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GROWTH OF SOLUTIONS TO LINEAR COMPLEX DIFFERENTIAL EQUATIONS IN AN ANGULAR REGION

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Abstract. In this article, we consider the growth of solutions of higher-order linear differential equations in an angular region instead of the complex plane.

1. Introduction and statement of results

We assume that the reader is familiar with the fundamental results and standard notations of the Nevanlinna theory in the unit disk $\Delta = \{z : |z| < 1\}$ and in the complex plane $\mathbb{C}$ (see [5, 7, 11]), such as $T(r, f), N(r, f), m(r, f), \delta(a, f)$. The order and lower order of f in $\mathbb{C}$ or in Δ are defined as follows:

$$\rho_{\mathbb{C}}(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \rho_{\Delta}(f) = \limsup_{r\to 1-} \frac{\log T(r, f)}{-\log(1-r)},$$
$$\mu_{\mathbb{C}}(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \mu_{\Delta}(f) = \liminf_{r\to 1-} \frac{\log T(r, f)}{-\log(1-r)}.$$

The meromorphic functions in the unit disk can be divided into the following three classes:

(1) bounded type: $T(r, f) = O(1)$ as $r \to 1-$;
(2) rational type: $T(r, f) = O(\log(1-r)^{-1})$ and $f(z)$ does not belong to (1);
(3) admissible in Δ:
$$\limsup_{r\to 1-} \frac{T(r, f)}{-\log(1-r)} = \infty.$$

Meromorphic functions in the complex plane can also be divided into the following three classes:

(1) bounded type: $T(r, f) = O(1)$ as $r \to \infty$;
(2) rational type: $T(r, f) = O(\log r)$ and $f(z)$ does not belong to (1);
(3) admissible in $\mathbb{C}$:
$$\limsup_{r\to\infty} \frac{T(r, f)}{\log r} = \infty.$$

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The growth of solutions to higher-order linear differential equations in $\mathbb{C}$ and in Δ has been investigated by many authors. Gundersen [4] and Heittokangas [6] considered the growth of solutions of the second-order linear differential equations and obtained a theorem in $\mathbb{C}$ and in Δ respectively as follows.

Theorem 1.1 ([4, 6])**.** *Let $B(z)$ and $C(z)$ be the analytic coefficients of the equation*

$$g'' + B(z)g' + C(z)g = 0 \tag{1.1}$$

in $\mathbb{C}$ (or in Δ). If either (i) $\rho(B) < \rho(C)$ or (ii) $B(z)$ is non-admissible while $C(z)$ is admissible, then all solutions $g \not\equiv 0$ of (1.1) are of infinite order of growth.

Chen [1] generalized Theorem 1.1 as follows.

Theorem 1.2 ([1])**.** *Let $A_0(z), \dots, A_k(z)$ be the analytic coefficients of the equation*

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0 \tag{1.2}$$

in $\mathbb{C}$ (or in Δ). If either (i) $\max_{1\le j\le k} \rho(A_j) < \rho(A_0)$, or (ii) $A_j(z)$ $(j = 1, 2, \dots, k)$ are non-admissible while $A_0(z)$ is admissible, then all solutions $f \not\equiv 0$ of (1.2) are of infinite order of growth.

In 1994, Wu [8, 9] used the Nevanlinna theory in an angle to study the growth of solutions of the second-order linear differential equation in an angular region and obtained the following two theorems.

Theorem 1.3 ([8])**.** *Let $A(z)$ and $B(z)$ be meromorphic in $\mathbb{C}$ with $\rho(A) < \rho(B)$ and $\delta(\infty, B) > 0$. Then every nontrivial meromorphic solution f of the equation*

$$f'' + A(z)f' + B(z)f = 0 \tag{1.3}$$

has infinite order. Furthermore, if $\rho(B) \le 1/2$ and $\delta(\infty, B) = 1$, then $\rho_{\alpha,\beta}(f) = +\infty$ for every angular region $\Omega(\alpha, \beta)$.

Theorem 1.4 ([9])**.** *Let $A(z)$ and $B(z)$ be analytic on $\overline{\Omega}(\alpha, \beta)$. If for any $K > 0$, the measure of*

$$\Big\{\theta : \alpha < \theta < \beta, \liminf_{r\to\infty} \frac{(|A(re^{i\theta})| + 1)r^K}{|B(re^{i\theta})|} = 0\Big\}$$

is larger than zero, then any solution $f \not\equiv 0$ of (1.3) has $\varrho_{\alpha,\beta}(f) = +\infty$.

