/p}.$$

In this article by a cut-off function we understand any real-valued $\omega(x_1) \in C_0^\infty(\mathbb{B})$ which equals 1 near $\partial\mathbb{B}$.

**Definition 2.3.** Let $\mathbb{B}$ be the stretched manifold associated with $B$. Then $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ denotes the subspace of all $u \in W_{loc}^{m,p}(\operatorname{int}\mathbb{B})$, such that

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{u \in W_{\rm loc}^{m,p}(\operatorname{int}\mathbb{B}); \omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)\}$$

for any cut-off function $\omega$, supported by a collar neighbourhood in $\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined as follows:

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + [1-\omega]W_0^{m,p}(\operatorname{int}\mathbb{B}),$$

where $W_0^{m,p}(\operatorname{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\operatorname{int}\mathbb{B})$ in the Sobolev spaces $W^{m,p}(\widetilde{X})$ when $\widetilde{X}$ is a closed compact $C^\infty$ manifold of dimension $n$ that containing $\mathbb{B}$ as a submanifold with boundary.

We then recall the cone Sobolev inequality and Poincaré inequality. For details we refer to [2, 3].

**Proposition 2.4** (Cone Sobolev Inequality)**.** *Assume that* $1 \le p < n$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, *and* $\gamma \in \mathbb{R}$. *Let* $\mathbb{R}_+^n := \mathbb{R}_+ \times \mathbb{R}^{n-1}, x_1 \in \mathbb{R}_+$ *and* $x' = (x_2, \dots, ) \in \mathbb{R}^{n-1}$. *Then the estimate*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^n)} \le c_1\|u\|_{L_p^\gamma(\mathbb{R}_+^n)} + (c_1 + \alpha c_2)\sum_{i=2}^n \|\partial_{x_i} u\|_{L_p^\gamma(\mathbb{R}_+^n)} + c_2\|u\|_{L_p^\gamma(\mathbb{R}_+^n)} \tag{2.2}$$

*holds for all* $u \in C_0^\infty(\mathbb{R}_+^n)$, *where* $\gamma^* = \gamma - 1$, $c_1 = \frac{(n-1)p}{n(n-p)}$, $\alpha = \frac{(n-1)p}{n-p}$ *and* $c_2 = \frac{|n - \frac{(\gamma-1)(n-1)p}{n-p}|^{\frac{1}{n}}}{n}$. *Moreover, if* $u \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{R}_+^n)$, *we have*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^n)} \le c\|u\|_{\mathcal{H}_p^{1,\gamma}(\mathbb{R}_+^n)}, \tag{2.3}$$

*where the constant* $c = c_1 + c_2$, *and* $c_1, \alpha$ *and* $c_2$ *are given in* (2.2)*.*

**Proposition 2.5** (Poincaré inequality)**.** *Let* $\mathbb{B} = [0,1) \times X$ *be a bounded subset in* $\mathbb{R}_+^n$, *and* $1 < p < +\infty, \gamma \in \mathbb{R}$. *If* $u(x_1, x') \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{B})$, *then*

$$\|u(x_1,x')\|_{L_p^\gamma(\mathbb{B})} \le c\|\nabla_{\mathbb{B}} u(x_1,x')\|_{L_p^\gamma(\mathbb{B})}, \tag{2.4}$$

*where* $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$, *and the constant* $c$ *depending only on* $\mathbb{B}$ *and* $p$.

**Proposition 2.6.** *For* $2 < p < 2^*$, *the embedding* $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \hookrightarrow \mathcal{H}_{p,0}^{0,\frac{n}{p}}(\mathbb{B})$ *is compact.*

It is easy to see that there exist two constant $c, \widetilde{c}$ such that the estimate

$$\|u\|_{L_p^{\frac{n}{p}}(\mathbb{B})} = \|u\|_{\mathcal{H}_{p,0}^{0,\frac{n}{p}}(\mathbb{B})} \le c\|u\|_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \le \widetilde{c}\|\nabla_{\mathbb{B}} u\|_{L_2^{n/2}(\mathbb{B})}$$

holds, so we will use the standard form $\|u\|_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} = \|\nabla_{\mathbb{B}} u\|_{L_2^{n/2}(\mathbb{B})}$. Let

$$S(\mathbb{B}) = \inf_{u\in\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \Big(\frac{\|\nabla_{\mathbb{B}} u\|_{L_2^{n/2}(\mathbb{B})}}{\|u\|_{L_{2^*}^{n/2^*}(\mathbb{B})}}\Big)^2.$$

We obtain the following results.

**Proposition 2.7.** *For any $\mathbb{B}$, we have $S(\mathbb{B}) = S(\mathbb{R}^n_+)$.*

*Proof.* For any domain $\mathbb{B}$, we extend a function $u \in C_0^\infty(\mathbb{B})$ by 0 outside $\mathbb{B}$. We may regard $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ as a subset of $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{R}^n_+)$. Hence we have $S(\mathbb{B}) \geq S(\mathbb{R}^n_+)$. Conversely, if $\{u_m\} \subset \mathcal{H}^{1,n/2}_{2,0}(\mathbb{R}^n_+)$ is a minimizing sequence for $S(\mathbb{R}^n_+)$. By density of $C_0^\infty(\mathbb{R}^n_+)$ in $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{R}^n_+)$, we may assume that $u_m \in C_0^\infty(\mathbb{R}^n_+)$. After translation and scaling

$$u_m \mapsto u_{R_m,\overline{x}_m}(x) = {R_m}^{-n/2^*} u_m(\overline{x}_{m,1}(\frac{x_1}{\overline{x}_{m,1}})^{1/R_m}, \overline{x}'_m + \frac{x'-\overline{x}'_m}{R_m}),$$

where $R_m > 0, \overline{x}_m = (\overline{x}_{m,1}, \dots, \overline{x}_{m,n}) = (\overline{x}_{m,1}, \overline{x}'_m)$, we can achieve that $v_m = u_{R_m,\overline{x}_m}(x) \in C_0^\infty(\mathbb{B})$. Then

$$\|\nabla_{\mathbb{B}} v_m\|_{L_2^{n/2}(\mathbb{B})} = \|\nabla_{\mathbb{B}} u_m\|_{L_2^{n/2}(\mathbb{B})}, \quad \|v_m\|_{L_2^{n/2^*}(\mathbb{B})} = \|u_m\|_{L_2^{n/2^*}(\mathbb{B})}.$$

Indeed, let $y_1 = \overline{x}_{m,1}(\frac{x_1}{\overline{x}_{m,1}})^{1/R_m}, y' = \overline{x}'_m + \frac{x'-\overline{x}'_m}{R_m}$. Then we have

$$\frac{dy_1}{y_1} = \frac{1}{R_m}\frac{dx_1}{x_1}, dy' = \frac{1}{R_m^{n-1}}dx', \quad x_1\partial_{x_1} = \frac{1}{R_m}y_1\partial_{y_1}.$$

It is easy to obtain

$$\begin{aligned}
\|\nabla_{\mathbb{B}} v_m\|^2_{L_2^{n/2}(\mathbb{B})} &= \int_{\mathbb{B}} |\nabla_{\mathbb{B}} v_m|^2 \frac{dx_1}{x_1}dx' \\
&= \int_{\mathbb{B}} |(x_1\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})v_m|^2 \frac{dx_1}{x_1}dx' \\
&= \int_{\mathbb{R}^n_+} |\nabla_{\mathbb{B}} u_m|^2 \frac{dy_1}{y_1}dy' \\
&= \|\nabla_{\mathbb{B}} u_m\|^2_{L_2^{n/2}(\mathbb{R}^n_+)}.
\end{aligned}$$

In an analogous manner, we can get $\|v_m\|_{L_{2^*}^{n/2^*}(\mathbb{B})} = \|u_m\|_{L_{2^*}^{n/2^*}(\mathbb{R}^n_+)}$. Thus $S(\mathbb{B}) \leq S(\mathbb{R}^n_+)$, and so we denote $S := S(\mathbb{B}) = S(\mathbb{R}^n_+)$. This completes the proof. □

**Remark 2.8.** It is easy to check that $S$ is achieved by the function

$$U(x_1, x') = \frac{c}{(1 + |\ln x_1|^2 + |x'|^2)^{(n-2)/2}}.$$

For convenience, we denote the extremal function for $S$ by

$$u_\varepsilon(x) = \frac{\varepsilon^{(n-2)/2}}{(\varepsilon^2 + |\ln x_1|^2 + |x'|^2)^{(n-2)/2}}$$

for $\varepsilon > 0$. Moreover, for each $\varepsilon > 0$,

$$v_\varepsilon(x) = \frac{[n(n-2)\varepsilon^2]^{(n-2)/4}}{(\varepsilon^2 + |\ln x_1|^2 + |x'|^2)^{(n-2)/2}}$$

is a positive solution of critical problem

$$-\Delta_{\mathbb{B}} u = |u|^{2^*-2} u \quad \text{in} \quad \mathbb{R}^n_+$$

with

$$\int_{\mathbb{R}^n_+} |\nabla_{\mathbb{B}} v_\varepsilon|^2 \frac{dx_1}{x_1} dx' = \int_{\mathbb{R}^n_+} |v_\varepsilon|^{2^*} \frac{dx_1}{x_1} dx' = S^{n/2}.$$

For completeness, we also introduce the $(PS)$-sequence, $(PS)_c$ sequence, and $(PS)$ condition.

