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OSCILLATION OF SOLUTIONS TO SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS OF GENERALIZED EULER TYPE

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Abstract. We are concerned with the oscillatory behavior of the solutions of a generalized Euler differential equation where the nonlinearities satisfy smoothness conditions which guarantee the uniqueness of solutions of initial value problems, however, no conditions of sub(super) linearity are assumed. Some implicit necessary and sufficient conditions and some explicit sufficient conditions are given for all nontrivial solutions of this equation to be oscillatory or nonoscillatory. Also, it is proved that solutions of the equation are all oscillatory or all nonoscillatory and cannot be both.

1. Introduction

The Euler equation arises in some practical problems in physics and engineering. In particular there are many results in the literature on the existence of oscillatory, periodic and almost periodic solutions of Euler equation.

The oscillation problem for second-order nonlinear differential equations has been studied in many papers; see the references in this article. In this article, we consider the second-order nonlinear differential equation of generalized Euler type

$$t^2\ddot{u} + f(u)t\dot{u} + g(u) = 0 \quad t > 0, \tag{1.1}$$

and give some implicit necessary and sufficient conditions and some explicit sufficient conditions for all nontrivial solutions of this equation to be oscillatory. Here, $f(u)$ and $g(u)$ satisfy smoothness conditions which guarantee the uniqueness of solutions of initial value problems and

$$ug(u) > 0 \quad \text{if } u \neq 0. \tag{1.2}$$

We suppose that all solutions of (1.1) are continuable in the future. A nontrivial solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, the solution is said to be *nonoscillatory.* For brevity, equation (1.1) is called oscillatory (respectively nonoscillatory) if all nontrivial solutions are oscillatory (respectively nonoscillatory). Because of Sturm's separation theorem, the solutions of second order linear differential equations are either all oscillatory or all nonoscillatory, but not both. Thus, we can classify second order linear differential equations

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into two types. However, the oscillation problem for (1.1) is not easy, because $g(u)$ and $f(u)$ are nonlinear.

If we let $f(u) = 0$ and $g(u) = \lambda u$, then (1.1) is called Euler differential equation. In this case, the number 1/4 is called the oscillation constant and it is the lower bound for all nontrivial solutions of (1.1) to be oscillatory. In fact, it is well known that if $\lambda > 1/4$, then all nontrivial solutions of (1.1) are oscillatory and otherwise they are nonoscillatory. Other results on the oscillation constant for linear differential equations can be found in [12, 13, 14, 15, 17, 19] and the references cited therein.

Several authors consider oscillation of solutions of second-order ordinary differential equations and some results can be found in [11, 12, 18, 20, 27]. Wong [27] studied the equation

$$u'' + a(t)g(u) = 0, \quad t > 0, \tag{1.3}$$

which includes the Emden Fowler differential equation. He used Sturm's comparison theorem and proved the following two theorems:

Theorem 1.1. *Assume that $a(t)$ is continuously differentiable and satisfies*

$$t^2 a(t) \geq 1, \tag{1.4}$$

for t sufficiently large, and that there exists a λ with $\lambda > 1/4$ such that

$$\frac{g(u)}{u} \geq \frac{1}{4} + \frac{\lambda}{(\log|u|)^2}, \tag{1.5}$$

for $|u|$ sufficiently large. Then all nontrivial solutions of (1.3) are oscillatory.

Theorem 1.2. *Assume that $a(t)$ is continuously differentiable and satisfies*

$$0 \leq t^2 a(t) \leq 1, \tag{1.6}$$

for t sufficiently large and

$$A(t) := \frac{a'(t)}{2a^{\frac{3}{2}}(t)} + 1 = o(t) \quad \text{as } t \to \infty. \tag{1.7}$$

If, in addition, $A(t) \leq 0$ and there exists a λ with $0 < \lambda \leq 1/16$ such that

$$\frac{g(u)}{u} \leq \frac{1}{4} + \frac{\lambda}{(\log|u|)^2}, \tag{1.8}$$

for $u > 0$ or $u < 0$, $|u|$ sufficiently large, then all nontrivial solutions of (1.3) are nonoscillatory.

The oscillation problem for (1.1) (when $f(u) = 0$) has been solved when

$$\limsup_{|u|\to\infty} \frac{g(u)}{u} < \frac{1}{4} \quad \text{or} \quad \liminf_{|u|\to\infty} \frac{g(u)}{u} > \frac{1}{4}.$$

In [7] the authors gave sufficient conditions for all nontrivial solutions of (1.1) to be oscillatory (when $f(u) = 0$) which can be applied in the case:

$$\liminf_{|u|\to\infty} \frac{g(u)}{u} \leq \frac{1}{4} \leq \limsup_{|u|\to\infty} \frac{g(u)}{u}.$$

In the next section, we will introduce a Liénard system which is equivalent to (1.1). To study the oscillation problem for (1.1) the significant point is to find conditions for deciding whether all orbits intersect the vertical isocline $y = F(u)$ and the y-axis in the equivalent Liénard system.