In 2009, Xu and Yi [10] generalized Theorem 1.4 to the case of linear higher order differential equation and obtained the following theorem.

Theorem 1.5. [10] *Let $A_j(z)(j = 0, 1, \dots, k-1)$ be analytic on $\Omega(\alpha, \beta)(0 < \beta-\alpha \le 2\pi)$, if for any $K > 0$ the θ's which satisfy $\alpha \le \theta \le \beta$ and*

$$\liminf_{r\to\infty} \frac{(|A_1(re^{i\theta})| + \dots + |A_{k-1}(re^{i\theta})| + 1)r^K}{|A_0(re^{i\theta})|} = 0 \tag{1.4}$$

form a set of positive measure. Then for every solution $f \not\equiv 0$ of (1.4) we have $\varrho_{\alpha,\beta}(f) = +\infty$.

Remark 1.6. The order $\varrho_{\alpha,\beta}(f)$ in Theorems 1.4 and 1.5 is defined by

$$\varrho_{\alpha,\beta}(f) = \limsup_{r\to\infty} \frac{\log\log M(r, \overline{\Omega}, f)}{\log r},$$

where $M(r, \overline{\Omega}, f) = \max_{\alpha\le\theta\le\beta} |f(re^{i\theta})|$ and $f \not\equiv 0$ is a function analytic on the set $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \le \arg z \le \beta\}(0 < \beta - \alpha \le 2\pi)$. The order $\rho_{\alpha,\beta}(f)$ in this paper is different from $\varrho_{\alpha,\beta}(f)$.

It is natural to pose the following question:

> How does the solutions of linear differential equations with analytic or meromorphic coefficients grow in an angular region?

Before stating our results, we give some notation and definitions of a meromorphic function in an angular region $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$. In this article, Ω usually denotes the angular region $\Omega(\alpha,\beta)$ and $\Omega_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$, where $0<\varepsilon<(\beta-\alpha)/2$. Let $f(z)$ be a meromorphic function on $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le\arg z\le\beta\}$. Recall the definition of Ahlfors-Shimizu characteristic in an angular region (see [7]). Set $\Omega(r)=\Omega(\alpha,\beta)\cap\{z:0<|z|<r\}=\{z:\alpha<\arg z<\beta,0<|z|<r\}$. Define

$$\mathcal{S}(r,\Omega,f)=\frac{1}{\pi}\iint_{\Omega(r)}\Big(\frac{|f'(z)|}{1+|f(z)|^2}\Big)^2d\sigma,$$
$$\mathcal{T}(r,\Omega,f)=\int_0^r\frac{\mathcal{S}(t,\Omega,f)}{t}dt.$$

The order and lower order of f on Ω are defined as follows (see pp.93 in [13]):

$$\rho_\Omega(f)=\limsup_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega,f)}{\log r},\quad \mu_\Omega(f)=\liminf_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega,f)}{\log r}.$$

We remark that the order $\rho_\Omega(f)$ of a meormorphic function f in an angular region given here is reasonable, because $\mathcal{T}(r,\mathbb{C},f)=T(r,f)+O(1)$ (see pp.20 in [3]).

From [13, Theorem 2.7.6], we know that if $\varrho_{\alpha,\beta}(f)=+\infty$, then $\rho_{\alpha,\beta}(f)=+\infty$; if $\rho_{\alpha+\varepsilon,\beta-\varepsilon}(f)=+\infty$ for some $0<\varepsilon<\frac{\beta-\alpha}{2}$, then $\varrho_{\alpha,\beta}(f)=+\infty$.

Now, we state our results in the following theorems.

Theorem 1.7. *Let $A(z)$ be analytic in an angular region $\Omega=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$ satisfying*

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r,\Omega_\varepsilon,A)}{r^\omega\log r}=\infty, \tag{1.5}$$

where $\Omega_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$, $0<\varepsilon<\frac{\beta-\alpha}{2}$, $\omega=\pi/(\beta-\alpha)$. Then, all solutions $f\not\equiv0$ of the equation $f^{(k)}+A(z)f=0$ have the order $\rho_\Omega(f)=+\infty$.