**Definition 2.9.** Let $E$ be a Banach space, $J \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We say that a sequence $\{u_n\} \subset E$ is a $(PS)_c$ sequence if it satisfies $J(u_n) \to c$ and $\|J'(u_n)\|_{E'} \to 0$, where $J'(\cdot)$ is the Fréchet differentiation of $J$ and $E'$ is the dual space of $E$. Moreover, if any $(PS)_c$ sequence has a subsequence $\{u_{n_j}\}$ which is convergent in $E$, then we say that $J$ satisfies $(PS)_c$ condition. If $(PS)_c$ condition holds for any $c \in \mathbb{R}$, we say that $J$ satisfies $(PS)$ condition.

## 3. Existence of a ground-state solution

Now, as in [8], we introduce the "Nehari" manifold associated with (1.1) and give some properties. We call

$$N_\lambda = \{u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}) \setminus \{0\}; \langle J'_\lambda(u), u\rangle = 0\}$$

the "Nehari" manifold, which the name "Nehari" manifold is borrowed from [14]. It is obvious that $u \in N_\lambda$ if and only if

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' = 0.$$

Define

$$\varphi_\lambda(u) = \langle J'_\lambda(u), u\rangle = \|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - \int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx'.$$

Thus for each $u \in N_\lambda$, we have

$$\begin{aligned}
\langle \varphi'_\lambda(u), u\rangle &= 2\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - q\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' - 2^*\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' \\
&= -\frac{4}{n-2}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - (q-2^*)\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' \qquad (3.1) \\
&= (2-q)\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - (2^*-q)\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx'. \qquad (3.2)
\end{aligned}$$

We split $N_\lambda$ into three parts:

$$\begin{aligned}
N^+_\lambda &= \{u \in N_\lambda; \langle \varphi'_\lambda(u), u\rangle > 0\}, \\
N^0_\lambda &= \{u \in N_\lambda; \langle \varphi'_\lambda(u), u\rangle = 0\}, \\
N^-_\lambda &= \{u \in N_\lambda; \langle \varphi'_\lambda(u), u\rangle < 0\}.
\end{aligned}$$

Thus we have the following results.

**Lemma 3.1.** *The energy functional $J_\lambda$ is coercive and bounded below on $N_\lambda$.*

*Proof.* For $u \in N_\lambda$, by Young's inequalities and Propositions 2.4 and 2.6, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{n}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - (\frac{1}{q} - \frac{1}{2^*})\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' \\ &\geq \frac{1}{n}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - \lambda\frac{2^*-q}{q2^*}\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}}\|u\|^q_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \\ &\geq \frac{1}{n}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - \frac{1}{n}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - D\lambda^{\frac{2}{2-q}} \\ &= -D_0\lambda^{\frac{2}{2-q}}, \end{aligned} \tag{3.3}$$

where $q^* = \frac{2^*}{2^*-q}$ and $D_0$ is a positive constant depending on $q, N, S$ and $\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})}$. Thus $J_\lambda$ is coercive and bounded below on $N_\lambda$. □

**Lemma 3.2.** *Suppose that $u_0$ is a local minimizer for $J_\lambda$ on $N_\lambda$ and $u_0 \notin N^0_\lambda$. Then $J'_\lambda(u_0) = 0$ in $\mathcal{H}^{-1,-\frac{n}{2}}_{2,0}(\mathbb{B})$. Furthermore, if $u_0$ is a non-trivial function in $\mathbb{B}$, then $u_0$ is a positive solution of* (1.1).

*Proof.* If $u_0$ is a local minimizer for $J_\lambda$ on $N_\lambda$, then $u_0$ is a solution of the optimization problem

$$\text{minimize } J_\lambda(u) \text{ subject to } \{u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}); \varphi_\lambda(u) = 0\}.$$

Hence by the theory of Lagrange multipliers, there exists a $\theta \in \mathbb{R}$ such that $J'_\lambda(u_0) = \theta\varphi'_\lambda(u_0)$ in $\mathcal{H}^{-1,-\frac{n}{2}}_{2,0}(\mathbb{B})$. Thus $\langle J'_\lambda(u_0), u_0\rangle = \theta\langle\varphi'_\lambda(u_0), u_0\rangle$.

Moreover, since $u_0 \notin N^0_\lambda$, we get $\langle\varphi'_\lambda(u_0), u_0\rangle \neq 0$, and so $\theta = 0$. Now if $u_0$ is a non-trivial function in $\mathbb{B}$, we can apply the so-called cone maximum principles due to [7] in order to get $u_0$ is positive in $\mathbb{B}$. This completes the proof. □

**Lemma 3.3.** *For each $\lambda > 0$, we have the following:*

(1) *for any $u \in N^+_\lambda$, we have $\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' > 0$;*
(2) *for any $u \in N^0_\lambda$, we have $\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' > 0$ and $\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' > 0$;*
(3) *for any $u \in N^-_\lambda$, we have $\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' > 0$.*

We omit the proof of Lemma 3.3 since it is easy to obtain this result from (3.1) and (3.2).

**Lemma 3.4.** *There exists $\Lambda_1 > 0$ such that $N^0_\lambda = \emptyset$ for $\lambda \in (0, \Lambda_1)$.*

*Proof.* Suppose that $N^0_\lambda \neq \emptyset$ for all $\lambda > 0$. If $u \in N^0_\lambda$, then from (3.1), (3.2), Proposition 2.6 and condition (H3), we obtain

$$\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \leq \lambda\frac{n-2}{4(2^*-q)}\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}}\|u\|^q_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})},$$

$$\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \leq \frac{2^*-q}{2-q}\|g\|_{L^\infty} S^{-\frac{2^*}{2}}\|u\|^{2^*}_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}.$$

Therefore,

$$c_1 \leq \|u\|_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \leq \lambda^{\frac{1}{1-q}} c_2,$$

where $c_1, c_2 > 0$ and are independent of the choice of $u$ and $\lambda$. For $\lambda$ sufficient small, this is a contradiction. Hence, there exists $\Lambda_1 > 0$ such that for $\lambda \in (0, \Lambda_1)$, we have $N^0_\lambda = \emptyset$. □

Now we can write $N_\lambda = N_\lambda^+ \bigcup N_\lambda^-$ and define $\alpha_\lambda = \inf_{u\in N_\lambda} J_\lambda(u)$, $\alpha_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u)$ and $\alpha_\lambda^- = \inf_{u\in N_\lambda^-} J_\lambda(u)$.

**Lemma 3.5.** *We have the following:*

(1) $\alpha_\lambda^+ < 0$ *for all* $\lambda \in (0, \Lambda_1)$.

(2) *there exists* $\Lambda_2 \in (0, \Lambda_1)$ *such that* $\alpha_\lambda^- > d_0$ *for some* $d_0 > 0$ *and* $\lambda \in (0, \Lambda_2)$.

*In particular,* $\alpha_\lambda^+ = \inf_{u\in N_\lambda} J_\lambda(u)$ *for all* $\lambda \in (0, \Lambda_2)$.

*Proof.* (1) Let $u \in N_\lambda^+$, then

$$\frac{2-q}{2^*-q}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} > \int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'$$

and

$$\begin{aligned} J_\lambda(u) &= (\frac{1}{2}-\frac{1}{q})\|u\|^2_{\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})} + (\frac{1}{q}-\frac{1}{2^*})\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx' \\ &< (\frac{1}{2}-\frac{1}{q})\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} + \frac{2-q}{2^*q}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \\ &= -\frac{2-q}{nq}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} < 0. \end{aligned}$$

Thus $\alpha_\lambda \le \alpha_\lambda^+ < 0$ for all $\lambda \in (0, \Lambda_1)$.

(2) Let $u \in N_\lambda^-$, then

$$\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \le \frac{2^*-q}{2-q}\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx' \le \frac{2^*-q}{2-q}S^{-\frac{2^*}{2}}\|g\|_{L^\infty(\mathbb{B})}\|u\|^{2^*}_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}.$$

This implies

$$\|u\|_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} > \Big(\frac{2-q}{2^*-q}\frac{S^{\frac{2^*}{2}}}{\|g\|_{L^\infty(\mathbb{B})}}\Big)^{\frac{1}{2^*-2}} \tag{3.4}$$

for any $u \in N_\lambda^-$. From (3.3), we obtain that

$$J_\lambda(u) \ge \|u\|^q_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}\Big[\frac{1}{n}\|u\|^{2-q}_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - \lambda\frac{2^*-q}{2^*q}\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})}S^{-\frac{q}{2}}\Big]. \tag{3.5}$$

Hence by (3.4) and (3.5), we obtain assertion (2). □

For each $u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})\setminus\{0\}$ with $\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx' > 0$, we write

$$t_{\max} = \Big(\frac{(2-q)\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{(2^*-q)\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'}\Big)^{\frac{n-2}{4}} > 0.$$

Then we have the following Lemma.

**Lemma 3.6.** *For each* $u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})\setminus\{0\}$*, there exists* $\Lambda_3 \in (0, \Lambda_2)$ *such that we have the following results:*

(1) *if* $\int_{\mathbb{B}} f_\lambda|u|^q\frac{dx_1}{x_1}dx' \le 0$*, then there is a unique* $t^- = t^-(u) > t_{\max}$ *such that* $t^-u \in N_\lambda^-$ *and* $J_\lambda(tu)$ *is increasing on* $(0, t^-)$ *and decreasing on* $(t^-, \infty)$*. Moreover,* $J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu)$.