2. Equivalent Liénard System and Property of (X^+)

The change of variable $t = e^s$ reduces (1.1) to the equation

$$\ddot{u} + \dot{u}(f(u) - 1) + g(u) = 0, \quad s \in \mathbb{R}, \tag{2.1}$$

where $\dot{} = \frac{d}{ds}$. Equation (2.1) is usually studied by means of an equivalent plane differential system. The most common one is

$$\begin{aligned} \dot{u} &= y - F(u) \\ \dot{y} &= -g(u), \end{aligned} \tag{2.2}$$

where $F(u) = \int_0^u f(\eta)d\eta - u$. This system is of Liénard type. Hereafter we denote s by t again.

Definition 2.1. System (2.2) has property (X^+) in the right half plane (resp. in the left half plane), if for every point $P = (u_0, y_0)$ with $y_0 > F(u_0)$ and $u_0 \geq 0$ (resp. $y_0 < F(u_0)$ and $u_0 \leq 0$), the positive semitrajectory of (2.2) passing through P denoted by $\gamma^+(P) := \{(u(t), y(t)) | t > t_0,\ (u(t_0), y(t_0)) = P\}$ crosses the vertical isocline $y = F(u)$.

Several interesting sufficient conditions for property (X^+) have been presented in [2, 5, 6, 7, 13, 14, 15, 25]. To study the oscillation problem for (1.1) we must find conditions for deciding whether all orbits intersect the vertical isocline $y = F(u)$ in the equivalent Liénard system (2.2) or not. Let

$$G(u) = \int_0^u g(\xi)d\xi.$$

Recently, in [6] the authors presented some sufficient conditions for property (X^+) in the right and left half-plane for system (2.2).

Theorem 2.2 ([2, Theorem 2.3])**.** *Assume that $G(+\infty) = +\infty$. Then, system (2.2) has property (X^+) in the right half-plane if*

$$\limsup_{u\to+\infty} \Big(\int_b^u \Big(\frac{F(\eta)g(\eta)}{(2G(\eta))^{3/2}} + \frac{g(\eta)}{G(\eta)} \Big) d\eta + \frac{F(u)}{\sqrt{2G(u)}} \Big) = +\infty, \tag{2.3}$$

for some $b > 0$.

Theorem 2.3 ([2, Theorem 4.3])**.** *Assume that $G(-\infty) = +\infty$. Then, system (2.2) has property (X^+) in the left half-plane if*

$$\liminf_{u\to-\infty} \int_u^b \Big(-\frac{F(\eta)g(\eta)}{(2G(\eta))^{3/2}} + \frac{g(\eta)}{G(\eta)} \Big) d\eta + \frac{F(u)}{\sqrt{2G(u)}} = -\infty, \tag{2.4}$$

for some $b < 0$.

The following theorem, which is a modification of Theorem 2.4 in [5], gives a necessary and sufficient condition for system (2.2) to have property (X^+) in the right half-plane.

Theorem 2.4 ([14])**.** *System (2.2) fails to have property (X^+) in the right half-plane if and only if there exist a constant $b \geq 0$ and a function $\varphi \in C^1([b, +\infty))$ such that $\varphi(u) > 0$ for $u \geq b$ and*

$$g(u) \leq -\varphi(u)(F'(u) + \varphi'(u)), \quad \text{for } u \geq b. \tag{2.5}$$

Similarly, the following analogous result is obtained with respect to property (X^+) in the left half-plane for system (2.2).

Theorem 2.5. *System* (2.2) *fails to have property* (X^+) *in the left half-plane if and only if there exist a constant* $b \le 0$ *and a function* $\varphi \in C^1((-\infty, b])$ *such that* $\varphi(u) > 0$*, for* $u \le b$ *and*

$$g(u) \ge -\varphi(u)(-F'(u) + \varphi'(u)), \quad \text{for } u \le b. \tag{2.6}$$

Definition 2.6. Equation (1.1) has property (X^+) in the right half plane (resp. in the left half plane), if system (2.2), which is equivalent with (1.1), has property (X^+) in the right half plane (resp. in the left half plane).

Thus we have the following two theorems.

Theorem 2.7. *Equation* (1.1) *fails to have property* (X^+) *in the right half-plane if and only if there exist a constant* $b \ge 0$ *and a function* $\varphi \in C^1([b, +\infty))$ *such that* $\varphi(u) > 0$ *for* $u \ge b$ *and*

$$g(u) \le -\varphi(u)(f(u) - 1 + \varphi'(u)), \quad \text{for } u \ge b. \tag{2.7}$$

Theorem 2.8. *Equation* (1.1) *fails to have property* (X^+) *in the left half-plane if and only if there exist a constant* $b \le 0$ *and a function* $\varphi \in C^1((-\infty, b])$ *such that* $\varphi(u) > 0$*, for* $u \le b$ *and*

$$g(u) \ge -\varphi(u)(1 - f(u) + \varphi'(u)), \quad \text{for } \ u \le b. \tag{2.8}$$

3. Explicit Conditions for Property of (X^+)

In this section we present some explicit sufficient conditions for equation (1.1) having property (X^+) in the right half-plane.

