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MULTIPLICITY OF SOLUTIONS FOR DISCRETE PROBLEMS WITH DOUBLE-WELL POTENTIALS

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Abstract. This article presents some multiplicity results for a general class of nonlinear discrete problems with double-well potentials. Variational techniques are used to obtain the existence of saddle-point type critical points. In addition to simple discrete boundary-value problems, partial difference equations as well as problems involving discrete p-Laplacian are considered. Also the boundedness of solutions is studied and possible applications, e.g. in image processing, are discussed.

1. Introduction

In this article, we present the existence and multiplicity results for a general class of discrete problems with the so-called double-well nonlinearities. In the first part of this work we study the problem

$$Ax + F(x) = 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

in which A is a symmetric positive semidefinite matrix and F is a continuous nonlinear vector function whose entries have the form $f_i(x_i) := g_i'(x_i)$ with g_i's being the double-well potentials (one could list $g(s) = (1 - s^2)^2$ as a typical example). Later, we generalize (1.1) and replace A by discrete p-Laplacian.

Problems of this type are known as Allen-Cahn or bi-stable equations and have been extensively analyzed in the continuous settings. Their history reaches back to the pioneering work by Allen, Cahn [3] which examined the coarsening of binary alloys. Such a process is represented by a boundary-value problem

$$\begin{gathered} u_t(x,t) = \varepsilon^2 \Delta u(x,t) - g'(u(x,t)), \quad x \in \Omega,\ t > 0, \\ \frac{\partial u(x,t)}{\partial n} = 0, \quad \text{for } x \text{ on } \partial\Omega,\ t > 0, \end{gathered} \tag{1.2}$$

in which u is a concentration rate of one of two components in the alloy and the parameter ε corresponds to the interfacial energy. Later, similar problems have appeared in the models of phase changes at the transition temperatures (see Fusco, Hale [11]), in the analysis of crystals' growth (e.g. Wheeler et al. [18]) and most recently in the image processing (e.g. Choi et al. [9]).

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The image processing application mentioned above serves as important motivation for the study of discrete counterparts of (1.2). In this area, similar models are studied and applied for the object identification, the so-called level set segmentation. The function u corresponds to the grayscale values (rescaled to the interval $[-1,1]$) in an $N \times N$-pixel image with discrete coordinates (x, y) (see e.g. Beneš et al. [5] for a nice description of this process). Apparently, stationary points of such discrete evolutionary equation are solutions of (1.1).

To authors' best knowledge, discrete problems with double-well potentials have not been analyzed so far. Recently, several papers have studied the solvability of nonlinear discrete problems (1.1), either in this general form or with specific operators A (e.g. Bai, Zhang [4], Galewski, Smejda [13], Mihăilescu, Rădulescu, Tersian [14] or Yang, Zhang [20]). Applying mountain-pass or linking theorems, they get conditions on the limit behaviour of functions f_i's near the origin and in the infinity. Other techniques have been used to get interesting results for general (e.g. partial) difference operators (e.g. Bereanu, Mawhin [6], Galewski, Orpel [12] or Stehlík [17]). One could find similar manuscripts in which the linear discrete operator A is replaced by p-Laplacian, e.g. Agarwal et al. [2] or Cabada et al. [7].

Working with a special (and thus narrower) class of problems we are able to get finer assumptions on the nonlinearities. At the same time, we are trying to preserve the generality by considering a wide class of discrete operators. Our main tools include the Saddle point theorem and coercivity variational results.

First, we briefly introduce basic notation and the necessary functional-analytic support (Sections 2 and 3). Then, in Section 4, we prove the existence of at least three solutions of the problem (1.1) with general double-well potentials. Consequently, we extend this result to the existence of at least five solutions for a class of double-wells with special behaviour at the origin (Section 5). Finally, we generalize these results for the case in which the operator A is replaced by p-Laplacian (Section 6). Several examples are included to illustrate the existence results.

2. Preliminaries

We study variational formulations of discrete problems and our main tool is the Saddle point theorem. Thus, we restrict our attention to functionals satisfying the Palais-Smale condition.

Definition 2.1 ([19, Definition 1.16])**.** Let X be Banach space, $\mathcal{J} \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. Then the functional $\mathcal{J}$ satisfies the *Palais-Smale condition* if any sequence $\{u_n\} \subset X$ such that

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0 \tag{2.1}$$

has a convergent subsequence. We call this sequence PS-sequence.

We use original version of the Saddle point theorem proven by Paul Henry Rabinowitz in 1978.

Theorem 2.2. *Let $X = Y \oplus Z$ be real Banach space with $\dim(Y) < \infty$ equipped with the norm $\|\cdot\|$. For some $\rho > 0$, we define*

$$M = \{u \in Y : \|u\| \le \rho\}, \quad M_0 = \{u \in Y : \|u\| = \rho\}.$$

Let $\mathcal{J}$ be a real functional, $\mathcal{J} \in C^1(X, \mathbb{R})$, such that

$$\inf_{u \in Z} \mathcal{J}(u) > \max_{u \in M_0} \mathcal{J}(u). \tag{2.2}$$

If $\mathcal{J}$ satisfies (2.1) *with*

$$c := \inf_{\gamma\in\Gamma} \max_{u\in M} \mathcal{J}(\gamma(u)), \quad \Gamma := \big\{\gamma \in C(M, X) : \gamma|_{M_0} = I\big\}$$

then c is the critical value of $\mathcal{J}$.