(2) *if* $\int_{\mathbb{B}} f_\lambda|u|^q\frac{dx_1}{x_1}dx' > 0$*, then there is a unique* $0 < t^+ = t^+(u) < t_{\max} < t^-$ *such that* $t^-u \in N_\lambda^-, t^+u \in N_\lambda^+, J_\lambda(tu)$ *is decreasing on* $(0, t^+)$*, increasing on* $(t^+, t^-)$ *and decreasing on* $(t^-, \infty)$*. Moreover,* $J_\lambda(t^+u) = \inf_{0\le t\le t_{\max}} J_\lambda(tu); J_\lambda(t^-u) = \sup_{t\ge t^+} J_\lambda(tu)$.

*Proof.* Fix $u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus \{0\}$. Let

$$s(t) = t^{2-q} \|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - t^{2^*-q} \int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' \quad \text{for } t \geq 0.$$

We have $s(0) = 0$, and $s(t) \to -\infty$ as $t \to \infty$. The function $s(t)$ achieves its maximum at $t_{\max}$, increasing in $[0, t_{\max})$ and decreasing in $(t_{\max}, \infty)$. Moreover, we get

$$\begin{aligned}
s(t_{\max}) &= \Big( \frac{(2-q)\|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})}}{(2^*-q)\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx'} \Big)^{\frac{2-q}{2^*-2}} \|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \\
&\quad - \Big( \frac{(2-q)\|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})}}{(2^*-q)\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx'} \Big)^{\frac{2^*-q}{2^*-2}} \int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' \\
&= \|u\|^q_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \Big[ \Big( \frac{2-q}{2^*-q} \Big)^{\frac{2-q}{2^*-2}} - \Big( \frac{2-q}{2^*-q} \Big)^{\frac{2^*-q}{2^*-2}} \Big] \Big( \frac{\|u\|^{2^*}_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})}}{\int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx'} \Big)^{\frac{2-q}{2^*-2}} \\
&\geq \|u\|^q_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \big( \frac{2^*-2}{2^*-q} \big) \big( \frac{2-q}{2^*-q} \big)^{\frac{2-q}{2^*-2}} D(S,g),
\end{aligned} \tag{3.6}$$

where $D(S,g) > 0$ is a constant depends on $S$ and $g$. We consider two cases now.

(1) $\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' \leq 0$. There is a unique $t^- > t_{\max}$ such that $s(t^-) = \int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx'$ and $s'(t^-) < 0$, which implies $t^- u \in N_\lambda^-$. Because of $t > t_{\max}$, we have

$$(2-q)\|tu\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - (2^*-q) \int_{\mathbb{B}} g|tu|^{2^*} \frac{dx_1}{x_1} dx' < 0$$

and

$$\begin{aligned}
&\frac{d}{dt} J_\lambda(tu)\big|_{t=t^-} \\
&= \Big\{ t\|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - t^{q-1} \int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' - t^{2^*-1} \int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' \Big\}\Big|_{t=t^-} = 0.
\end{aligned}$$

Thus $J_\lambda(tu)$ is increasing on $(0, t^-)$ and decreasing on $(t^-, \infty)$. Moreover, $J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu)$.

(2) $\int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' > 0$. By (3.6), we know that there exists $\Lambda_3 > 0$ such that

$$\begin{aligned}
s(0) = 0 < \lambda \int_{\mathbb{B}} f_+ |u|^q \frac{dx_1}{x_1} dx' &\leq \lambda \|f_+\|_{L_{q^*}^{\frac{n}{q^*}}(\mathbb{B})} S^{-\frac{q}{2}} \|u\|^q_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \\
&\leq \|u\|^q_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \big( \frac{2^*-2}{2^*-q} \big) \big( \frac{2-q}{2^*-q} \big)^{\frac{2-q}{2^*-2}} D(S,g) \leq s(t_{\max})
\end{aligned}$$

for $\lambda \in (0, \Lambda_3)$. It follows that there are a unique $t^+$ and a unique $t^-$ such that for $0 < t^+ < t_{\max} < t^-$,

$$s(t^+) = \int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' = s(t^-)$$

and $s'(t^+) > 0 > s'(t^-)$.

As in case (1), we have $t^+ u \in N_\lambda^+$, $t^- u \in N_\lambda^-$, and $J_\lambda(t^- u) \geq J_\lambda(tu) \geq J_\lambda(t^+ u)$ for each $t \in [t^+, t^-]$. Furthermore, we can get $J_\lambda(t^+ u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. In other words, $J_\lambda(tu)$ is decreasing on $(0, t^+)$, increasing on $(t^+, t^-)$ and decreasing on $(t^-,\infty)$ again. Moreover, $J_\lambda(t^+u)=\inf_{0\le t\le t_{\max}} J_\lambda(tu)$, $J_\lambda(t^-u)=\sup_{t\ge t^+} J_\lambda(tu)$. This completes the proof. □

For $c>0$, we define

$$\begin{gathered}
J_0^c(u)=\frac{1}{2}\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}-\frac{c}{2^*}\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx',\\
N_0^c=\{u\in\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})\setminus\{0\};\langle (J_0^c)'(u),u\rangle=0\}.
\end{gathered}$$

**Lemma 3.7.** *Let $q^*=\frac{2^*}{2^*-q}$. Then for each $u\in N_\lambda^-$, we have the following:*

(1) *there is a unique $t^c(u)>0$ such that $t^c(u)u\in N_0^c$ and*
$$\sup_{t\ge0} J_0^c(tu)=J_0^c(t^c(u)u)=\frac{1}{n}\Big(\frac{\|u\|^{2^*}_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{c\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'}\Big)^{(n-2)/2}.$$

(2) $J_\lambda(u)\ge(1-\lambda)^{n/2}J_0^1(t_uu)-\frac{\lambda(2-q)}{2q}(\|f_+\|_{L_{q^*}^{\frac{n}{q^*}}(\mathbb{B})}S^{-\frac{q}{2}})^{\frac{2}{2-q}}$.

*Proof.* (1) For each $u\in N_\lambda^-$, let
$$f(t)=J_0^c(tu)=\frac{1}{2}t^2\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}-\frac{1}{2^*}t^{2^*}c\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'.$$
Then by Lemma 3.3, we have

- $f(t)\to-\infty$ as $t\to\infty$,
- $f'(t)=t\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}-t^{2^*-1}c\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'$,
- $f''(t)=\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}-(2^*-1)t^{2^*-2}c\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'$.

Let
$$t^c(u):=\Big(\frac{\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{c\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'}\Big)^{\frac{1}{2^*-2}}>0.$$
Then $f'(t^c(u))=0, t^c(u)u\in N_0^c$ and
$$\begin{aligned}
f''(t^c(u))&=\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}-(2^*-1)\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}\\
&=(2-2^*)\|u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}<0.
\end{aligned}$$
Thus there is a unique $t^c(u)>0$ such that $t^c(u)u\in N_0^c$ and
$$\max_{t\ge0} J_0^c(tu)=J_0^c(t^c(u)u)=\frac{1}{n}\Big(\frac{\|u\|^{2^*}_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{c\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'}\Big)^{(n-2)/2}.$$

(2) For each $u\in N_\lambda^-$, let $c=\frac{1}{1-\lambda}$. Then from the previous argument, we know that there exist $t^c=t^c(u)>0$ and $t_u>0$ such that $t^cu\in N_0^c$ and $t_uu\in N_0^1$. By Propositions 2.4 and 2.6, Hölder inequality, and Young's inequality, we obtain
$$\begin{aligned}
\int_{\mathbb{B}} f_+|t^cu|^q\frac{dx_1}{x_1}dx'&\le\|f_+\|_{L_{q^*}^{\frac{n}{q^*}}(\mathbb{B})}S^{-\frac{q}{2}}\|t^cu\|^q_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}\\
&\le\frac{2-q}{2}(\|f_+\|_{L_{q^*}^{\frac{n}{q^*}}(\mathbb{B})}S^{-\frac{q}{2}})^{\frac{2}{2-q}}+\frac{q}{2}\|t^cu\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}.
\end{aligned}$$

Then from this inequality and Part (1), we obtain

$$\begin{aligned}
&\sup_{t\geq 0} J_\lambda(tu)\\
&\geq J_\lambda(t^c u)\\
&\geq \frac{1-\lambda}{2}\|t^c u\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - \frac{1}{2^*}\int_{\mathbb{B}} g|t^c u|^{2^*}\frac{dx_1}{x_1}dx' - \frac{\lambda(2-q)}{2q}(\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}\\
&= (1-\lambda)J_0^{\frac{1}{1-\lambda}}(t^c u) - \frac{\lambda(2-q)}{2q}(\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}\\
&= (1-\lambda)^{n/2}\frac{1}{n}\Big(\frac{\|u\|^{2^*}_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{\int_{\mathbb{B}} g|u|^{2^*}\frac{dx_1}{x_1}dx'}\Big)^{(n-2)/2} - \frac{\lambda(2-q)}{2q}(\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}\\
&\geq (1-\lambda)^{n/2}J_0^1(t_u u) - \frac{\lambda(2-q)}{2q}(\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}.
\end{aligned}$$

Since $\sup_{t\geq 0} J_\lambda(tu) = J_\lambda(u)$, we have

$$J_\lambda(u) \geq (1-\lambda)^{\frac{2^*}{2^*-2}} J_0^1(t_u u) - \frac{\lambda(2-q)}{2q}(\|f_+\|_{L^{\frac{n}{q^*}}_{q^*}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}.$$

This completes the proof. □

Next, we establish the existence of a local minimum for $J_\lambda$ on $N_\lambda^+$.