Corollary 3.1. *Suppose that* $G(+\infty) = +\infty$ *and* $\liminf_{u\to\infty} f(u) > 1$. *Then equation* (1.1) *has property* (X^+) *in the right half-plane.*

Proof. By way of contradiction, we suppose that equation (1.1) fails to have property (X^+) in the right half-plane. Therefore by Theorem 2.7 there exist a constant $b \ge 0$ and a function $\varphi \in C^1([b, +\infty))$ such that $\varphi(u) > 0$ for $u \ge b$ and

$$g(u) \le -\varphi(u)(f(u) - 1 + \varphi'(u)). \tag{3.1}$$

However, since $\liminf_{u\to\infty} f(u) > 1$ there exists $b' > 0$ such that $f(u) > 1$ for $u > b'$. Now from (3.1) for $u > max\{b, b'\} = b''$ we have

$$-g(u) \ge \varphi(u)\varphi'(u).$$

By integration we have

$$-G(u) + G(b'') \ge \frac{\varphi^2(u)}{2} - \frac{\varphi^2(b'')}{2},$$

for u sufficiently large. This is a contradiction since $G(+\infty) = +\infty$ and the proof is complete. $\square$

Corollary 3.2. *Suppose that* $\limsup_{u\to\infty} f(u) = \lambda < 1$. *Then equation* (1.1) *fails to have property* (X^+) *in the right half-plane if*

$$\limsup_{u\to\infty} \frac{g(u)}{u} < \frac{(1-\lambda)^2}{4}. \tag{3.2}$$

Proof. By (3.2) there exist β with $\lambda < \beta < 1$ and $b > 0$ such that

$$\frac{g(u)}{u} < \frac{(1-\beta)^2}{4} \quad \text{for } u \ge b.$$

However, since $\limsup_{u\to\infty} f(u) = \lambda$ there exists a $b' > 0$ such that $f(u) < \beta$ for $u > b'$. Now let $\varphi(u) = (\frac{1-\beta}{2})u$. For $u \ge \max\{b, b'\}$ we have

$$\begin{aligned}
\frac{-\varphi(u)(f(u)-1+\varphi'(u))}{u} &= -\frac{(1-\beta)}{2}\big(-1+f(u)+\frac{1-\beta}{2}\big)\\
&\ge -\frac{(1-\beta)}{2}\big(-1+\beta+\frac{1-\beta}{2}\big)\\
&= \frac{(1-\beta)^2}{4} > \frac{g(u)}{u}.
\end{aligned}$$

From Theorem 2.7 this equation fails to have property (X^+) in the right half-plane. ☐

Example 3.3. Consider (1.1) with

$$\begin{gathered}
f(u) = \beta\sin u + \frac{1}{u^2+1}, \quad \beta < 1,\\
g(u) = u^\gamma \tan^{-1} u + \alpha u, \quad \alpha < \frac{(1-\beta)^2}{4},\ 0 < \gamma < 1.
\end{gathered}$$

Since $\limsup_{u\to\infty} f(u) = \beta < 1$ and

$$\limsup_{u\to\infty} \frac{g(u)}{u} = \limsup_{u\to\infty} \frac{tan^{-1}u}{u^{1-\gamma}} + \alpha = \alpha < \frac{(1-\beta)^2}{4},$$

by Corollary 3.2 this equation does not have property (X^+) in the right half-plane.

Corollary 3.4. *Suppose that* $\limsup_{u\to\infty} f(u) < \beta < 1$. *Then* (1.1) *fails to have property* (X^+) *in the right half-plane, if there exist* λ *with* $0 \le \lambda < (1-\beta)^2/16$ *and a* $k > 0$ *such*

$$\frac{g(u)}{u} \le \frac{(1-\beta)^2}{4} + \frac{\lambda}{(log(ku))^2}, \tag{3.3}$$

for u sufficiently large.

Proof. Define $\varphi(u) = \frac{1-\beta}{2}u + \frac{\alpha u}{log(ku)}$ where $k > 0$ and α is a constant such that $\lambda < \frac{1-\beta}{2}\alpha - \alpha^2 < \frac{(1-\beta)^2}{16}$, (notice that $\max_{\alpha\in\mathbb{R}}(\frac{1-\beta}{2}\alpha - \alpha^2) = \frac{(1-\beta)^2}{16}$). For u sufficiently large we have

$$\begin{aligned}
&\frac{-\varphi(u)(f(u)-1+\varphi'(u))}{u}\\
&= \Big(\frac{1-\beta}{2} + \frac{\alpha}{\log(ku)}\Big)\Big(1 - \frac{1-\beta}{2} - \frac{\alpha\log(ku)-\alpha}{(\log(ku))^2} - f(u)\Big)\\
&\ge \Big(\frac{1-\beta}{2} + \frac{\alpha}{\log(ku)}\Big)\Big(1 - \frac{1-\beta}{2} - \frac{\alpha\log(ku)-\alpha}{(\log(ku))^2} - \beta\Big)\\
&= \frac{(1-\beta)^2}{4} + \frac{\frac{1-\beta}{2}\alpha - \alpha^2}{(\log(ku))^2} + \frac{\alpha^2}{(\log(ku))^3}\\
&> \frac{(1-\beta)^2}{4} + \frac{\lambda}{(\log(ku))^2} \ge \frac{g(u)}{u}.
\end{aligned}$$