The proof of the above lemma can be found in [19, Theorem 2.11].

We also use the following statement about weakly coercive functionals in $\mathbb{R}^N$ whose proof could be found in [16, Thoerem 1.9].

Theorem 2.3. *Let $F : \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous and weakly coercive functional. Then F is bounded from below on H and there exists a minimizer $u_0 \in H$ such that $F(u_0) = \min_{u\in H} F(u)$. Moreover, if the Fréchet derivative $F'(u_0)$ exists then $F'(u_0) = o$.*

3. Notation and assumptions - survey

Let $x = (x_1, x_2, \dots, x_N)^T$ be a vector in $\mathbb{R}^N$, equipped with the standard Euclidean norm

$$\|x\| = \Big(\sum_{i=1}^{N} |x_i|^2\Big)^{1/2}.$$

Vector o denotes the trivial solution; i.e., $o = (0, 0, \dots, 0)^T$. We study problem (1.1). Throughout the paper we assume that an $N \times N$ matrix A satisfies assumptions:

- (A1) A is symmetric and positive semidefinite,
- (A2) multiplicity of eigenvalue $\lambda_1 = 0$ is one,
- (A3) the corresponding first eigenvector is $e_1 = (1, 1, \dots, 1)^{\mathrm{T}}$.

The vector function $F : \mathbb{R}^N \mapsto \mathbb{R}^N$ has components $f_i(x_i)$

$$F(x) = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ \dots \\ f_N(x_N) \end{pmatrix} = \begin{pmatrix} f(1, x_1) \\ f(2, x_2) \\ \dots \\ f(N, x_N) \end{pmatrix},$$

with integrable f_i, whose potentials are denoted by g_i,

$$g_i'(s) = f_i(s).$$

Consequently, the potential $G : \mathbb{R}^N \mapsto \mathbb{R}$ corresponding to F is

$$\nabla G(x) = F(x), \quad G(x) = \sum_{i=1}^{N} g_i(x_i).$$

As mentioned above, we work with the double-well potentials; i.e., we assume that:

- (G1) (evenness) for all $is \in \mathbb{R}$: $g_i(s) = g_i(-s)$,
- (G2) for all i: $g_i(0) > 0$, $g_i(\pm 1) = 0$,
- (G3) (non-negativity) for all $i, s \in \mathbb{R}$: $g_i(s) \geq 0$,
- (G4) (weak coercivity) for all i: $\lim_{s\to\infty} g_i(s) = \infty$,
- (G5) for all i: $g_i \in C^1(\mathbb{R})$.

Under these assumptions solutions of the problem (1.1) are critical points of the functional $\mathcal{J} : \mathbb{R}^n \mapsto \mathbb{R}$,

$$\mathcal{J}(u) := \frac{1}{2}\langle Au, u\rangle + \sum_{i=1}^{N} g_i(u_i) = \frac{1}{2} u^{\mathrm{T}} Au + G(u). \tag{3.1}$$

Thus, we study the existence and multiplicity of the critical points of $\mathcal{J}$ in Sections 4–5. To illustrate the content of the set of matrices which satisfy (A1)–(A3) let us include a couple of examples at this stage.

Example 3.1. Let us consider a second-order periodic discrete problem

$$\begin{gathered} -\Delta^2 x_{i-1} + f_i(x_i) = 0, \quad i = 1, \dots, N \\ x_0 = x_N, \quad \Delta x_0 = \Delta x_N. \end{gathered} \tag{3.2}$$

One could rewrite this problem as an equation in $\mathbb{R}^N : Ax + F(x) = 0$ with

$$A = \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix}, \quad F(x) = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ \dots \\ f_N(x_N) \end{pmatrix} \tag{3.3}$$

Matrix A satisfies assumptions (A1)–(A3) (see e.g. [1]).

The second, slightly more complicated, example considers a partial difference equation on a square.

Example 3.2. *Let us study a two-dimensional nonlinear Poisson equation coupled with Neumann boundary conditions*

$$\begin{gathered} -\Delta_1^2 y_{k-1,l} - \Delta_2^2 y_{k,l-1} + f(y_{k,l}) = 0, \quad k, l = 1, \dots, N \\ 0 = \Delta_1 y_{0,l} = \Delta_1 y_{N,l} = \Delta_2 y_{k,0} = \Delta_2 y_{k,N}, \end{gathered} \tag{3.4}$$

where Δ_i denotes the standard difference operator with respect to the i-th variable. Rearranging $y_{k,l}$ to the vector $x = (y_{1,1}, \dots, y_{1,N}, y_{2,1}, \dots, y_{N-1,N}, y_{N,1}, \dots, y_{N,N})^T$ one could rewrite the boundary problem (3.4) to the matrix equation (1.1) with A being a block tridiagonal matrix having dimension $N^2 \times N^2$,

$$A = \begin{bmatrix} B_1 & -I & & & \\ -I & B_2 & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B_{N-1} & -I \\ & & & -I & B_N \end{bmatrix},$$

where I is the $N \times N$ identity matrix and $N \times N$ matrices B_i have the form

$$B_1 = B_N = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 2 \end{bmatrix},$$

$$B_k = \begin{bmatrix} 3 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 3 \end{bmatrix},$$

for $k = 2, \dots, N-1$. Matrix A satisfies (A1)–(A3) (see e.g. [8]).