**Theorem 3.8.** *For each $\lambda < \Lambda_3$, the functional $J_\lambda$ has a minimizer $u_\lambda^+$ in $N_\lambda^+$ which satisfies*

(1) *$u_\lambda^+$ is a positive solution of* (1.1)*;*
(2) *$J_\lambda(u_\lambda^+) \to 0$ as $\lambda \to 0$;*
(3) *$J_\lambda(u_\lambda^+) = \alpha_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u)$.*

*Proof.* As in [8, Lemma 4.7], we can obtain a $(PS)_{\alpha_\lambda}$-sequence for $J_\lambda$ defined$\{u_k\} \subset N_\lambda$, then by Proposition 2.6 and (3.3), there exists a subsequence still denoted by $\{u_k\}$, and a solution $u_\lambda^+ \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ of the equation (1.1) such that $u_k \rightharpoonup u_\lambda^+$ weakly in $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ and $u_k \to u_\lambda^+$ strongly in $L_q^{\frac{n}{q}}(\mathbb{B})$ as $k\to\infty$.

First, we claim that $\int_{\mathbb{B}} f_\lambda |u_\lambda^+|^q \frac{dx_1}{x_1}dx' \neq 0$. If not, by Proposition 2.6, we can conclude that

$$\int_{\mathbb{B}} f_\lambda |u_\lambda^+|^q\frac{dx_1}{x_1}dx' = 0, \quad \int_{\mathbb{B}} f_\lambda |u_k|^q\frac{dx_1}{x_1}dx' \to 0 \quad \text{as } k\to\infty.$$

Thus

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_k|^2\frac{dx_1}{x_1}dx' = \int_{\mathbb{B}} g|u_k|^{2^*}\frac{dx_1}{x_1}dx' + o(1),$$

and

$$\begin{aligned}
\frac{1}{n}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_k|^2\frac{dx_1}{x_1}dx' &= \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_k|^2\frac{dx_1}{x_1}dx' - \frac{1}{q}\int_{\mathbb{B}} f_\lambda|u_k|^q\frac{dx_1}{x_1}dx'\\
&\quad - \frac{1}{2^*}\int_{\mathbb{B}} g|u_k|^{2^*}\frac{dx_1}{x_1}dx' + o(1)\\
&= \alpha_\lambda + o(1).
\end{aligned}$$

This contradicts to $\alpha_\lambda < 0$ by Lemma 3.5. Thus $\int_{\mathbb{B}} f_\lambda |u_\lambda^+|^q \frac{dx_1}{x_1} dx' \neq 0$. In particular $u_\lambda^+$ is a nontrivial solution of (1.1). We now prove $u_k \to u_\lambda^+$ strongly in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ as $k \to \infty$. Supposing the contrary, then

$$\|u_\lambda^+\|_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} < \lim_{k\to\infty} \inf \|u_k\|_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})}.$$

Thus

$$\begin{aligned}
&\|u_\lambda^+\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - \int_{\mathbb{B}} g|u_\lambda^+|^{p+1} \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} f_\lambda |u_\lambda^+|^q \frac{dx_1}{x_1} dx' \\
&< \lim_{k\to\infty} \inf \Big( \|u_k\|^2_{\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})} - \int_{\mathbb{B}} g|u_k|^{2^*} \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} f_\lambda |u_k|^q \frac{dx_1}{x_1} dx' \Big) = 0.
\end{aligned}$$

This contradicts to the fact that $u_\lambda^+ \in N_\lambda$. Hence $u_k \to u_\lambda^+$ strongly in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ as $k \to \infty$ and $J_\lambda(u_\lambda^+) = \alpha_\lambda$. It follows that $u_\lambda^+ \in N_\lambda^+$ and $J_\lambda(u_\lambda^+) = \alpha_\lambda^+ = \alpha_\lambda$ from Lemma 3.6. Since $J_\lambda(u_\lambda^+) = J_\lambda(|u_\lambda^+|)$ and $|u_\lambda^+| \in N_\lambda^+$, by Lemma 3.2, we may assume that $u_\lambda^+$ is a nonnegative (nontrivial) solution of (1.1). Then we can apply the the so-called cone maximum principles due to [7] in order to get $u_\lambda^+$ is positive in $\mathbb{B}$. Moreover, by Lemma 3.1 and Lemma 3.5, we obtain

$$0 > J_\lambda(u_\lambda^+) \geq -D_0 \lambda^{\frac{2}{2-q}}.$$

Thus $J_\lambda(u_\lambda^+) \to 0$ as $\lambda \to 0$. □

## 4. Existence of multiple solutions

In this section, we use the idea of category to get multiple positive solutions of (1.1) in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ and give the proof of Theorem 1.2. Initially, we give the definition of category.

**Definition 4.1.** Let $M$ be a topological space and consider a closed subset $A \subset M$. We say that $A$ has category $k$ relative to $M(\operatorname{cat}_M(A) = k)$, if $A$ is covered by $k$ closed sets $A_j, 1 \leq j \leq k$, which are contractible in $M$, and if $k$ is minimal with this property. If no such finite covering exists, we let $\operatorname{cat}_M(A) = \infty$.

For the properties of $\operatorname{cat}_M(A)$ we refer to [16]. Next we need two Propositions related to the category.

**Proposition 4.2.** *Let $H$ be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space) and assume $h \in C^1(H, \mathbb{R})$ bounded from below. Let $-\infty < \inf_H h < a < b < +\infty$. Suppose that $h$ satisfies Palais-Smale condition on the sublevel $\{u \in H; h(u) \leq b\}$ and that $a$ is not a critical level for $h$. Then*

$$\sharp\{u \in h^a; \nabla h(u) = 0\} \geq \operatorname{cat}_{h^a}(h^a),$$

*where $h^a \equiv \{u \in H; h(u) \leq a\}$.*

For a proof of the above proposition, see [5, Theorem 2.1].

**Proposition 4.3.** *Let $Q, \Omega^+$ and $\Omega^-$ be closed sets with $\Omega^- \subset \Omega^+$; Let $\beta : Q \to \Omega^+$, $\psi : \Omega^- \to Q$ be two continuous maps such that $\beta \circ \psi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$. Then $\operatorname{cat}_Q(Q) \geq \operatorname{cat}_{\Omega^+}(\Omega^-)$.*

For a proof of the above proposition, see [5, Lemma 2.2]. The proof of Theorem 1.2 is based on Proposition 4.2 and 4.3. Now we first define a cut-off function. Let $\eta \in C_0^\infty(\mathbb{R}_+^n)$ such that $0 \le \eta \le 1$, $|\nabla_{\mathbb{B}}\eta| \le c$ and

$$\eta(x) = \begin{cases} 1, & (|\ln x_1|^2 + |x'|^2)^{1/2} \le \frac{r_0}{2}, \\ 0, & (|\ln x_1|^2 + |x'|^2)^{1/2} \ge r_0. \end{cases}$$

Define

$$w_{\varepsilon,z} = \eta(\frac{x_1}{z_1}, x'-z')v_\varepsilon(\frac{x_1}{z_1}, x'-z').$$

**Theorem 4.4.** *For any $z \in M$, we have $\|w_{\varepsilon,z}\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} = S^{n/2} + O(\varepsilon^{n-2})$.*

*Proof.* First we have

$$\begin{aligned}
&\|w_{\varepsilon,z}\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} \\
&= \int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_{\varepsilon,z}|^2 \frac{dx_1}{x_1} dx' \\
&= \int_{\mathbb{B}} |\nabla_{\mathbb{B}}(\eta(\frac{x_1}{z_1}, x'-z')v_\varepsilon(\frac{x_1}{z_1}, x'-z'))|^2 \frac{dx_1}{x_1} dx' \\
&= \int_{\Omega_{r_0}(z_1,z')} |\nabla_{\mathbb{B}}(\eta(\frac{x_1}{z_1}, x'-z')v_\varepsilon(\frac{x_1}{z_1}, x'-z'))|^2 \frac{dx_1}{x_1} dx' \\
&= \int_{\Omega_{r_0}(1,0)} |\nabla_{\mathbb{B}}\eta(x_1,x') \cdot v_\varepsilon(x_1,x') + \eta(x_1,x') \cdot \nabla_{\mathbb{B}} v_\varepsilon(x_1,x')|^2 \frac{dx_1}{x_1} dx' \\
&= \int_{\Omega_{r_0}(1,0)} |\nabla_{\mathbb{B}}\eta|^2 v_\varepsilon^2 + \eta^2 |\nabla_{\mathbb{B}} v_\varepsilon|^2 + 2\eta v_\varepsilon \nabla_{\mathbb{B}}\eta \cdot \nabla_{\mathbb{B}} v_\varepsilon \frac{dx_1}{x_1} dx',
\end{aligned}$$