From Theorem 2.7 this equation fails to have property (X^+) in the right half-plane. □

Corollary 3.5. *Suppose that* $\limsup_{u\to\infty} f(u) = 0$. *Then* (1.1) *fails to have property* (X^+) *in the right half-plane, if there exist* λ *with* $0 < \lambda < 1/16$, *and a* $k > 0$ *such that*

$$\frac{g(u)}{u} \le \frac{1}{4} + \frac{\lambda}{(\log(ku))^2}, \tag{3.4}$$

for u *sufficiently large.*

Corollary 3.6. *Suppose that there exists a constant* $b > 0$ *such that* $g(u)$ *is in* $C^1([b, +\infty))$. *Then* (1.1) *fails to have property* (X^+) *in the right half-plane, if*

$$g'(u) \le -f(u),$$

for u *sufficiently large.*

Proof. For u sufficiently large let $\varphi(u) = g(u)$. Note that from (1.2), $g'(u) \le -f(u)$ implies $g(u) \le -g(u)(f(u) - 1 + g'(u))$, so

$$g(u) \le -\varphi(u)(f(u) - 1 + \varphi'(u)),$$

for u sufficiently large. Thus, if $g'(u) \le -f(u)$ by Theorem 2.7 this equation fails to have property (X^+) in the right half-plane. □

In [5] the authors proved the following lemma.

Lemma 3.7. *Let* $\lambda > 1/16$ *and* $k > 0$. *Then there does not exist a function* $\varphi(u)$ *such that* $\varphi \in C^1(\mathbb{R})$, $\varphi(u) > 0$ *for* u *sufficiently large, and*

$$\varphi(u)(1 - \varphi'(u)) \ge \frac{1}{4}u + \frac{\lambda u}{(\log(ku))^2}, \tag{3.5}$$

for u *sufficiently large.*

From the above lemma, immediately we have the following result.

Lemma 3.8. *Let* $\lambda > 1/16$, $k > 0$ *and* $f(u) \ge 0$ *for* u *sufficiently large. Then there does not exist a function* $\varphi(u)$ *such that* $\varphi \in C^1(\mathbb{R})$, $\varphi(u) > 0$ *for* u *sufficiently large, and*

$$\varphi(u)(1 - f(u) - \varphi'(u)) \ge \frac{1}{4}u + \frac{\lambda u}{(\log(ku))^2}, \tag{3.6}$$

for u *sufficiently large.*

Corollary 3.9. *Let* $\lambda > \frac{1}{16}$, $k > 0$ *and* $f(u) \ge 0$ *for* u *sufficiently large. Then equation* (1.1) *has property* (X^+) *in the right half-plane if*

$$\frac{g(u)}{u} \ge \frac{1}{4} + \frac{\lambda}{(\log(ku))^2}, \tag{3.7}$$

for u *sufficiently large.*

Remark 3.10. Corollaries 3.1, 3.2, 3.4, 3.5, 3.6 and 3.9 can be formulated for property (X^+) in the left half-plan.

We say that system (2.2) has property (Z_1^+) (resp., (Z_3^+)) if there exists a point $P(u_0, y_0)$ with $y_0 = F(u_0)$ and $u_0 \ge 0$ (resp., $u_0 \le 0$) such that the positive semi-trajectory of (2.2) starting at P approaches the origin through only the first (resp., third) quadrant. Now we consider the following theorem and corollary concerning property (Z_1^+) which was established in [4].

Theorem 3.11 ([4]). *Suppose that there exists a $\delta > 0$ such that $F(u) > 0$ for $0 < u < \delta$. If for every $k \in [0,1]$ there exists a constant $\gamma_k > 0$ such that*
$$\liminf_{u\to 0^+}\Big(\frac{\int_0^u \frac{g(\eta)}{F(\eta)}d\eta}{(1-k+\gamma_k)(k+\gamma_k)F(u)}\Big) > 1, \tag{3.8}$$
then (2.2) fails to have property (Z_1^+).

Corollary 3.12 ([4]). *Suppose that there exists a $\delta > 0$ such that $F(u) > 0$ for $0 < u < \delta$. If there exists a ξ with $1 < \xi \le 2$ such that*
$$\liminf_{u\to 0^+}\Big(\frac{\int_0^u \frac{g(\eta)}{F(\eta)}d\eta}{\xi F(u)}\Big) > 1, \tag{3.9}$$
then the system (2.2) fails to have property (Z_1^+).

Using Theorem 3.11, we prove the following lemma about the asymptotic behavior of system (2.2).