4. Saddle-point geometry - three solutions

To establish the saddle-point geometry from Theorem 2.2, we decompose $\mathbb{R}^N$:

$$\mathbb{R}^N = Y \oplus Z.$$

The subspaces Y and Z are generated by eigenvectors e_i of matrix A in the following way

$$Y = \operatorname{span}\{e_1\}, \quad Z = \operatorname{span}\{e_2, e_3 \dots, e_N\}. \tag{4.1}$$

Lemma 4.1. *Let A be a matrix satisfying* (A1)–(A3), *$G : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying* (G1)–(G5)*. Then there exist at least three solutions of equation* (1.1)*, with at least one being the saddle-point type critical point of the functional $\mathcal{J}$.*

Proof. The functional $\mathcal{J}$ is weakly coercive and non-negative. Indeed, the positive semidefiniteness (A1) of A and the limit behavior (G4) of G imply $\mathcal{J}(u) \leq \sum_{i=1}^N G(u_i) \to \infty$. The non-negativity of $\mathcal{J}$ follows from the semi-definiteness (A1) of A and the non-negativity (G3) of G.

Thanks to the first eigenvector assumption (A3) and the fact that $g_i(\pm 1) = 0$ (assumption (G2)), one can observe that the constant functions $e_1 = (1, 1, \dots, 1)^{\mathrm{T}}$ and $-e_1$ are global minimizers of $\mathcal{J}$ and consequently trivial solutions of (1.1), $\mathcal{J}(e_1) = 0$. Obviously, $e_1, -e_1 \in Y$, we define $M_0 := \{-e_1, e_1\}$.

We prove the saddle-point geometry of functional $\mathcal{J}$ by contradiction. Due to the continuity and weak coercivity of $\mathcal{J}$ there exists a minimizer $\tilde{u} \in Z$ such that $\mathcal{J}(\tilde{u}) = \inf_{u \in Z} \mathcal{J}(u)$, cf. Theorem 2.3. Let us assume that $\mathcal{J}(\tilde{u}) \leq \mathcal{J}(e_1)$. But by the definition of Z and (G2), $\mathcal{J}(\tilde{u}) \geq \frac{1}{2}\langle A\tilde{u}, \tilde{u}\rangle > 0$ if $\tilde{u} \in Z \setminus \{o\}$. Moreover, $\mathcal{J}(o) = \sum_{i=1}^N g_i(0) > 0$ holds true. This implies that $\mathcal{J}(\tilde{u}) > \mathcal{J}(e_1)$.

It remains to show that the Palais-Smale condition (2.1) holds. This is a straightforward consequence of the weak coercivity of $\mathcal{J}$. Let us assume that $\{u_n\}$ is a PS-sequence. Then there exists some $K > 0$ such that $\|u_n\| < K$. By compactness of $\{u_n\}$ there exists a subsequence u_{n_k} converging to a critical point.

Consequently, all assumptions of Theorem 2.2 are satisfied and we get the existence of the third extremal value of $\mathcal{J}$, or equivalently, the existence of the third solution of (1.1). □

Remark 4.2. The contribution of this result does not lie in the fact that it provides the existence of at least three solutions. Note that assumptions (G1)–(G5) directly imply that o, e_1 and $-e_1$ are solutions of (1.1). It is the saddle-point type geometry of at least one of the solutions which we use in the following section.

5. Saddle-point geometry - five solutions

In this section, we focus on nonlinearities with special behaviour in the neighbourhood of 0. We assume that

- (G6) thee exist $\delta > 0$, $K > 0$ $\beta \in (1, 2)$ such that for all $s \in \mathbb{R}$ and all i: $|s| < \delta$ implies $g_i(s) \leq g_i(0) - K|s|^\beta$.

This assumption ensures the maximality of $\mathcal{J}$ at the origin $o = (0, 0, \dots, 0)^T$.

Lemma 5.1. *Let G satisfy* (G6)*. Then o is a local maximizer of $\mathcal{J}$ in $\mathbb{R}^N$.*

Proof. Let us choose a fixed nonzero $u \in \mathbb{R}^N$ with $\|u\| = 1$. Then

$$\begin{aligned}\mathcal{J}(tu) &\le \frac{1}{2}t^2\langle Au, u\rangle + \sum_{i=1}^{N}\Big(g_i(0) - Kt^\beta |u_i|^\beta\Big)\\ &= c_1 t^2 - c_2 t^\beta + c_3 =: h(t),\end{aligned}$$

where the constants are such that $c_1 \ge 0$, $c_2, c_3 > 0$. Then $h'(t) < 0$ is equivalent to

$$\frac{2c_1}{\beta c_2} < t^{\beta-2}. \tag{5.1}$$

But the fact $\beta < 2$ implies that there exists some $t_0 > 0$ such that the inequality (5.1) is satisfied for all $t \in (0, t_0)$.