where $(1,0) \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$. Then from the definition of $v_\varepsilon$ we obtain

$$\begin{aligned}
\int_{\Omega_{r_0}(1,0)} |\nabla_{\mathbb{B}}\eta|^2 v_\varepsilon^2 \frac{dx_1}{x_1} dx' &\le c \int_{\Omega_{r_0}(1,0)\setminus\Omega_{\frac{r_0}{2}}(1,0)} \frac{[n(n-2)\varepsilon^2]^{(n-2)/2}}{[\varepsilon^2 + |\ln x_1|^2 + |x'|^2]^{n-2}} \frac{dx_1}{x_1} dx' \\
&= \int_{B_{r_0}\setminus B_{\frac{r_0}{2}}} \frac{[n(n-2)\varepsilon^2]^{(n-2)/2}}{[\varepsilon^2 + |z_1|^2 + |z'|^2]^{n-2}} dz_1 dz' \\
&\le c \int_{\frac{r_0}{2}}^{r^0} r^{n-1} \frac{[n(n-2)\varepsilon^2]^{(n-2)/2}}{[\varepsilon^2 + r^2]^{n-2}} dr \\
&\le c \int_{\frac{r_0}{2}}^{r^0} r^{n-1-2n+4} \varepsilon^{n-2} dr = O(\varepsilon^{n-2}),
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \int_{\Omega_{r_0}(1,0)} 2\eta v_\varepsilon \nabla_{\mathbb{B}}\eta \cdot \nabla_{\mathbb{B}} v_\varepsilon \frac{dx_1}{x_1} dx' \Big| \\
&\le c \int_{\Omega_{r_0}(1,0)\setminus\Omega_{\frac{r_0}{2}}(1,0)} \eta |v_\varepsilon| |\nabla_{\mathbb{B}} v_\varepsilon| \frac{dx_1}{x_1} dx' \\
&\le c \int_{\Omega_{r_0}(1,0)\setminus\Omega_{\frac{r_0}{2}}(1,0)} \eta \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2_{\mathbb{B}}]^{(n-2)/2}} [n(n-2)\varepsilon^2]^{\frac{n-2}{4}}
\end{aligned}$$

$$
\begin{aligned}
&\times (\frac{n-2}{2}) \frac{2|x|_{\mathbb{B}}}{[\varepsilon^2+|x|_{\mathbb{B}}^2]^{n/2}} \frac{dx_1}{x_1} dx' \\
&\leq c \int_{\Omega_{r_0}(1,0)\setminus \Omega_{\frac{r_0}{2}}(1,0)} \eta \frac{|x|_{\mathbb{B}} \varepsilon^{n-2}}{|x|_{\mathbb{B}}^{2n-2}} \frac{dx_1}{x_1} dx' \\
&\leq c \int_{B_{r_0}\setminus B_{\frac{r_0}{2}}} \frac{1}{|x|^{2n-3}} \varepsilon^{n-2} dx = O(\varepsilon^{n-2}).
\end{aligned}
$$

Moreover, since $\int_{\mathbb{R}_+^n} |\nabla_{\mathbb{B}} v_\varepsilon|^2 \frac{dx_1}{x_1} dx' = S^{n/2}$ (see Remark 2.8) and

$$
\begin{aligned}
&\Big| \int_{\Omega_{r_0}(1,0)} \eta^2 |\nabla_{\mathbb{B}} v_\varepsilon|^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{R}_+^n} |\nabla_{\mathbb{B}} v_\varepsilon|^2 \frac{dx_1}{x_1} dx' \Big| \\
&= \int_{\mathbb{R}_+^n \setminus \Omega_{\frac{r_0}{2}}(1,0)} (1-\eta^2) |\nabla_{\mathbb{B}} v_\varepsilon|^2 \frac{dx_1}{x_1} dx' \\
&\leq c \int_{\mathbb{R}_+^n \setminus \Omega_{\frac{r_0}{2}}(1,0)} (1-\eta^2) [n(n-2)\varepsilon^2]^{(n-2)/2} \frac{|x|_{\mathbb{B}}^2}{[\varepsilon^2+|x|_{\mathbb{B}}^2]^n} \frac{dx_1}{x_1} dx' \\
&\leq c \int_{\mathbb{R}^n \setminus B_{\frac{r_0}{2}}} [n(n-2)\varepsilon^2]^{(n-2)/2} \frac{|x|^2}{[\varepsilon^2+|x|^2]^n} dx_1 dx' \\
&\leq c\varepsilon^{n-2} \int_{\frac{r_0}{2}}^{+\infty} \frac{r^{n+1}}{r^{2n}} dr = O(\varepsilon^{n-2}),
\end{aligned}
$$

we obtain $\|w_{\varepsilon,z}\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} = S^{n/2} + O(\varepsilon^{n-2})$. □

**Theorem 4.5.** *We have* $\inf_{u\in N_0^1} J_0^1(u) = \inf_{u\in N_0} J_0(u) = \inf_{u\in N^\infty} J^\infty(u) = \frac{1}{n} S^{n/2}$, *where* $J^\infty(u) = \frac{1}{2}\|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - \frac{1}{2^*}\int_{\mathbb{B}} |u|^{2^*} \frac{dx_1}{x_1} dx'$ *and* $N^\infty = \{u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus \{0\}; \langle (J^\infty)'(u), u\rangle = 0\}$. *Furthermore,* (1.1) *with* $\lambda = 0$ *does not admit any solution* $u_0$ *such that* $J_0(u_0) = \frac{1}{n} S^{n/2}$.

*Proof.* Define $\overline{g} : \mathbb{R}_+^n \to \mathbb{R}$ by

$$
\overline{g}(x) = \begin{cases} g(x), & x \in \overline{\mathbb{B}}, \\ 0, & \text{elsewhere}. \end{cases}
$$

as an extension of $g$. Then from Lemma 3.6 we know that there is a unique $t_0(w_{\varepsilon,z}) > 0$ such that $t_0(w_{\varepsilon,z}) w_{\varepsilon,z} \in N_0 (N_\lambda$ for $\lambda = 0)$ for all $\varepsilon > 0$. By the definition of $w_{\varepsilon,z}$ and Remark 1.1, we have

$$
\|t_0(w_{\varepsilon,z}) w_{\varepsilon,z}\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} = \int_{\mathbb{B}} g |t_0(w_{\varepsilon,z}) w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx',
$$

and so

$$
[t_0(w_{\varepsilon,z})]^{\frac{4}{n-2}} = \frac{\int_{\mathbb{B}} g |w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx'}{\|w_{\varepsilon,z}\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})}}.
$$

With the definition of $v_\varepsilon$, we get

$$
\int_{\mathbb{B}} g |w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' = \int_{\Omega_{r_0(z)}} g(x) \Big| \eta(\frac{x_1}{z_1}, x'-z') v_\varepsilon(\frac{x_1}{z_1}, x'-z') \Big|^{2^*} \frac{dx_1}{x_1} dx'
$$

$$= \int_{\mathbb{R}^n_+} \frac{[n(n-2)\varepsilon^2]^{n/2}\overline{g}(x_1z_1, x'+z')\eta^{2^*}(x)}{(\varepsilon^2+|\ln x_1|^2+|x'|^2)^n}\frac{dx_1}{x_1}dx'.$$

Thus by condition (H2) and Remark 1.1, we obtain

$$\begin{aligned}
0 &\le \frac{1}{[n(n-2)\varepsilon^2]^{n/2}}\Big[\int_{\mathbb{R}^n_+}|v_\varepsilon|^{2^*}\frac{dx_1}{x_1}dx' - \int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*}\frac{dx_1}{x_1}dx'\Big] \\
&= \int_{\mathbb{R}^n_+\setminus\Omega_{\frac{r_0}{2}}(1,0)} \frac{[1-\overline{g}(x_1z_1, x'+z')\eta^{2^*}(x)]}{(\varepsilon^2+|\ln x_1|^2+|x'|^2)^n}\frac{dx_1}{x_1}dx' \\
&\quad + \int_{\Omega_{\frac{r_0}{2}}(1,0)} \frac{[1-\overline{g}(x_1z_1, x'+z')\eta^{2^*}(x)]}{(\varepsilon^2+|\ln x_1|^2+|x'|^2)^n}\frac{dx_1}{x_1}dx' \\
&\le \int_{\mathbb{R}^n_+\setminus\Omega_{\frac{r_0}{2}}(1,0)} \frac{1}{|x|_{\mathbb{B}}^{2n}}\frac{dx_1}{x_1}dx' + c_0\int_{\Omega_{\frac{r_0}{2}}(1,0)}\frac{|x|_{\mathbb{B}}^{\rho}}{(\varepsilon^2+|x|_{\mathbb{B}}^2)^n}\frac{dx_1}{x_1}dx' \qquad (4.1)\\
&= \int_{\mathbb{R}^n\setminus B_{\frac{r_0}{2}}} \frac{1}{|x|^{2n}}dx_1dx' + c_0\int_{B_{\frac{r_0}{2}}}\frac{|x|^{\rho}}{(\varepsilon^2+|x|^2)^n}dx_1dx' \\
&\le n\omega_n\int_{\frac{r_0}{2}}^{+\infty} r^{-(n+1)}dr + \frac{c_0n\omega_n}{\varepsilon^2}\int_0^{\frac{r_0}{2}} r^{\rho-n+1}dr \\
&= \omega_n(\frac{r_0}{2})^{-n} + \frac{c_0n\omega_n}{\varepsilon^2(\rho-(n-2))}(\frac{r_0}{2})^{\rho-(n-2)} \\
&\le c_1 + \frac{c_2}{\varepsilon^2}
\end{aligned}$$

for all $z \in M$, where $\omega_n$ is the volume of the unit ball $B_1 \subset \mathbb{R}^n$. Then

$$\lim_{\varepsilon\to 0}\int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*}\frac{dx_1}{x_1}dx' = S^{n/2} \quad \text{uniformly in } z \in M. \qquad (4.2)$$