Lemma 3.13. *For each point $P(p, F(p))$ with $p > 0$, the positive semitrajectory of (2.2) starting at P crosses the negative y-axis if one of the following conditions hold.*

(i) *There exists a $\delta > 0$ such that $F(u) < 0$ for $0 < u < \delta$ or $F(u)$ has an infinite number of positive zeroes clustering at $u = 0$.*
(ii) *If there exists a $\delta > 0$ such that $F(u) > 0$ for $0 < u < \delta$ and the conditions of Theorem 3.11 hold.*

Proof. Suppose that there exists a point $P(p, F(p))$ with $p > 0$ such that the positive semitrajectory of (2.2) starting at P does not intersect the negative y-axis. Let $(u(t), y(t))$ be a solution of (2.2) defined on an interval $[t_0,\infty)$ with $(u(t_0), y(t_0)) = P$. Then the solution $(u(t), y(t))$ corresponds to the positive semitrajectory of (2.2) starting at P. At $t = t_0$ we have
$$\dot u(t_0) = 0, \quad \dot y(t_0) = -g(p) < 0.$$
So, this positive trajectory enters the region $D := \{(u, y) : y < F(u),\ u > 0\}$. Taking into account the vector field of (2.2) we have
$$\dot u < 0 \quad \text{in } D.$$
Thus, by the assumption that this trajectory does not intersect the negative y-axis we have
$$0 < u(t) \le u(t_0) \quad \text{for } t \ge t_0.$$
On the other hand, $\dot y < 0$ in D. So, if $y(t)$ does not approach $-\infty$ it must tends to 0 (note that the origin is the only equilibrium point of (2.2)), which is impossible in both cases (i) and (ii). Therefore,
$$y(t) \to -\infty \quad \text{as } t \to +\infty.$$
Hence, it follows from the first equation of (2.2) that
$$\dot u(t) \to -\infty \quad \text{as } t \to \infty,$$
and therefore, there exists a $t_1 > t_0$ such that
$$\dot u(t) \le -1 \quad \text{for } t \ge t_1.$$

Integration leads to

$$-u(t_1) < u(t) - u(t_1) \leq t_1 - t \to -\infty \quad \text{as } t \to \infty.$$

This is a contradiction, the proof is complete. □

Turning our attention to the left half-plane and using the change of variables $(u, y) \to (-u, -y)$, we can formulate Theorem 3.11 and Corollary 3.12 for property (Z_3^+) as follows.

Theorem 3.14. *Suppose that there exists a $\delta > 0$ such that $F(u) < 0$ for $-\delta < u < 0$. If for every $k \in [0, 1]$ there exists a constant $\gamma_k > 0$ such that*

$$\liminf_{u \to 0^-} \Big(\frac{\int_0^u \frac{g(\eta)}{F(\eta)} d\eta}{(1 - k + \gamma_k)(k + \gamma_k)F(u)} \Big) > 1, \tag{3.10}$$

then the system (2.2) fails to have property (Z_3^+).

Corollary 3.15. *Suppose that there exists a $\delta > 0$ such that $F(u) < 0$ for $-\delta < u < 0$. If there exists a ξ with $1 < \xi \leq 2$ such that*

$$\liminf_{u \to 0^-} \Big(\frac{\int_0^u \frac{g(\eta)}{F(\eta)} d\eta}{\xi F(u)} \Big) > 1, \tag{3.11}$$

then (2.2) fails to have property (Z_3^+).

Lemma 3.16. *For each point $P(-p, F(-p))$ with $p > 0$, the positive semitrajectory of (2.2) starting at P crosses the positive y-axis if one of the following conditions hold.*

- (i) *There exists a $\delta > 0$ such that $F(u) > 0$ for $-\delta < u < 0$ or $F(u)$ has infinite number of negative zeroes clustering at $u = 0$.*
- (ii) *If there exists a $\delta > 0$ such that $F(u) < 0$ for $-\delta < u < 0$ and the conditions of Theorem 3.14 hold.*

4. Oscillation Theorems

In this section we present implicit necessary and sufficient condition for all nontrivial solutions of this system to be oscillatory or nonoscillatory. To do this, we need the following two lemmas. In the proof of these lemmas we follow the ideas used in [22, Lemmas 4.1 and 4.2].

Lemma 4.1. *Every solution of (2.2) is unbounded, except the zero solution, if $uF(u) < 0$.*

Proof. By way of contradiction, we suppose that there exists a bounded solution $(u(\zeta), y(\zeta))$ of (2.2) initiating at $\zeta = \zeta_0 > 0$; that is,

$$|u(\zeta)| + |y(\zeta)| \leq A \quad \text{for } \zeta \geq \zeta_0, \tag{4.1}$$

with $A > 0$. Then the solution $(u(\zeta), y(\zeta))$ circles clockwise around the origin. Let $\zeta_i > \zeta_0$ and $b_i > 0$ with

$$(u(\zeta_i), y(\zeta_i)) = (0, b_i) \quad \text{for } i = 1, 2.$$

Define a Liapunov function

$$V(u, y) = \frac{1}{2} y^2 + G(u).$$

From $uF(u) < 0$ and (1.2), we have

$$\dot{V}_{(2.2)}(u,y) = -F(u)g(u) > 0 \quad \text{if } u \neq 0.$$

If $\zeta_1 < \zeta_2$, then $b_1 < b_2$. In fact, we have

$$\frac{1}{2}b_1^2 = V(u(\zeta_1), y(\zeta_1)) < V(u(\zeta_2), y(\zeta_2)) = \frac{1}{2}b_2^2.$$