Let us define $\gamma(u) = \frac{2c_1}{\beta c_2}$ for given u. Then the function γ is continuous and attains its maximal value $\gamma_{\max}$ on the compact set $\mathcal{S} = \{u \in X : \|u\| = 1\}$. If we choose t_0 to be a positive constant satisfying $t_0 < \gamma_{\max}$ then the estimate (5.1) holds uniformly for all $t \in (0, t_0)$ and for all $u \in \mathcal{S}$. Consequently, the origin o is a local maximizer of $\mathcal{J}$. □

This result yields directly the existence of at least five solutions.

Theorem 5.2. *Let A be a matrix satisfying* (A1)–(A3) *and let G satisfy* (G1)–(G6). *Then there exist at least five solutions of equation* (1.1).

Proof. Firstly, note that Lemma 5.1 implies that o is a local maximizer of $\mathcal{J}$.

Secondly, recall that $e_1 = (1, 1, \dots, 1)^{\mathrm{T}}$ and $-e_1$ are global minimizers of $\mathcal{J}$.

Finally, we could apply the arguments from the proof of Lemma 4.1 to get the existence of the saddle-point type critical point $\tilde{u}$. Considering Lemma 5.1, we see that $\tilde{u} \ne o$. Taking into account the evenness of $\mathcal{J}$, $-\tilde{u}$ is a saddle-point type critical point of $\mathcal{J}$ as well. □

Again, we provide a simple example to demonstrate this result, especially the assumption (G6).

Example 5.3. For the sake of brevity and generality, we abstract from a specific discrete operator (those from Examples 3.1 and 3.2 could be used immediately as well as many others) and concentrate on the nonlinearity. In order to illustrate the assumption (G6) we consider potentials

$$g_i(s) = |1 - |s|^a|^b, \tag{5.2}$$

with $a, b > 1$.

Differentiating, we see that these correspond to $f_i(s) = -ab|s|^{a-1}\operatorname{sign}(s)\big|1 - |s|^a\big|^{b-1}\operatorname{sign}(1 - |s|^a)$. Obviously, G satisfies (G1)–(G5), hence Lemma 4.1 yields the existence of at least three solutions of equation (1.1) for any difference operator generating a matrix A satisfying (A1)–(A3).

Let us examine the condition (G6) for this case. We choose $\delta = 1/2$. Since the functions g_i's are even, we consider only $s \in [0, 1/2)$ which simplifies the corresponding derivatives. Then $g_i'(s) = f_i(s) = -ab(1 - s^a)^{b-1}s^{a-1}$ and the inequality $g_i(s) \le g_i(0) - K|s|^\beta$ holds if $f_i(s) \le -K\beta s^{\beta-1}$ for all $s \in [0, 1/2)$. But one could make the following estimate for $a \in (1, 2)$ and $b > 1$:

$$\begin{aligned}g_i'(s) = f_i(s) &= -ab(1 - s^a)^{b-1}s^{a-1}\\ &\le -ab(1 - 2^{-a})^{b-1}s^{a-1} \le -K\beta s^{\beta-1},\end{aligned}$$

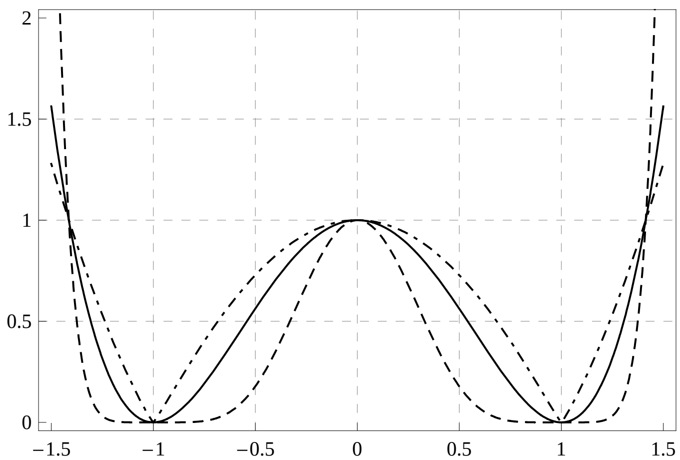

FIGURE 1. Functions $g_i(s) = |1 - |s|^a|^b$ with $a = 2$ and $b = 1.1$ (dot-dashed line), $b = 2$ (solid line) and $b = 6$ (dashed line)

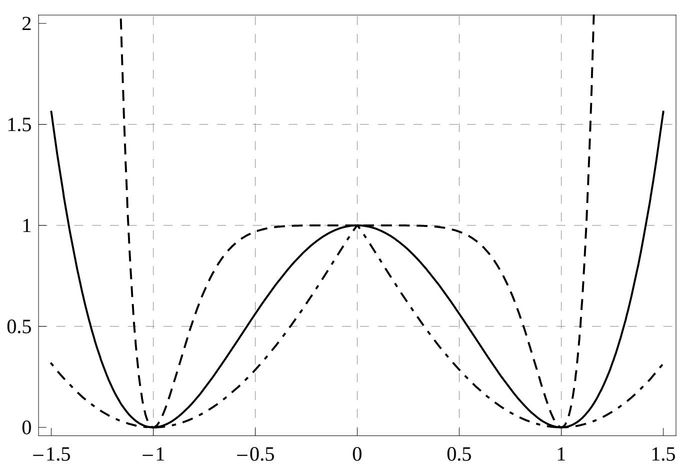

FIGURE 2. Functions $g_i(s) = |1 - |s|^a|^b$ with $b = 2$ and $a = 1.1$ (dot-dashed line), $a = 2$ (solid line) and $a = 6$ (dashed line)