Thus from Theorem 4.4 and (4.2), we obtain

$$\lim_{\varepsilon\to 0} t_0(w_{\varepsilon,z}) = 1, \quad \lim_{\varepsilon\to 0}\|t_0(w_{\varepsilon,z})w_{\varepsilon,z}\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} = S^{n/2}$$

uniformly in $z \in M$. Then we obtain

$$\inf_{u\in N_0} J_0(u) \le J_0(t_0(w_{\varepsilon,z})w_{\varepsilon,z}) \to \frac{1}{n}S^{n/2}, \quad \text{as } \varepsilon \to 0,$$

and so $\inf_{u\in N_0} J_0(u) \le \inf_{u\in N^\infty} J^\infty(u) = \frac{1}{n}S^{n/2}$. Let $u \in N_0$. Then by Lemma 3.6(1), we have $J_0(u) = \sup_{t\ge 0} J_0(tu)$.

Moreover, there is a unique $t_u > 0$ such that $t_uu \in N^\infty$, and then

$$J_0(u) \ge J_0(t_uu) \ge J^\infty(t_uu) \ge \frac{1}{n}S^{n/2}.$$

This implies $\inf_{u\in N_0} J_0(u) \ge \frac{1}{n}S^{n/2}$. Therefore,

$$\inf_{u\in N_0} J_0(u) = \inf_{u\in N^\infty} J^\infty(u) = \frac{1}{n}S^{n/2}.$$

Similarly, we have $\inf_{u\in N_0^1} J_0^1(u) = \frac{1}{n}S^{n/2}$.

Next we will show that (1.1) with $\lambda = 0$ does not admit any solution $u_0$ such that $J_0(u_0) = \inf_{u\in N_0} J_0(u)$. We argue by contradiction. Suppose that there exists $u_0 \in N_0$ such that $J_0(u_0) = \inf_{u\in N_0} J_0(u)$. Since $J_0(u_0) = J_0(|u_0|)$ and $|u_0| \in N_0$, by Lemma 3.2, we may assume that $u_0$ is a positive solution of (1.1) with $\lambda = 0$.

Moreover, by Lemma 3.6 (1), we obtain $J_0(u_0) = \sup_{t\geq 0} J_0(tu_0)$. Thus there is a unique $t_{u_0} > 0$ such that $t_{u_0}u_0 \in N^\infty$ and so

$$\begin{aligned}\frac{1}{n}S^{n/2} &= \inf_{u\in N_0} J_0(u) = J_0(u_0) \geq J_0(t_{u_0}u_0),\\ &\geq J^\infty(t_{u_0}u_0) + \frac{{t_{u_0}}^{2^*}}{2^*}\int_{\mathbb{B}}(1-g)|u_0|^{2^*}\frac{dx_1}{x_1}dx'\\ &\geq \frac{1}{n}S^{n/2} + \frac{{t_{u_0}}^{2^*}}{2^*}\int_{\mathbb{B}}(1-g)|u_0|^{2^*}\frac{dx_1}{x_1}dx'.\end{aligned}$$

This implies $\int_{\mathbb{B}}(1-g)|u_0|^{2^*}\frac{dx_1}{x_1}dx' = 0$. But this is a contradiction since $u_0$ is positive. We obtain the assertion. □

**Theorem 4.6.** *Suppose that $\{u_k\}$ is a minimizing sequence for $J_0^1(\cdot)$ to $N_0^1$, then we have*

$$\int_{\mathbb{B}}(1-g)|u_k|^{2^*}\frac{dx_1}{x_1}dx' = o(1).$$

*Furthermore, $\{u_k\}$ is a $(PS)_{\frac{1}{n}S^{n/2}}$-sequence for $J^\infty(\cdot)$ in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$.*

*Proof.* For each $k$, there is a unique $t_k > 0$ such that $t_ku_k \in N^\infty$; that is,

$$t_k^2\int_{\mathbb{B}}|\nabla_{\mathbb{B}}u_k|^2\frac{dx_1}{x_1}dx' = t_k^{2^*}\int_{\mathbb{B}}|u_k|^{2^*}\frac{dx_1}{x_1}dx'.$$

Then by Lemma 3.7,

$$J_0^1(u_k) \geq J_0^1(t_ku_k) = J^\infty(t_ku_k) + \frac{t_k^{2^*}}{2^*}\int_{\mathbb{B}}(1-g)|u_k|^{2^*}\frac{dx_1}{x_1}dx' \tag{4.3}$$

$$\geq \frac{1}{n}S^{n/2} + \frac{t_k^{2^*}}{2^*}\int_{\mathbb{B}}(1-g)|u_k|^{2^*}\frac{dx_1}{x_1}dx'. \tag{4.4}$$

From Theorem 4.5, we have $J_0^1(u_k) = \frac{1}{n}S^{n/2} + o(1)$ and

$$\frac{t_k^{2^*}}{2^*}\int_{\mathbb{B}}(1-g)|u_k|^{2^*}\frac{dx_1}{x_1}dx' = o(1).$$

We will show that there exists $c_0 > 0$ such that $t_k > c_0$ for all $n$. We argue by contradiction. Then we may assume $t_k \to 0$ as $k \to \infty$. Since $J_0^1(u_k) = \frac{1}{n}S^{n/2} + o(1)$ and $J^\infty(t_ku_k) = \frac{1}{n}t_k^2\|u_k\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} + o(1)$, by (3.3), $\|u_k\|_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})}$ is uniformly bounded and so $\|t_ku_k\|_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \to 0$ or $J^\infty(t_ku_k) \to 0$. This contradicts to the fact $J^\infty(t_ku_k) \geq \frac{1}{n}S^{n/2} > 0$. Thus $\int_{\mathbb{B}}(1-g)|u_k|^{2^*}\frac{dx_1}{x_1}dx' = o(1)$. In an analogous manner as in [8, Lemma 4.7], we have $\{u_k\}$ is a $(PS)_{\frac{1}{n}S^{n/2}}$-sequence for $J^\infty$ in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$. This completes the proof. □

For the positive number $d$, we consider the filtration of the "Nehari" manifold $N_0^1$ as follows:

$$N_0^1(d) = \{u \in N_0^1; J_0^1(u) \leq \frac{1}{n}S^{n/2} + d\}.$$

Let $\Phi : \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \to \mathbb{R}_+^n$ be the barycenter map defined by $\Phi(u) = \frac{\int_{\mathbb{B}} x|u|^{2^*}\frac{dx_1}{x_1}dx'}{\int_{\mathbb{B}}|u|^{2^*}\frac{dx_1}{x_1}dx'}$, then we have the following result.

**Theorem 4.7.** *For each positive number $\delta < r_0$, there exists $d_\delta > 0$ such that $\Phi(u) \in M_\delta$ for all $u \in N_0^1(d_\delta)$.*

*Proof.* Suppose the contrary. Then there exists a sequence $\{u_k\} \in N_0^1$ and $\delta_0 < r_0$ such that $J_0^1(u_k) = \frac{1}{n}S^{n/2} + o(1)$ and $\Phi(u_k) \notin M_{\delta_0}$ for all $k$. By Theorem 4.6, we know $\{u_k\}$ is a $(PS)_{\frac{1}{n}S^{n/2}}$-sequence for $J^\infty$ in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$. It follows from (3.3) that there exists a subsequence (still denoted by $\{u_k\}$) and $u_0 \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ such that $u_k \rightharpoonup u_0$ in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$. By the so-called cone concentration compactness principle (see [4, Proposition 2.8], there exist two sequences $\{x_k\} \subset \mathbb{B}$, $\{R_k\} \subset \mathbb{R}^+$, $x_0 \in \overline{\mathbb{B}}$ and a positive solution $v_0 \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{R}_+^n)$ of critical problem $-\Delta_{\mathbb{B}} u = |u|^{2^*-2}u$ in $\mathbb{R}_+^n$ with $J^\infty(v_0) = \frac{1}{n}S^{n/2}$ such that $x_k \to x_0$ and $R_k \to \infty$ as $k \to \infty$, and $\|u_k(x) - R_k^{(n-2)/2} v_0((\frac{x_1}{x_{k,1}})^{R_k}, x_k' + R_k(x'-x_k'))\|_{\mathcal{H}_{2,0}^{1,n/2}} \to 0$ as $k \to \infty$. Then

$$
\begin{aligned}
\Phi(u_k) &= \frac{\int_{\mathbb{B}} x|u_k|^{2^*}\frac{dx_1}{x_1}dx'}{\int_{\mathbb{B}} |u_k|^{2^*}\frac{dx_1}{x_1}dx'} \\
&= \frac{\int_{\mathbb{B}} x\big|R_k^{(n-2)/2} v_0((\frac{x_1}{x_{k,1}})^{R_k}, x_k' + R_k(x'-x_k'))\big|^{2^*}\frac{dx_1}{x_1}dx' + o(1)}{\int_{\mathbb{B}} \big|R_k^{(n-2)/2} v_0((\frac{x_1}{x_{k,1}})^{R_k}, x_k' + R_k(x'-x_k'))\big|^{2^*}\frac{dx_1}{x_1}dx' + o(1)}, \\
&= \frac{\int_{\mathbb{R}_+^n}(x_{k,1}x_1^{\frac{1}{R_k}}, \frac{x'-x_k'}{R_k} + x_k')|v_0(x)|^{2^*}\frac{dx_1}{x_1}dx'}{\int_{\mathbb{R}_+^n}|v_0(x)|^{2^*}\frac{dx_1}{x_1}dx'} + o(1) \\
&= x_0 + o(1).
\end{aligned}
$$