Thus, if this solution starts at a point $(0, b_1)$ on the y-axis, it circles clockwise around the origin and intersects the y-axis again at a point $(0, b_2)$ with $b_2 > b_1$. Thus, it can not tend to the origin. Now by Poincare-Bendixson theorem [10] its ω-limit set is a periodic orbit. Therefore, we can conclude that, there exists a simple closed curve C surrounding the origin such that

$$\operatorname{dist}\big\{(u(\zeta), y(\zeta)), C\big\} \to 0 \quad \text{as } \zeta \to \infty. \tag{4.2}$$

Let $\delta_0 > 0$ be sufficiently small and define

$$M = \max\big\{g(u) : \delta_0 \leq u \leq A\big\}, \quad m = \min\big\{-F(u)g(u) : \delta_0 \leq u \leq A\big\}.$$

By (4.2) the solution $(u(\zeta), y(\zeta))$ does not stay in $\big\{(u,y) : |u| < \delta_0\big\}$. Hence, using the fact that $(u(\zeta), y(\zeta))$ circles clockwise around the origin and tends to C, there exist sequences $\{t_n\}$ and $\{t_n'\}$ with $\zeta_0 < t_n < t_n' < t_{n+1}$ and $t_n \to \infty$ as $n \to \infty$ such that

$$u(t_n) = u(t_n') = \delta_0, \quad y(t_n) > \delta_0, y(t_n') < -\delta_0$$

and $u(\zeta) > \delta_0$ for $t_n < \zeta < t_n'$. We have

$$-2\delta_0 > y(t_n') - y(t_n) = -\int_{t_n}^{t_n'} g(u(\zeta))d\zeta \geq -M(t_n' - t_n),$$

and therefore, for $\zeta > t_n'$,

$$\begin{aligned} V(u(\zeta), y(\zeta)) - V(u(\zeta_0), y(\zeta_0)) &= -\int_{\zeta_0}^{\zeta} F(u(\eta))g(u(\eta))d\eta \\ &\geq \sum_{k=1}^{n} \int_{t_k}^{t_k'} -F(u(\zeta))g(u(\zeta))d\zeta \\ &\geq m\sum_{k=1}^{n}(t_k' - t_k) \geq \frac{2m\delta_0}{M}n, \end{aligned}$$

which tends to ∞ as $n \to \infty$. Thus, $V(u(\zeta), y(\zeta)) \to +\infty$ as $\zeta \to +\infty$. This contradicts (4.1) and completes the proof. $\square$

Consider the Liapunov function

$$V(u,y) = \frac{1}{2}y^2 + G(u)$$

and consider the curve

$$V(u,y) = V(u_0, y_0),$$

where $u_0 > 0$.

It is obvious that if $uF(u) < 0$, $F'(u) \leq 0$ and from (1.2) that $V(u, F(u))$ is increasing for $u > 0$ and decreasing for $u < 0$, and $V(0,0) = 0$. Thus, the equation

$$V(u, F(u)) = V(u_0, y_0)$$

has exactly two roots. Therefore, there exist two points of intersection of this curve with the curve $y = F(u)$.

Let $(-a, F(-a))$ and $(-b, F(-b))$ be the intersection points, where $a > 0$ and $b > 0$. Define

$$S = \{(u, y) : -a \le u \le c, V(u, y) \le V(u_0, y_0)\}$$

in which $c = \max\{b, u_0\}$. Then it is clear that S is a bounded set. Lemma 4.1 shows that every solution of (2.2) starting in $S \setminus \{0\}$ does not remain in S. Note that

$$\dot{V}_{(2.2)}(u, F(u)) = -F(u)g(u) > 0 \quad \text{if } u \neq 0.$$

Then we also see that every solution of (2.2) starting in S^c, the complement of S in $\mathbb{R}^2$, stays in S^c for all future time. Therefore, we have the following lemma.

Lemma 4.2. *Suppose*

$$uF(u) < 0, \quad F'(u) \le 0. \tag{4.3}$$

Then every solution of (2.2) *starting in* $S \setminus \{0\}$ *enters* S^c *which is a positive invariant set with respect to* (2.2)*.*

Remark 4.3. Lemma 4.2 holds even if the condition $F'(u) \le 0$ is replaced by $F'(u) < \frac{-g(u)}{F(u)}$.

The main theorem of this section is as follows.