if we choose $K \le b(1 - 2^{-a})^{b-1}$ and $\beta = a$, holds for all $s \in [0, 1/2)$. Hence the assumption (G6) is satisfied for $a \in (1, 2)$ and $b > 1$. Consequently, Theorem 5.2 provides the existence of at least five solutions of problem (1.1) with $g_i(s) = |1 - |s|^a|^b$ with $a \in (1, 2)$, $b > 1$.

Under additional assumptions on the constant K in (G6), we can extend this result also to the case with $\beta = 2$.

Theorem 5.4. *Let A be a matrix satisfying* (A1)–(A3) *and let G satisfy* (G1)–(G5) *and*

(G6') *Thee exist $\delta > 0$, $K > \frac{\lambda_{\max}}{2}$ such that for all $s \in \mathbb{R}$ and all i: $|s| < \delta$ implies $g_i(s) \le g_i(0) - K|s|^2$,* *with $\lambda_{\max}$ denoting the largest eigenvalue of A. Then there exist at least five solutions of the equation* (1.1).

Proof. Following the proof of Lemma 5.1 we can make the following estimate for arbitrary vector u, $\|u\| = 1$.

$$\begin{aligned} \mathcal{J}(tu) &\leq \frac{1}{2} t^2 \langle Au, u\rangle + \sum_{i=1}^{N} \Big(g_i(0) - K t^2 |u_i|^2 \Big) \\ &\leq \frac{1}{2} t^2 \lambda_{\max} - K t^2 + \sum_{i=1}^{N} g_i(0) \\ &= \left(\frac{\lambda_{\max}}{2} - K \right) t^2 + \sum_{i=1}^{N} g_i(0) =: h(t). \end{aligned}$$

Under the assumption (G6)$'$, the coefficient by t^2 is negative which implies that $h(t)$, and $\mathcal{J}(tu)$ have local maximizers at the origin o. The arbitrary choice of u implies the existence of a pair of non-trivial saddle-point type critical points $\tilde{u}$ and $-\tilde{u}$. □

Remark 5.5. In the spirit of Remark 4.2 one could rephrase the statements of Theorems 5.2 and 5.4 in the following way. Under the assumptions (G6) or (G6)$'$ there exist, aside from trivial solutions o, e_1 and $-e_1$, at least two nontrivial solutions, both being saddle points of $\mathcal{J}$.

Example 5.6. In this example, we study the behaviour of functions g_i's at 0 and its consequences for solution multiplicity for $\beta = 2$. Let us assume that nonlinear terms g_i are defined as $g_i(s) = \frac{1}{\varepsilon^2} |1 - |s|^a|^b$ with $\varepsilon > 0$. For a given $\delta > 0$ we could repeat the procedure of Example 5.3 to show that (G6)$'$ is satisfied for

$$K \leq \frac{b}{\varepsilon^2} (1 - \delta^a)^{b-1} \quad \beta = a = 2.$$

Considering the arbitrary choice of δ, we could see that the assumption (G6)$'$ is satisfied for any

$$\varepsilon < \sqrt{\frac{2b}{\lambda_{\max}}}, \quad \text{or equivalently} \quad b > \frac{\lambda_{\max} \varepsilon^2}{2}$$

which, if satisfied, guarantees the existence of at least five solutions of problem (1.1).

We conclude this section with the study of a specific boundary-value problem.

Example 5.7. Let us consider the one-dimensional discrete problem with periodic boundary conditions (see (3.2))

$$\begin{gathered} -\Delta^2 x_{i-1} + g_i'(x_i) = 0, \quad i = 1, \dots, N \\ x_0 = x_N, \ \Delta x_0 = \Delta x_N, \end{gathered} \tag{5.3}$$

with $g_i(s) = \frac{1}{\varepsilon^2} |1 - |s|^a|^b$. The eigenvalues of the corresponding matrix A (cf. (3.3)) satisfy (see e.g. [1, Chapter 11])

$$\lambda_{\max} = \begin{cases} 4 & \text{if } N \text{ is even,} \\ 4 \sin^2 \left(\frac{N-1}{N} \frac{\pi}{2} \right) & \text{if } N \text{ is odd.} \end{cases}$$

Hence, the results listed in this section imply that problem (5.7) has at least five solutions if either $a \in (1,2)$ and $b > 1$ (Example 5.3) or $a = 2$ and $b > \frac{\lambda_{\max}\varepsilon^2}{2}$ (Example 5.6).