Now we will show that $x_0 \in M_{\delta_0}$. Since

$$
\begin{aligned}
&\int_{\mathbb{B}} g|u_k|^{2^*}\frac{dx_1}{x_1}dx' \\
&= \int_{\mathbb{B}} g(x)|R_k^{(n-2)/2} v_0((\frac{x_1}{x_{k,1}})^{R_k}, x_k' + R_k(x'-x_k'))|^{2^*}\frac{dx_1}{x_1}dx' + o(1) \\
&= \int_{\mathbb{R}_+^n} g(x_{k,1}x_1^{\frac{1}{R_k}}, \frac{x'-x_k'}{R_k} + x_k')|v_0(x)|^{2^*}\frac{dx_1}{x_1}dx' + o(1) \\
&= g(x_0)S^{n/2} + o(1),
\end{aligned}
$$

we have $g(x_0) = \max_{x\in\overline{\mathbb{B}}} g(x) = 1$, and so $x_0 \in M$. This is a contradiction. We obtain the assertion. □

Now, we consider the filtration of the manifold $N_\lambda^-$ as follows. Let

$$
N_\lambda(c) = \{u \in N_\lambda^-; J_\lambda(u) < c\}
$$

and denote

$$
\overline{w}_{\varepsilon,z} = [n(n-2)\varepsilon^2]^{-\frac{n-2}{4}} w_{\varepsilon,z}.
$$

Then we have the following results.

**Theorem 4.8.** *Let $\Lambda_3 > 0$ be as in Lemma 3.6 and $\varepsilon = \lambda^{\frac{2}{(2-q)(n-2)}}$. Then there exists $0 < \Lambda_* \le \Lambda_3$ such that for $\lambda < \Lambda_*$, we have*

$$
\sup_{t\ge 0} J_\lambda(t\overline{w}_{\varepsilon,z}) < c_\lambda = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}}D_0 \tag{4.5}
$$

*uniformly in $z \in M$, where $D_0$ is a positive constant defined in Lemma 3.1. Furthermore, there exists $t_z^- > 0$ such that $t_z^- \overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ and $\Phi(t_z^- \overline{w}_{\varepsilon,z}) \in M_\delta$ for all $z \in M$.*

*Proof.* By (4.1) and $\int_{\mathbb{R}_+^n} |v_\varepsilon|^{2^*} \frac{dx_1}{x_1} dx' = S^{n/2} > 0$ for all $\varepsilon > 0$, we have

$$0 \le 1 - S^{-n/2} \int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' \le (c_1 + \frac{c_2}{\varepsilon^2}) S^{-n/2} [n(n-2)\varepsilon^2]^{n/2}$$

for all $z \in M$; i.e.,

$$1 - (c_1 + \frac{c_2}{\varepsilon^2}) S^{-n/2} [n(n-2)\varepsilon^2]^{n/2} \le S^{-n/2} \int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' \le 1$$

for all $z \in M$. Since $\varepsilon = \lambda^{\frac{2}{(2-q)(n-2)}}$ and $n \ge 3$, there exists a positive number $\Lambda_4$ such that

$$0 < 1 - (c_1 + \frac{c_2}{\varepsilon^2}) S^{-n/2} [n(n-2)\varepsilon^2]^{n/2} < 1$$

for all $\lambda \in (0, \Lambda_4)$. Then we can deduce that

$$\begin{aligned} 1 - (c_1 + \frac{c_2}{\varepsilon^2}) S^{-n/2} [n(n-2)\varepsilon^2]^{n/2} &< (1 - (c_1 + \frac{c_2}{\varepsilon^2}) S^{-n/2} [n(n-2)\varepsilon^2]^{n/2})^{2/2^*} \\ &\le (S^{-n/2} \int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx')^{2/2^*} \le 1 \end{aligned}$$

for all $z \in M$, which implies that

$$\Big( \int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' \Big)^{2/2^*} = S^{(n-2)/2} + O(\varepsilon^{n-2}) \tag{4.6}$$

for all $z \in M$. Thus from Theorem 4.4 and (4.6) we obtain

$$\begin{aligned} \Psi(\overline{w}_{\varepsilon,z}) &= \frac{\|\overline{w}_{\varepsilon,z}\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{\Big( \int_{\mathbb{B}} g|\overline{w}_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' \Big)^{2/2^*}} \\ &= \frac{\|w_{\varepsilon,z}\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})}}{\Big( \int_{\mathbb{B}} g|w_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' \Big)^{2/2^*}} \\ &= \frac{S^{n/2} + O(\varepsilon^{n-2})}{S^{(n-2)/2} + O(\varepsilon^{n-2})} \end{aligned}$$

for all $z \in M$. Hence

$$\Psi(\overline{w}_{\varepsilon,z}) - S = \frac{S^{n/2} + o(\varepsilon^{n-2})}{S^{(n-2)/2} + o(\varepsilon^{n-2})} - S = O(\varepsilon^{n-2})$$

for all $z \in M$. Using the fact $\max_{t\ge0}(\frac{t^2}{2}a - \frac{t^{2^*}}{2^*}b) = \frac{1}{n}(\frac{a}{b^{2/2^*}})^{n/2}$ for all $a, b > 0$, we can deduce that

$$\sup_{t\ge0} J_0^1(t\overline{w}_{\varepsilon,z}) = \frac{1}{n} (\Psi(\overline{w}_{\varepsilon,z}))^{n/2}.$$

Then we get $\sup_{t\ge0} J_0^1(t\overline{w}_{\varepsilon,z}) = \frac{1}{n} S^{n/2} + O(\varepsilon^{n-2})$ for all $z \in M$.

Now, we will show that (4.5) holds. Let $\Lambda_5 \le \min\{\Lambda_3, \Lambda_4\}$ be a positive number such that $\frac{1}{n} S^{n/2} - \lambda^{\frac{2}{2-q}} D_0 > 0$ for all $\lambda \in (0, \Lambda_5)$. Since

$$J_\lambda(t\overline{w}_{\varepsilon,z}) = \frac{t^2}{2} \|\overline{w}_{\varepsilon,z}\|^2_{\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})} - \frac{t^q}{q} \int_{\mathbb{B}} f_\lambda |\overline{w}_{\varepsilon,z}|^q \frac{dx_1}{x_1} dx' - \frac{t^{2^*}}{2^*} \int_{\mathbb{B}} g|\overline{w}_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx'$$

and $\int_{\mathbb{B}} f_\lambda |\overline{w}_{\varepsilon,z}|^q \frac{dx_1}{x_1} dx' > 0$, we have $J_\lambda(t\overline{w}_{\varepsilon,z}) < J_0^1(t\overline{w}_{\varepsilon,z})$ for all $t \geq 0$ and $\lambda > 0$. Then there exists $t_0 > 0$ such that

$$\sup_{0\leq t\leq t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0$$

for all $\lambda \in (0, \Lambda_5)$. Now, we only need to show that $\sup_{t\geq t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0$ for all $z \in M$. First we have

$$\begin{aligned}
\sup_{t\geq t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) &= \sup_{t\geq t_0}[J_0^1(t\overline{w}_{\varepsilon,z}) - \frac{t^q}{q}\int_{\mathbb{B}} f_\lambda |\overline{w}_{\varepsilon,z}|^q \frac{dx_1}{x_1} dx'] \\
&\leq \frac{1}{n}S^{n/2} + O(\varepsilon^{n-2}) - \frac{\lambda t_0^q}{q} f_{\min} \int_{\Omega_{r_0}(z)} |\overline{w}_{\varepsilon,z}|^q \frac{dx_1}{x_1} dx',
\end{aligned}$$

where $f_{\min} = \min\{f(x); x \in \overline{M}_{r_0}\} > 0$. Let $0 < \lambda \leq (\frac{r_0}{2})^{\frac{(2-q)(n-2)}{2}}$. Then we have

$$0 < \varepsilon = \lambda^{\frac{2}{(2-q)(n-2)}} \leq \frac{r_0}{2}$$

and

$$\begin{aligned}
\int_{\Omega_{\frac{r_0}{2}}(z)} |\overline{w}_{\varepsilon,z}|^q \frac{dx_1}{x_1} dx' &= \int_{\Omega_{\frac{r_0}{2}}(z)} \frac{1}{(\varepsilon^2 + |\ln \frac{x_1}{z_1}|^2 + |x'-z'|^2)^{\frac{q(n-2)}{2}}} \frac{dx_1}{x_1} dx' \\
&= \int_{\Omega_{\frac{r_0}{2}}(1,0)} \frac{1}{(\varepsilon^2 + |\ln y_1|^2 + |y'|^2)^{\frac{q(n-2)}{2}}} \frac{dy_1}{y_1} dy' \\
&= \int_{B_{\frac{r_0}{2}}} \frac{1}{(\varepsilon^2 + |z_1|^2 + |z'|^2)^{\frac{q(n-2)}{2}}} dz_1 dz' \\
&\geq \int_{B_{\frac{r_0}{2}}} \frac{1}{r_0^{q(n-2)}} dz_1 dz' = D_1(n,q,r_0)
\end{aligned}$$

for all $z \in M$, where $D_1(n,q,r_0)$ is a positive constant depends on $n, q, r_0$.