Theorem 1.8. *Let $A_0,\dots,A_k$ be analytic in an angular region $\Omega=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$. If either (i) for any small $0<\varepsilon<\frac{\beta-\alpha}{2}$, we have $\rho_\Omega(A_j)<\rho_{\Omega_\varepsilon}(A_0)-\omega$ $(j=1,2,\dots,k)$, or (ii)$A_j(z)$ $(j=1,2,\dots,k)$ satisfy $\mathcal{T}(r,\Omega,A_j)=O(\log r)$ while $A_0(z)$ satisfies*

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r,\Omega_\varepsilon,A_0)}{r^\omega\log r}=\infty, \tag{1.6}$$

where $\Omega_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$, $\omega=\pi/(\beta-\alpha)$, then all solutions $f\not\equiv0$ of (1.2) have the order $\rho_\Omega(f)=+\infty$.

2. Preliminaries

In this section, we give some auxiliary results for the proof of the theorems. The following result was proved in [12].

Lemma 2.1 ([12]). *The transformation*

$$\zeta(z)=\frac{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)}-1}{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)}+1}\quad\Big(\theta_0=\frac{\alpha+\beta}{2}\Big) \tag{2.1}$$

maps the angular region $X=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$ *conformally onto the unit disk* $\{\zeta:|\zeta|<1\}$ *in the* $\zeta-$*plane, and maps* $z=e^{i\theta_0}$ *to* $\zeta=0$*. The image of* $X_\varepsilon(r)=\{z:1\le|z|\le r,\alpha+\varepsilon\le\arg z\le\beta-\varepsilon\}(0<\varepsilon<\frac{\beta-\alpha}{2})$ *in the* $\zeta-$*plane is contained in the disk* $\{\zeta:|\zeta|\le h\}$*, where*

$$h=1-\frac{\varepsilon}{\beta-\alpha}r^{-\frac{\pi}{\beta-\alpha}}.$$

On the other hand, the inverse image of the disk $\{\zeta:|\zeta|\le h\}(h<1)$ *in the* $z-$*plane is contained in* $X\cap\{z:|z|\le r\}$*, where*

$$r=\Big(\frac{2}{1-h}\Big)^{(\beta-\alpha)/\pi}.$$

The inverse transformation of (2.1) *is*

$$z=e^{i\theta_0}\Big(\frac{1+\zeta}{1-\zeta}\Big)^{(\beta-\alpha)/\pi}. \tag{2.2}$$

Using Lemma 2.1, we will prove the following lemma, which to the best of our knowledge has not been published before.

Lemma 2.2. *Let* $f(z)$ *be meromorphic in an angular region* $\Omega=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$*. For any small* $\varepsilon>0$*, write* $\omega=\frac{\pi}{\beta-\alpha},\eta=\frac{\varepsilon}{\beta-\alpha}$*. Then the following inequalities hold:*

$$\mathcal{T}(r,\mathbb{C},f(z(\zeta)))\le 2\mathcal{T}\Big(\big(\frac{2}{1-r}\big)^{1/\omega},\Omega,f(z)\Big)+O(1), \tag{2.3}$$

$$\mathcal{T}(r,\Omega_\varepsilon,f(z))\le\frac{r^\omega}{\omega\eta}\mathcal{T}(1-\eta r^{-\omega},\mathbb{C},f(z(\zeta)))+O(1), \tag{2.4}$$

where $z=z(\zeta)$ *is the inverse transformation of* (2.1)*. Consequently,*

$$\rho_\Delta(f(z(\zeta)))\le\frac{1}{\omega}\rho_\Omega(f(z)),\quad \rho_{\Omega_\varepsilon}(f(z))\le(\rho_\Delta(f(z(\zeta)))+1)\omega. \tag{2.5}$$

Proof. By Lemma 2.1, for the inverse of the transformation (2.1) it follows that

$$z(\Delta_h)\subset\Omega\cap\Big\{z:|z|\le\big(\frac{2}{1-h}\big)^{1/\omega}\Big\},\quad\text{where } \Delta_h=\{z:|z|<h\}.$$

Since the term $\mathcal{S}$ is a conformal invariant, we derive

$$\mathcal{S}(t,\mathbb{C},f(z(\zeta)))\le\mathcal{S}\Big(\big(\frac{2}{1-t}\big)^{1/\omega},\Omega,f(z)\Big).$$