6. Application to the discrete p-Laplacian

In this section we extend the results to the problems with p-Laplacian; i.e., we consider a situation in which the left-hand side discrete operator is not linear. Let $p > 1$, and define Banach space

$$X = \big\{u = (u_0, u_1, \dots, u_{N+1})^T : u_0 = u_1,\ u_N = u_{N+1}\big\} \subset \mathbb{R}^{N+2}$$

equipped with norm

$$\|u\|_p = \Big(\sum_{i=0}^{N+1} |u_i|^p\Big)^{1/p}.$$

We define a nonlinear functional $\mathcal{J}_p : u \in X \mapsto \mathcal{J}_p(u) \in \mathbb{R}$ by

$$\mathcal{J}_p(u) := \sum_{i=1}^{N}\Big(\frac{|\Delta u_{i-1}|^p}{p} + g_i(u_i)\Big) + \frac{|\Delta u_N|^p}{p}. \tag{6.1}$$

The critical point u of (6.1) corresponds to the solution of the problem

$$\begin{gathered} -\Delta(\varphi_p(\Delta u_{i-1})) + f_i(u_i) = 0 \quad \text{for } i = 1, \dots, N, \\ \Delta u_0 = \Delta u_N = 0, \end{gathered} \tag{6.2}$$

where $\varphi_p : s \mapsto |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) := 0$.

Remark 6.1. One can observe that $e_1 = (1, 1, \dots, 1)^T$, $-e_1$ and o are solutions of (6.2).

As in the linear case we proceed by proving the existence of at least three solutions under the assumption (G1)–(G5). In order to obtain the existence of at least five solutions, we modify (G6) and assume that

- (G6P) There exist $\delta > 0$, $K > 0$, $\beta \in (1, p)$ such that for all $s \in \mathbb{R}$ and all i: $|s| < \delta$ implies $g_i(s) \le g_i(0) - K|s|^\beta$

holds instead. Assumption (G6P) plays equivalent role as (G6) in the linear case. It ensures that $\mathcal{J}_p$ attains its local maximum at o.

Lemma 6.2. *Let G satisfy* (G6P). *Then o is a local maximizer of $\mathcal{J}_p$ in X.*

Proof. We follow the steps of Lemma 5.1. First, we choose a fixed nonzero $u \in X$. Then

$$\begin{aligned} \mathcal{J}_p(tu) &\le \sum_{i=1}^{N}\Big(t^p\frac{|\Delta u_i|^p}{p} + g_i(0) - Kt^\beta|u_i|^\beta\Big) + t^p\frac{|\Delta u_N|^p}{p} \\ &= c_1t^p - c_2t^\beta + c_3 =: h(t), \end{aligned}$$

with $c_1 \ge 0$, $c_2, c_3 > 0$. Rewriting $h'(t) < 0$ we obtain

$$\gamma(u) := \frac{c_1 p}{c_2 \beta} < t^{\beta - p}. \tag{6.3}$$

The inequality $p > \beta$ implies that there exists some $t_0 > 0$ such that inequality (6.3) is satisfied for all $t \in (0, t_0)$.

Since the function γ is continuous on the compact set $\mathcal{S} = \{u \in X : \|u\|_p = 1\}$, it attains its maximal value $\gamma_{\max}$. If we choose t_0 as $0 < t_0 < \gamma_{\max}$ then estimate (6.3) holds uniformly for all $t \in (0, t_0)$ and for all $u \in \mathcal{S}$ and the zero function o is a local maximizer of $\mathcal{J}_p$ in X. □

Consequently, we are ready to prove the existence result for p-Laplacian which corresponds directly to Lemma 4.1 and Theorem 5.2.

Theorem 6.3. *Let us assume that G satisfies* (G1)–(G5)*. Then there exist at least three solutions of* (6.2)*, with at least one being the saddle-point type critical point of the functional $\mathcal{J}_p$. Moreover, if G satisfies* (G6P) *there exist at least five solutions of* (6.2).

Proof. We follow the proof of Lemma 4.1 and Theorem 5.2. It is easy to see that the functional $\mathcal{J}_p$ is weakly coercive and nonnegative. The constant solutions e_1 and $-e_1$ are global minimizers of $\mathcal{J}_p$ on X. Let us put $X = Y \oplus Z$ with $Y = \operatorname{span}\{e_1\}$. Let us denote $M_0 = \{-e_1, e_1\}$. Then $\mathcal{J}|_{M_0} = 0$ holds.

The continuity and weak coercivity of $\mathcal{J}$ on X yield that there exists $\tilde{z} \in Z$ such that $\mathcal{J}_p(\tilde{z}) = \inf_{z\in Z} \mathcal{J}_p(z)$. Due to the assumption (G3) we have $\mathcal{J}_p(\tilde{z}) \geq \sum_{i=1}^{N} \frac{|\Delta z_i|^p}{p} > 0$ ($\tilde{z}$ is a nonconstant function). Thus $\mathcal{J}_p$ satisfies the saddle-point geometry. To show that the Palais-Smale condition holds true, we literally follow the proof of Lemma 4.1. Finally, the direct application of the abstract Theorem 2.2 gives a critical point u of saddle-point type and the existence of at least three solutions.