Theorem 4.4. *Suppose* (3.8) *holds. All nontrivial solutions of* (2.2) *are nonoscillatory if there exist a constant* $R > 0$ *and a function* $\varphi \in C^1(\mathbb{R} - (-R, R))$ *such that*

$$\varphi(|u|) > 0, \quad \text{and} \quad \frac{g(u)}{u} \le \frac{-\varphi(|u|)(F'(u) + \varphi'(|u|))}{|u|}, \tag{4.4}$$

for $u > R$ *or* $u < -R$*. Otherwise, all nontrivial solutions of* (2.2) *are oscillatory. Therefore, the solutions of second-order nonlinear differential equation* (1.1) *are all oscillatory or all nonoscillatory and cannot be both.*

Proof. First suppose that (4.4) does not hold. Then (2.2) which is equivalent to (1.1) has property (X^+) in the right and left half-plan. Thus, it follows from Lemmas 3.13 and 3.16 that every solution of (2.2) keeps on rotating around the origin except the zero solution. Hence, all nontrivial solutions of (1.1) are oscillatory.

Now suppose that (4.4) holds. Then by Theorems 2.4 or 2.5, system (2.2) fails to have property (X^+) in the right or left half-plane. We consider only the case that (2.2) fails to have property (X^+) in the right half-plane, because the proof in the other case is similar. Hence, there exists a point $P_0(u_0, y_0)$ with $u_0 \ge 0$ and $y_0 > F(u_0)$ such that positive semitrajectory of (2.2) runs to infinity without intersecting the curve $y = F(u)$.

Suppose (2.2) has an oscillatory solution. Let $\gamma^+(Q)$ be the positive semitrajectory which corresponds to the oscillatory solution of (1.1) starting from point Q. By virtue of Lemma 4.2 we see that $\gamma^+(Q)$ eventually goes around the set S infinitely many times. Therefore, it crosses the half-line $\{(u, y) : u = u_0 \text{ and } y > y_0\}$ at a point $P_1(u_0, y_1)$ with $y_1 > y_0$. From the uniqueness of solution for the initial value problem, it turns out that

(i) $\gamma^+(Q)$ coincides with $\gamma^+(P_1)$ except for the arc QP_1
(ii) $\gamma^+(P_1)$ lies above $\gamma^+(P_0)$

Hence, $\gamma^+(Q)$ runs to infinity without crossing the curve $y = F(u)$. This contradicts the fact that $\gamma^+(Q)$ circles the set S and completes the proof. $\square$

Example 4.5. Consider (2.2) with

$$F(u) = -u^3 - u\sin^2 u, \quad g(u) = \alpha|u|^m \operatorname{sgn}(u) + \sin u,$$

for $|u|$ sufficiently large with $\alpha > 1$ and $m \in \mathbb{R}$. It is obvious that $F(u)$ satisfies (4.3) and for $u > 0$,

$$\frac{g(u)}{u} = \alpha u^{m-1} + \frac{\sin u}{u}.$$

On the other hand, for $u > 0$ by choosing $\varphi(u) = u^\beta$ with $\beta < 3$ we have

$$\frac{-\varphi(|u|)(F'(u) + \varphi'(|u|))}{|u|} = 3u^{\beta+1} + u^{\beta-1}\sin^2 u + u^\beta \sin 2u - \beta u^{2\beta-2}.$$

Therefore, if $m < \beta + 2$ or if $m = \beta + 2$ and $\alpha < 3$, then there exist a constant $R > 0$ such that for $u > R$ condition (4.4) holds. Thus by Theorem 4.4 all nontrivial solutions of this system are nonoscillatory.

Remark 4.6. Theorem 4.4 holds even if the condition $F'(u) \le 0$ replaced by $F'(u) < -g(u)/F(u)$.

The following result follows from the above theorem.

Theorem 4.7. *All nontrivial solutions of* (1.1) *are nonoscillatory if there exist a constant $R > 0$ and a function $\varphi \in C^1(\mathbb{R} - (-R, R))$ such that*

$$\varphi(|u|) > 0 \quad \text{and} \quad \frac{g(u)}{u} \le \frac{-\varphi(|u|)((f(u) - 1) + \varphi'(|u|))}{|u|}, \tag{4.5}$$

for $u > R$ or $u < -R$. Otherwise, all nontrivial solutions of (1.1) *are oscillatory. Therefore, the solutions of second-order nonlinear differential equation* (1.1) *are all oscillatory or all nonoscillatory and cannot be both.*

Remark 4.8. Corollaries 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 can be formulated for the solutions of equation (1.1) to be nonoscillatory or oscillatory.

The following corollaries of Theorems 4.4 and 4.7 are very useful in applications.