Dividing the above by t and integrating from 0 to r gives

$$\begin{aligned}\mathcal{T}(r,\mathbb{C},f(z(\zeta))) &= \int_0^r \frac{\mathcal{S}(t,\mathbb{C},f(z(\zeta)))}{t}dt = \int_{1/2}^r \frac{\mathcal{S}(t,\mathbb{C},f(z(\zeta)))}{t}dt + O(1)\\ &\le 2\int_{1/2}^r \mathcal{S}(t,\mathbb{C},f(z(\zeta)))dt + O(1)\\ &\le 2\int_{1/2}^r \mathcal{S}\Big(\big(\frac{2}{1-t}\big)^{1/\omega},\Omega,f(z)\Big)dt + O(1)\\ &\le 2\int_1^{(\frac{2}{1-r})^{1/\omega}} \frac{\mathcal{S}(t,\Omega,f(z))}{t^{\omega+1}}dt + O(1)\\ &= 2\mathcal{T}\Big(\big(\frac{2}{1-r}\big)^{1/\omega},\Omega,f(z)\Big) + O(1).\end{aligned} \tag{2.6}$$

Secondly, for the transformation (2.1), we have

$$\zeta(\{z : 1 \le |z| \le r, \alpha+\varepsilon \le \arg z \le \beta-\varepsilon\}) \subset \Delta_{(1-\eta r^{-\omega})}.$$

Then,

$$\mathcal{S}(r,\Omega_\varepsilon,f(z)) \le \mathcal{S}\left(1-\eta r^{-\omega},\mathbb{C},f(z(\zeta))\right).$$

Divide the above by r and integrate from 1 to r:

$$\begin{aligned}\mathcal{T}(r,\Omega_\varepsilon,f(z)) &= \int_1^r \frac{\mathcal{S}(t,\Omega_\varepsilon,f(z))}{t}dt + O(1)\\ &\le \int_1^r \frac{\mathcal{S}(1-\eta t^{-\omega},\mathbb{C},f(z(\zeta)))}{t}dt + O(1)\\ &= \frac{1}{\omega}\int_{1-\eta}^{1-\eta r^{-\omega}} \frac{\mathcal{S}(x,\mathbb{C},f(z(\zeta)))}{1-x}dx + O(1)\\ &\le \frac{r^\omega}{\omega\eta}\int_{1-\eta}^{1-\eta r^{-\omega}} \mathcal{S}(x,\mathbb{C},f(z(\zeta)))dx + O(1)\\ &\le \frac{r^\omega}{\omega\eta}\int_{1-\eta}^{1-\eta r^{-\omega}} \frac{\mathcal{S}(x,\mathbb{C},f(z(\zeta)))}{x}dx + O(1)\\ &= \frac{r^\omega}{\omega\eta}\mathcal{T}(1-\eta r^{-\omega},\mathbb{C},f(z(\zeta))) + O(1).\end{aligned} \tag{2.7}$$

Using the definition of order, we obtain (2.5). The proof is complete. □

Lemma 2.3 ([2]). *Let $f(z)$ be meromorphic in $\Omega = \{z : \alpha < \arg z < \beta\}(0 < \beta-\alpha < 2\pi)$ and $z = z(\zeta)$ be the inverse transformation of (2.1). Write $F(\zeta) = f(z(\zeta)), \psi(\zeta) = f^{(l)}(z(\zeta))$. Then,*

$$\psi(\zeta) = \sum_{j=1}^{l} \alpha_j F^{(j)}(\zeta) \tag{2.8}$$

where the coefficients α_j are the polynomials (with numerical coefficients) in the variables $V(\zeta)(= \frac{1}{z'(\zeta)}), V'(\zeta), V''(\zeta), \dots$. Moreover, we have $T(r,\alpha_j) = O(\log(1-r)^{-1}), j = 1,2,\dots,l$.

Lemma 2.3 can be proved by the same method of [2, Lemma 1], where the lemma was stated for a different transformation:

$$\zeta = \frac{(ze^{-i\theta_0})^{\omega} - (ze^{-i\theta_0})^{-\omega} - \kappa}{(ze^{-i\theta_0})^{\omega} - (ze^{-i\theta_0})^{-\omega} + \kappa}, \quad \omega = \frac{\pi}{\beta - \alpha}, \tag{2.9}$$

where κ is a positive parameter. However, the conformal transformation (2.9) maps the sector $\{z : |z| > 1, \alpha < \arg ztheta < \beta\}(0 < \beta - \alpha < 2\pi)$ onto the unit disk $\{\zeta : |\zeta| < 1\}$ while the transformation (2.1) maps the angular region $\{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$ onto the unit disk $\{\zeta : |\zeta| < 1\}$. For completeness, we give the proof of Lemma 2.3 using the method of [2, Lemma 1].