Moreover, if G satisfies (G6P), Lemma 6.2 implies that the critical point o is a local maximizer thus there has to exist a critical point $\tilde{u} \neq o$. The evenness of $\mathcal{J}_p$ implies that also $-\tilde{u}$ is a critical point of the saddle-point type. □

Example 6.4. Focusing on the role of the parameter p in the p-Laplace operator, we choose the standard double-well function $g_i(s) = (1 - s^2)^2$. Consequently, the problem (6.2) has the form

$$\begin{gathered} -\Delta(|\Delta u_{i-1})|^{p-2} \operatorname{sign}(\Delta u_{i-1})) - 4u_i + 4u_i^3 = 0 \quad \text{for } i = 1, \dots, N, \\ \Delta u_0 = \Delta u_N = 0. \end{gathered} \tag{6.4}$$

Taking into account the first part of Theorem 6.3 we see that the problem (6.4) has at least three solutions for arbitrary $p > 1$. Moreover, following the procedure from Example 5.3 one could check that $g_i(s) \leq g_i(0) - K|s|^\beta$ with $\beta = 2$. Hence, the assumption (G6P) is satisfied for $p > 2$ and the problem (6.4) has at least five solutions.

As in the linear case, we are able to extend the existence of five solutions to the case with $p = \beta$.

Theorem 6.5. *Let $p > 1$, function G satisfy* (G1)–(G5) *and*

(G6P') *there exist $\delta > 0$, $K > \frac{2^{p+1}}{p}$ such that for all $s \in \mathbb{R}$ and all i: $|s| < \delta$ implies $g_i(s) \leq g_i(0) - K|s|^p$.*

Then there exist at least five solutions of equation (6.2).

Proof. We follow the proof of Lemma 5.1. Let us fix some u, $\|u\|_p \neq 0$. Then using Minkovski inequality, employing built-in Neumann conditions in the space X and the assumption (G6P') we get following upper bound of $\mathcal{J}_p$ on X:

$$\begin{aligned}
\mathcal{J}_p(tu) &= \sum_{i=1}^{N+1} \frac{|\Delta t u_{i-1}|^p}{p} + \sum_{i=1}^{N} g_i(tu_i) \\
&= \frac{t^p}{p} \sum_{i=1}^{N+1} |u_i - u_{i-1}|^p + \sum_{i=1}^{N} g_i(tu_i) \\
&\le \frac{t^p}{p} \Big(\Big(\sum_{i=1}^{N+1} |u_i|^p \Big)^{1/p} + \Big(\sum_{i=0}^{N} |u_i|^p \Big)^{1/p} \Big)^p + \sum_{i=1}^{N} g_i(tu_i) \\
&\le \frac{2^p t^p}{p} \sum_{i=0}^{N+1} |u_i|^p + \sum_{i=1}^{N} g_i(tu_i) \\
&= \frac{2^p t^p}{p} \Big(|u_1|^p + |u_N|^p + \sum_{i=1}^{N} |u_i|^p \Big) + \sum_{i=1}^{N} g_i(tu_i) \\
&\le \frac{2^{p+1} t^p}{p} \sum_{i=1}^{N} |u_i|^p + \sum_{i=1}^{N} g_i(tu_i) \\
&\le \frac{2^{p+1} t^p}{p} \sum_{i=1}^{N} |u_i|^p + \sum_{i=1}^{N} \big(g_i(0) - K t^p |u_i|^p \big) \\
&= t^p \Big(\frac{2^{p+1}}{p} - K \Big) \sum_{i=1}^{N} |u_i|^p + \sum_{i=1}^{N} g_i(0).
\end{aligned}$$

The assumption (G6P') implies the negativity of $\frac{2^{p+1}}{p} - K$ which guarantees that function $t \mapsto \mathcal{J}(tu)$ attains a local maximimum at $t = 0$. Since we can choose u arbitrarily, there exists at least one pair of non-trivial saddle-point type critical points $\tilde{u}$ and $-\tilde{u}$. Moreover, constant solutions o, e_1 and $-e_1$ also solve (6.2). This completes the proof. □

Remark 6.6. Note that the assumption (G6P') in the linear case $p = 2$ requires $K > 4$, which is stricter than the assumption $K > \frac{\lambda_{\max}}{2}$ from (G6)$'$. This implies that it could be possible to get a better lower bound on K by employing more subtle estimates in the proof of Theorem 6.5.

Example 6.7. We consider

$$g_i(s) = \frac{1}{\varepsilon^p} |1 - |s|^p|^b = \frac{1}{\varepsilon^2} \tilde{g}_i(s)$$

with $\varepsilon > 0$ as in Example 5.6 to illustrate the influence of parameter p on the solution multiplicity. Following the procedure employed in Example 5.6 we get $K \le \frac{b}{\varepsilon^p}(1 - \delta^p)^{b-1}$. The assumption (G6P') is then reduced to

$$\varepsilon < \frac{1}{2} \sqrt[p]{\frac{bp}{2}}.$$

Remark 6.8. In the continuous case, the multiplicity of solutions was studied in Otta [15]. For $p > \beta$ one can prove that there exists a sequence of solutions of the continuous version of (6.2). Whereas for $p \le \beta$ the set of solutions is finite and the minimal number of solutions is three – constant solutions ± 1, 0. On the one hand, the discrete results presented here studies only the case with $p \geq \beta$. On the other hand, the assumptions are weaker in the sense that no growth conditions on g_i's are required with the exception of $p = \beta$.