Corollary 4.9. *Suppose that all nontrivial solutions of* (1.1) *with g_1 are oscillatory (resp. nonoscillatory). If*

$$\frac{g_2(u)}{u} > \frac{g_1(u)}{u}, \quad \textit{(resp. } \frac{g_2(u)}{u} < \frac{g_1(u)}{u}, \textit{)}$$

for $|u| > R$ with a sufficiently large R, then all nontrivial solutions of (1.1) *with g_2 are oscillatory (resp. nonoscillatory).*

Corollary 4.10. *Suppose that all nontrivial solutions of* (1.1) *with f_1 are oscillatory (resp. nonoscillatory). If*

$$f_1(u) > f_2(u), \quad \textit{(resp. } f_1(u) < f_2(u), \textit{)}$$

for $|u| > R$ with a sufficiently large R, then all nontrivial solutions of (1.1) *with f_2 are oscillatory (resp. nonoscillatory).*

References

[1] R. P. Agarwal, A. Aghajani, V. Roomi; *Existence of Homoclinic Orbits for General Planer Dynamical System of Liénard Type,* Dynamics of Continuous, Discrete and Impulsive Systems Series **A:** Mathematical Analysis **18** (2012) 267-284.

[2] A. Aghajani, Y. Jalilian, V. Roomi; *Oscillation theorems for the generalized Liénard system,* Mathematical & Computer Modelling, **54** (2011) 2471-2478.

[3] A. Aghajani, A. Moradifam; *The generalized Liénard equations,* Glasgow Mathematical Journal, **51** (2009), 605-617 .

[4] A. Aghajani, A. Moradifam; *On the homoclinic orbits of the generalized Liénard equations,* Applied Mathematics Letters, **20** (2007) 345-351.

[5] A. Aghajani, A. Moradifam; *Oscilation of solutions of second-order nonlinear differential equations of Euler type*, J. Math. Anal. Appl. **326** (2007), 1076–1089.

[6] A. Aghajani, V. Roomi; *On the property of (X^+) for generalized Liénard system,* International Journal of Dynamical Systems and Differential Equations **1** (2008), 276–282.

[7] A. Aghajani, V. Roomi; *An oscilation theorems for second-order nonlinear differential equations of Euler type,* Turkish Journal of Mathematics, **36** (2012), 273-280.

[8] F. V. Atkinson; *On second order non-linear oscillations*, Pacific J. Math. **5** (1955), 643-647.

[9] L. Cesari; *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations,* Springer-Verlag, Berlin, 1963.

[10] J. Cronin; *Ordinary Differential Equations: Introduction and Qualitative Theory,* Third Edition, Chapman & Hall/CRC Pure and Applied Mathematics, 2007.

[11] S. R. Grace; *Oscillation theorems for nonlinear differential equations of second order,* J. Math. Anal. and Appl., **171** (1992), 220241.

[12] S. R. Grace, B. S. Lalli; *On the second order nonlinear oscillations,* Bull. Inst. Math. Acad. Sinica, **15** (1987), 297309.

[13] M. Gyllenberg, P. Yan, and J. Jiang; *The qualitative behavior of the second-order system with zero diagonal coefficient*, J. Math. Anal. Appl. **291** (2004), 322–340.

[14] M. Gyllenberg and P. Yan; *New conditions for the intersection of orbits with the vertical isocline of the Liénard system*, Mathematical and Computer Modelling **49** (2009), 906–911.

[15] T. Hara and T. Yoneyama; *On the global center of generalized Liénard equation and its application to stability problems*, Funkcial. Ekvac. **28** (1985), 171–192.

[16] E. Hille; *Non-oscillation theorems*, Tran. Amer. Math. Soc. **64** (1948), 234–252.

[17] A. Kneser; *Untersuchungen uber die reelen Nullstellen der Integrale linearer Differentialgleichungen*, Math. Ann. **42** (1893), 409–435.

[18] J. V. Manojlovic; *Oscillation criteria for sublinear differential equations with damping*, Acta Math. Hungar. **104** (12) (2004), 153169.

[19] Z. Nehari; *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. **85** (1957), 428–445.

[20] S. H. Saker, P. Y. H. Pang, R. P. Agarwal; *Oscillation Theorems For Second Order Nonlinear Functional Differential Equations with Damping,* Dynamic Sys. Appl. **12** (2003), 307-322.

[21] G. Sansone, R. Conti; *Equazioni Differenziali Non Lineari,* Cremonese, Roma, 1956.

[22] J. Sugie, T. Hara; *Nonlinear Oscillation of Second Order Differential Equations of Euler Type,* Proceedings of the American Mathematical Society, **124** (10) (1996), 3173-3181.

[23] C. A. Swanson; *Comparison and Oscillation Theory of Linear Differential Equations,* Academic Press, New York/London 1968.

[24] B. Van der Pol; *Sur les oscillations de relaxation,* The Philos. Magazine **7** (1926), 978–992.

[25] G. Villari and F. Zanolin; *On a dynamical system in the Liénard plane. Necessary and sufficient conditions for the intersection with the vertical isocline and applications,* Funkcial. Ekvac. **33** (1990), 19–38.

[26] J. S. W. Wong; *Remarks on nonoscillation theorems for a second order non-linear differential equation,* Proc. Amer. Math. Soc. **83** (1981), 541–546.

[27] J. S. W. Wong; *Oscillation theorems for second-order nonlinear differential equations of Euler type,* Methods Appl. Anal. **3** (1996), 476–485.

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