Thus for $\varepsilon = \lambda^{\frac{2}{(2-q)(n-2)}}$ and $\lambda \in (0, (\frac{r_0}{2})^{\frac{(2-q)(n-2)}{2}})$, we obtain

$$\sup_{t\geq t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) \leq \frac{1}{n}S^{n/2} + O(\lambda^{\frac{2}{(2-q)}}) - \frac{t_0^q f_{\min}}{q} D_1(n,q,r_0)\lambda.$$

Then we can choose $0 < \Lambda_* \leq \min\{\Lambda_5, (\frac{r_0}{2})^{\frac{(2-q)(n-2)}{2}}\}$ such that $\sup_{t\geq t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0$ for all $\lambda \in (0, \Lambda_*)$ and $\sup_{t\geq 0} J_\lambda(t\overline{w}_{\varepsilon,z}) = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0$ for all $z \in M$.

Finally, we will show that there exists $t_z^- > 0$ such that $t_z^- \overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ for all $z \in M$. By Lemma 3.6 and $\int_{\mathbb{B}} f_\lambda |\overline{w}_{\varepsilon,z}|^q \frac{dx_1}{x_1} dx' > 0$ and $\int_{\mathbb{B}} g|\overline{w}_{\varepsilon,z}|^{2^*} \frac{dx_1}{x_1} dx' > 0$, there exists $t_z^- > 0$ such that $t_z^- \overline{w}_{\varepsilon,z} \in N_\lambda^-$ and $J_\lambda(t_z^- \overline{w}_{\varepsilon,z}) < c_\lambda = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0$ for all $z \in M$. Thus $t_z^- \overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$. Moreover, we have $\Phi(t_z^- \overline{w}_{\varepsilon,z}) = \Phi(\overline{w}_{\varepsilon,z}) \in M_\delta$ for all $z \in M$ by the definition of $\overline{w}_{\varepsilon,z}$. We complete the proof. □

**Theorem 4.9.** *Let $\delta, d_\delta > 0$ be as in Theorem 4.7. Then there exists $0 < \Lambda_\delta \leq \Lambda_*$ such that for $\lambda < \Lambda_\delta$, we have $\Phi(u) \in M_\delta$ for all $u \in N_\lambda(c_\lambda)$.*

*Proof.* For $u \in N_\lambda(c_\lambda)$, by Lemma 3.7, there exists a unique $t_u > 0$ such that $t_u u \in N_0^1$ and

$$\begin{aligned} J_0^1(t_u u) &\le (1-\lambda)^{-n/2}(J_\lambda(u) + \frac{\lambda(2-q)}{2q}(\|f_+\|_{L_{q^*}^{\frac{n}{q^*}}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}) \\ &\le (1-\lambda)^{-n/2}(\frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0 + \frac{\lambda(2-q)}{2q}(\|f_+\|_{L_{q^*}^{\frac{n}{q^*}}(\mathbb{B})} S^{-\frac{q}{2}})^{\frac{2}{2-q}}). \end{aligned}$$

Then there exists $0 < \Lambda_\delta \le \Lambda_*$ such that for $\lambda < \Lambda_\delta$,

$$J_0^1(t_u u) \le \frac{1}{n}S^{n/2} + d_\delta$$

for all $u \in N_\lambda(c_\lambda)$. By Theorem 4.7, we have $t_u u \in N_0^1(d_\delta)$ and

$$\Phi(u) = \frac{\int_{\mathbb{B}} x|t_u u|^{2^*} \frac{dx_1}{x_1} dx'}{\int_{\mathbb{B}} |t_u u|^{2^*} \frac{dx_1}{x_1} dx'} = \Phi(t_u u) \in M_\delta$$

for all $u \in N_\lambda(c_\lambda)$. This completes the proof. □

Now, we want to show that $J_\lambda$ satisfies the $(PS)_c$ condition in $H_0^1(\Omega)$ for $c \in (-\infty, c_\lambda)$, where $c_\lambda$ is defined in Theorem 4.8.

**Theorem 4.10.** *$J_\lambda$ satisfies the $(PS)_c$ condition in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ for $c \in (-\infty, c_\lambda)$.*

*Proof.* Let $\{u_k\}$ be a $(PS)_c$ sequence in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ for $J_\lambda$. It is easy to see that $\{u_k\}$ is bounded in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ by a standard argument. Going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u$ weakly in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$. By Proposition 2.6, we know $u_k \to u$ a.e. in $\mathbb{B}$ and $u_k \to u$ strongly in $L_s^{\frac{n}{s}}(\mathbb{B})$ for any $1 \le s < 2^*$. Then we obtain

$$\begin{gathered} \int_{\mathbb{B}} f_\lambda |u_k|^q \frac{dx_1}{x_1} dx' = \int_{\mathbb{B}} f_\lambda |u|^q \frac{dx_1}{x_1} dx' + o(1), \\ \|u_k - u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} = \|u_k\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - \|u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} + o(1), \\ \int_{\mathbb{B}} g|u_k - u|^{2^*} \frac{dx_1}{x_1} dx' = \int_{\mathbb{B}} g|u_k|^{2^*} \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g|u|^{2^*} \frac{dx_1}{x_1} dx' + o(1) \end{gathered}$$

Moreover, we can obtain $J'_\lambda(u) = 0$ in $\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})$. Since $J_\lambda(u_k) = c + o(1)$ and $J'_\lambda(u_k) = o(1)$ in $\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})$, we deduce that

$$\frac{1}{2}\|u_k - u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - \frac{1}{2^*}\int_\Omega g|u_k - u|^{2^*} \frac{dx_1}{x_1} dx' = c - J_\lambda(u) + o(1) \tag{4.7}$$

and

$$\|u_k - u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} - \int_{\mathbb{B}} g|u_k - u|^{2^*} \frac{dx_1}{x_1} dx' = o(1).$$

Now, we may assume that

$$\|u_k - u\|^2_{\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})} \to l, \quad \int_{\mathbb{B}} |u_k - u|^{2^*} \frac{dx_1}{x_1} dx' \to l \quad \text{as } k \to \infty. \tag{4.8}$$

Suppose $l \ne 0$. Applying Theorem 4.5, we obtain

$$(\frac{1}{2} - \frac{1}{2^*})l \ge \frac{1}{n}S^{n/2}.$$

Then by Lemma 3.1, (4.7) and (4.8), we have

$$c = (\frac{1}{2} - \frac{1}{2^*})l + J_\lambda(u) \geq \frac{1}{n}S^{n/2} - D_0\lambda^{\frac{2}{2-q}} = c_\lambda,$$

which contradicts the definition of $c$. Hence $l = 0$; that is, $u_n \to u$ strongly in $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$. □

Now, by Theorems 4.6, 4.8, and 4.10, we can find $\Lambda_\delta > 0$ such that $J_\lambda$ satisfies the $(PS)$ condition on $N_\lambda(c_\lambda)$ and $\Phi(u) \in M_\delta$ for all $u \in N_\lambda(c_\lambda)$ and $\lambda < \Lambda_\delta$. Let $F_\varepsilon(z) = t_z^- \overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ as that in Theorem 4.7. Then we have the following result.

**Theorem 4.11.** *Let $\delta$, $\Lambda_\delta > 0$ be as in Theorems 4.7 and 4.9, then for each $\lambda < \Lambda_\delta, J_\lambda$ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $N_{\lambda,+}(c_\lambda) = \{u \in N_\lambda(c_\lambda); u \geq 0\}$.*

*Proof.* By Theorem 4.8, we can assume that for any such $\lambda$ and for any $z \in M$,

$$J_\lambda(F_\varepsilon(z)) < c_\lambda = \frac{1}{n}S^{n/2} - \lambda^{\frac{2}{(2-q)}} D_0.$$

Thus $F_\varepsilon(M) \subset N_\lambda(c_\lambda)$.

Moreover, by Theorem 4.9, we get $\Phi(N_\lambda(c_\lambda)) \subset M_\delta$. Then, by Theorem 4.8, the map $\Phi \circ F$ is homotopic to the inclusion $j : M \to M_\delta$, for any $\lambda < \Lambda_\delta$. Thus by Theorem 4.10 and Propositions 4.2, 4.3, we obtain $J_\lambda$ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $N_{\lambda,+}(c_\lambda)$. This completes the proof. □

*Proof of Theorem 1.2.* By Theorems 3.8 and 4.11 and by considering Lemmas 3.2 and 3.5, we complete the proof of Theorem 1.2. □

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