7. Boundedness of solutions

In this section, we focus on the boundedness of solutions to (6.2). Returning back to the application of similar problems in the image processing, there is a reasonable question of whether the solutions stay between -1 and 1 (recall that u described the gray scale between -1 and 1 there). Let us study solutions of initial-value problem

$$\begin{gathered} -\Delta(\varphi_p(\Delta u_{i-1})) + f_i(u_i) = 0 \quad \text{for } i = 1, \dots, N, \\ u_0 = \tilde{u}_0, \quad u_1 = \tilde{u}_1, \end{gathered} \tag{7.1}$$

under additional assumption on the nonlinear functions g_i:

(G7) for each i, $s \mapsto g_i(s)$ is an increasing function for $s > 1$.

Note that all g_i's mentioned above in this paper satisfy (G7). In the following, we use $\varphi_{p'}(\varphi_p(s)) = s$, where p and p' are conjugate exponents (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$).

Theorem 7.1. *Let us assume that g_i satisfy* (G1), (G2), (G3), (G7). *If u is a solution of* (6.2) *then $|u_i| \leq 1$ for all $i = 1, 2, \dots N$.*

Proof. To prove the boundedness of solutions to (6.2) we use solutions of related initial-value problem (7.1). Let us expand difference terms to get recursively defined solution u to (7.1) in the form

$$\begin{gathered} u_{i+1} = u_i + \varphi_{p'}(\varphi_p(u_i - u_{i-1}) + f_i(u_i)) \quad \text{for } i = 1, \dots, N, \\ u_0 = \tilde{u}_0, \quad u_1 = \tilde{u}_1. \end{gathered} \tag{7.2}$$

Assumption (G7) guarantees the positivity of $g_i'(s) = f_i(s)$ for all $s > 1$. We choose $\tilde{u}_0 = \tilde{u}_1 > 1$. Using assumptions (G1), (G2), (G3) and (G7) one can obtain following estimates:

$$\begin{gathered} u_2 = u_1 + \varphi_{p'}(\varphi_p(u_1 - u_0) + f_1(u_1)) = u_0 + \varphi_{p'}(f_1(u_1)) = u_0 + \delta, \\ u_3 = u_2 + \varphi_{p'}(\varphi_p(u_2 - u_1) + f_2(u_2)) \geq u_2 + \varphi_{p'}(\varphi_p(u_2 - u_1)) = u_2 + \delta, \\ \dots \\ \begin{aligned} u_{N+1} &= u_N + \varphi_{p'}(\varphi_p(u_N - u_{N-1}) + f_N(u_N)) \\ &\geq u_N + \varphi_{p'}(\varphi_p(u_N - u_{N-1})) = u_N + \delta. \end{aligned} \end{gathered}$$

Since $u_{N+1} \geq u_N + \delta$ the vector u can not satisfy homogenous Neumann boundary conditions at x_N and x_{N+1}. The same solution behavior can be observed for $\tilde{u}_0 = \tilde{u}_1 < -1$. This implies that a solution of the boundary-value problem (6.2) has to satisfy initial condition $u_0 = u_1 \in [-1, 1]$.

It remains to show that there exists no index k, $k \in \{1, 2, \dots, N\}$ such that $|u_k| > 1$. Let us consider u to be a solution of (6.2) and such index k do exist. Without any loss of generality, we can assume that $u_{k-1} \leq 1$, $u_k > 1$ hold. Then a solution satisfies (7.2) and

$$u_{k+1} = u_k + \varphi_{p'}(\varphi_p(u_k - u_{k-1}) + f_k(u_k)) \geq u_k + \delta$$

where $\delta = u_k - u_{k-1} > 0$. By induction, following steps from previous paragraph, we get $u_{N+1} \geq u_N + \delta$ which is a contradiction to u as a solution of (6.2). By similar arguments, one can show that there exists no index k such that $u_k < -1$. This completes the proof. ∎

Remark 7.2. In the continuous case, Drábek et al. [10] studied the following problem

$$\begin{gathered} \varepsilon^p(|u'(x)|^{p-2}u'(x))' - g'(u(x)) = 0, \quad x \in (0,1), \\ u'(0) = u'(1) = 0, \end{gathered} \tag{7.3}$$

with $g(s) = |1 - s^2|^b$, $b > 1$. Interestingly, they proved that for $p > b$ there exist dead-core solutions touching the values ± 1. Moreover, for $p = 4$ and $b = 2$, they proved that these solutions forms continua of saddle points of the related functional in $W^{1,p}(0,1)$.

Final remarks. Multiplicity results for discrete equations with double-well potentials offer many interesting questions. This paper contains some basic answers but there are many issues which could be followed further.

On the one hand, there is some space in improving the presented results. General multiplicities of the eigenvalue $\lambda = 0$ (see (A2)) or more intricate double well potentials (see (G2) and (G3)) could be considered.

On the other hand, different approaches could extend some of the results as well. More complicated operators (like multidimensional p-Laplacian) could be analyzed. Furthermore, the boundedness of solutions has been proven by iteration technique and is thus restricted to the one-dimensional problems. As our numerical experiments suggest some other techniques could generalize this to partial difference equations as well.

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