Proof. Put

$$V(\zeta) = \frac{1}{z'(\zeta)}.$$

By a simple calculation, we have

$$f'(z(\zeta)) = V(\zeta)F'(\zeta).$$

An obvious induction shows that

$$\psi(\zeta) = f^{(l)}(z(\zeta)) = \sum_{j=1}^{l} \alpha_j F^{(j)}(\zeta)$$

where the coefficients α_j are polynomials (with numerical coefficients) in the variables $V, V', V'', \dots$. Taking the derivative on both side of (2.2), we obtain that

$$\frac{dz}{d\zeta} = \frac{e^{i\theta_0}}{\omega}\Big(\frac{1+\zeta}{1-\zeta}\Big)^{\frac{1}{\omega}-1} \frac{2}{(1-\zeta)^2}, \quad \omega = \frac{\pi}{\beta - \alpha}, \theta_0 = \frac{\alpha+\beta}{2}.$$

Then

$$\Big|\frac{dz}{d\zeta}\Big| \le \frac{1}{\omega} \frac{2^{1/\omega}}{(1-|\zeta|)^{\frac{1}{\omega}+1}}.$$

Therefore,

$$T(r, z'(\zeta)) \le \log M(r, z'(\zeta)) \le \Big(\frac{1}{\omega} + 1\Big) \log \frac{2}{1-r} + \log \frac{1}{\omega},$$

By the first fundamental theorem,

$$T\Big(r, \frac{1}{z'(\zeta)}\Big) = T(r, z'(\zeta)) + \log \frac{1}{|z'(0)|} \le \Big(\frac{1}{\omega} + 1\Big) \log \frac{2}{1-r} + \log \frac{1}{\omega} + \log 2\omega.$$

Thus,

$$T(r, V(\zeta)) = T\Big(r, \frac{1}{z'(\zeta)}\Big) \le \Big(\frac{1}{\omega} + 1\Big) \log \frac{2}{1-r} + \log \frac{1}{\omega} + \log 2\omega,$$

$$\begin{aligned} T(r, V^{(k)}) &= m(r, V^{(k)}) + N(r, V^{(k)}) \le m\Big(r, \frac{V^{(k)}}{V}\Big) + m(r, V) + kN(r, V) \\ &\le m\Big(r, \frac{V^{(k)}}{V}\Big) + (k+1)T(r, V) \le O\Big(\log \frac{2}{1-r}\Big), \quad k = 1, 2, \dots. \end{aligned}$$

In view of the coefficients α_j are polynomials (with numerical coefficients) in the variables $V, V', V'', \dots$, we have

$$T(r, \alpha_j) \le O\Big(\log \frac{2}{1-r}\Big), \quad j = 1, 2, \dots, l.$$

The proof is complete. □

3. Proof of Theorem 1.7

Suppose that $f \not\equiv 0$ is a solution of $f^{(k)} + A(z)f = 0$ in Ω. Then $F(\zeta) = f(z(\zeta))$ is a solution of the differential equation

$$\alpha_k F^{(k)}(\zeta) + \alpha_{k-1} F^{(k-1)}(\zeta) + \cdots + \alpha_1 F'(\zeta) + B(\zeta)F(\zeta) = 0 \tag{3.1}$$

in Δ, where $B(\zeta) = A(z(\zeta)), \alpha_j (j = 1, 2, \dots, k)$ are described in Lemma 2.3. From the condition (1.5) and the inequality (2.4), we obtain that B is admissible in Δ. By Lemma 2.3, we get that $T(r, \alpha_j) = O(\log(1-r)^{-1})(j = 1, 2, \dots, k)$, so $\alpha_j (j = 1, 2, \dots, k)$ are non-admissible in Δ. By Theorem 1.2, we have $\rho_\Delta(F) = \infty$. Combining this with (2.5) leads to $\rho_\Omega(f) = \infty$. Theorem 1.7 follows.

4. Proof of Theorem 1.8

Suppose that $f \not\equiv 0$ is a solution of (1.2) in Ω. In view of (2.8), we have

$$\begin{aligned}
&\sum_{i=1}^{k} A_i(z(\zeta)) f^{(i)}(z(\zeta)) + A_0(z(\zeta)) f(z(\zeta)) \\
&= \sum_{i=1}^{k} A_i(z(\zeta)) \sum_{j=1}^{i} \alpha_j F^{(j)}(\zeta) + A_0(z(\zeta)) f(z(\zeta)) \\
&= \sum_{j=1}^{k} \alpha_j \sum_{i=j}^{k} A_i(z(\zeta)) F^{(j)}(\zeta) + A_0(z(\zeta)) f(z(\zeta)).
\end{aligned}$$

Then $F(\zeta) = f(z(\zeta))$ is a solution of the differential equation

$$B_k(\zeta)F^{(k)}(\zeta) + B_{k-1}(\zeta)F^{(k-1)}(\zeta) + \cdots + B_1(\zeta)F'(\zeta) + B_0(\zeta)F(\zeta) = 0 \tag{4.1}$$

in Δ, where $B_0(\zeta) = A_0(z(\zeta)), B_j(\zeta) = \alpha_j \sum_{i=j}^{k} A_i(z(\zeta))(j = 1, 2, \dots, k)$. Since $T(r, \alpha_j) = O(\log(1-r)^{-1})(j = 1, 2, \dots, k)$, it follows that

$$\begin{aligned}
T(r, B_j) &\le T(r, \alpha_j) + \sum_{i=j}^{k} T(r, A_i(z(\zeta))) + O(1) \\
&= \sum_{i=j}^{k} T(r, A_i(z(\zeta))) + O(\log(1-r)^{-1}), \quad j = 1, 2, \dots, k.
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{gathered}
\frac{1}{\omega} \rho_{\Omega_\varepsilon}(A_0) \le \rho_\Delta(B_0) + 1, \\
\rho_\Delta(B_j) \le \frac{1}{\omega} \max_{1 \le j \le k} \rho_\Omega(A_j), \quad \text{for } j = 1, 2, \dots, k.
\end{gathered}$$

Combining the above with the condition (i) gives

$$\rho_\Delta(B_j) < \rho_\Delta(B_0), \quad \text{for } j = 1, 2, \dots, k.$$

By Theorem 1.2, we have $\rho_\Delta(F) = \infty$. Combining this with (2.5) leads to $\rho_\Omega(f) = \infty$.

In view of $\mathcal{T}(r, \Omega, A_j) = O(\log r)$ it follows that $T(r, B_j) = O(\log(1-r)^{-1})$. From the condition (1.6) and the inequality (2.4), we obtain that B_0 is admissible in Δ. By Theorem 1.2, we have that $\rho_\Delta(F) = \infty$. This leads to $\rho_\Omega(f) = \infty$. Then Theorem 1.8 follows.

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References

[1] Z. X. Chen; *The properties of solutions of class of differential equations in the unit disk* (in Chinese), J. Jiangxi Norm. Univer., 2002, 26(3): 189-190.

[2] A. Edrei; *Meromorphic functions with three radially distributed values*, Trans. Amer. Math. Soc., 1955(78): 276-293.

[3] A. A. Goldberg, I. V. Ostrovskii; *Value Distribution of Meromorphic Functions*, Translations of Mathematical Monographs, Vol.236, 2008.

[4] G. Gundersen; *Finite order solutions of second differential equations*, Trans. Amer. Math. Soc., 1988, 305(1): 415-429.

[5] W. K. Hayman; *Meromorphic Functions*, Oxford, 1964.

[6] J. Heitiokangas; *On complex differential equations in the unit disc.* Ann Acad Sci, Fennica Math. Dissertations, 2000, 25(1): 1-54.

[7] M. Tsuji; *Potential theory in modern function theory*, Maruzen Co. LTD Tokyo, 1959.

[8] S. J. Wu; *Estimates for the logarithmic derivative of a meromorphic function in an angle, and their application.* Proceedings of the conference on complex analysis, Edited by Zhong Li, Fu Yao Ren, Lo Yang and Shun Yan Zhang, Conference Proceedings and Lecture Notes in Analysis, I, International Press, Cambridge, MA, 1994: 235-240.

[9] S. J. Wu; *On the growth of solutions of second order linear differential equations in an angle,* Complex Variables, 1994(24): 241-248.

[10] J. F. Xu, H. X. Yi; *Solutions of higher order linear differential equations in an angle*, Applied Mathematics Letters, 2009, 22(4): 484-489.

[11] L. Yang; *Value Distribution And New Research*, Springer-Verlag, Berlin, 1993.

[12] G. H. Zhang; *Theory of Entire and Meromorphic Functions–Deficient and Asymptotic Values and Singular Directions*(in Chinese), Science Press, Beijing, 1986. English translate in American Mathematical Society, Providence, RI, 1993.

[13] J. H. Zheng; *Value Distribution of Meromorphic Functions*, Tsinghua University Press and Springer-Verlag, Beijing and Berlin, 2010.

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