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QUASILINEAR SYSTEMS ASSOCIATED WITH SUPERCONDUCTIVITY

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ABSTRACT. In a previous article, Aramaki [4] considered a semilinear system with general nonlinearity in a three dimensional domain which arises in the mathematical theory of superconductivity. There the problem is reduced to the study of a quasilinear system. There it is assumed that the domain is simply-connected and without holes, and that the normal component of the curl of the boundary data vanishes. In this article, we these conditions are removed, and the analysis relies heavily on the recent work by Lieberman and Pan [16].

1. INTRODUCTION

In this artile, we consider the regularity of weak solutions for a quasilinear system arising from superconductivity theory. More precisely, to understand the nucleation of instability in the mathematical theory of superconductors, many authors considered a semilinear system

$$\begin{gathered} -\lambda^2 \operatorname{curl}^2 \mathbf{A} = (1-|\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \\ \lambda(\operatorname{curl} \mathbf{A})_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^3$, $\mathcal{H}^e$ is a given vector field on $\partial\Omega$, and $\lambda > 0$ is a parameter which means the penetration depth physically. Throughout this paper, for any vector field $\mathbf{v}$, $\mathbf{v}_T$ denotes the tangent component of $\mathbf{v}$ on $\partial\Omega$. If the solution $\mathbf{A}(x) = (A_1(x), A_2(x), A_3(x))$ of (1.1) satisfies

$$\|\mathbf{A}\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{3}}, \tag{1.2}$$

then it can be seen that $\mathbf{A}$ is locally stable. For any solution $\mathbf{A}$ of (1.1) satisfying (1.2), if we define $\mathbf{H} = \lambda \operatorname{curl} \mathbf{A}$, then it is known that $\mathbf{H}$ satisfies the quasilinear system

$$\begin{gathered} -\lambda^2 \operatorname{curl}[F_0(\lambda^2|\operatorname{curl} \mathbf{H}|^2) \operatorname{curl} \mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \\ \mathbf{H}_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

and

$$\lambda\|\operatorname{curl} \mathbf{H}\|_{L^\infty(\Omega)} < \sqrt{\frac{4}{27}}. \tag{1.4}$$

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The function F_0 in (1.3) is constructed implicitly by the equivalence relation

$$v = F_0(t^2)t \Leftrightarrow t = (1 - v^2)v, \tag{1.5}$$

and $F_0(0) = 1$. It is elementary to show that F_0 is uniquely defined for $0 \leq t \leq \sqrt{4/27}$, or equivalently for $0 \leq v \leq 1/\sqrt{3}$.

For two dimensional superconductors, a system of type (1.1) was derived by Chapman [8], and studied by Berestycki et al [6], Chapman [9], Pan and Kwek [21]. For three dimensional case, (1.1) and (1.3) were studied by Monneau [18] (with $\lambda = 1$), Bates and Pan [5]. Aramaki [2, 3, 4] studied the semilinear system with more general nonlinearity:

$$\begin{gathered} -\lambda^2 \operatorname{curl}^2 \mathbf{A} = f_0(|\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \\ \lambda(\operatorname{curl} \mathbf{A})_T = \mathcal{H}_T^e \quad \text{in } \partial\Omega, \end{gathered} \tag{1.6}$$

and the associated quasilinear system (1.3) where F_0 is a function constructed by f_0. In [5], the authors considered the regularity of weak solutions of (1.3) under the hypotheses that the domain is simply-connected and has no holes, and $\boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}_T^e = 0$ on $\partial\Omega$. Recently Lieberman and Pan [16] succeeded to remove the hypotheses.

In the case of anisotropic superconductors, the superconductivity is described by the anisotropic Ginzburg-Landau system

$$\begin{gathered} -\lambda^2 \operatorname{curl}^2 \mathbf{A} = [1 - g^Q(\mathbf{A})]Q\mathbf{A} \quad \text{in } \Omega, \\ \lambda(\operatorname{curl} \mathbf{A})_T = \mathcal{H}_T^e \quad \text{on } \partial\Omega \end{gathered} \tag{1.7}$$

where $Q = M^{-1}$ and M is a diagonal matrix called an effective mass tensor, $g^Q(\mathbf{A}) = \langle Q\mathbf{A}, \mathbf{A}\rangle$. Hereafter, for any vectors $\mathbf{a}, \mathbf{b}$, $\langle \mathbf{a}, \mathbf{b}\rangle = \mathbf{a} \cdot \mathbf{b}$ denotes the Euclidean inner product in $\mathbb{R}^3$. If $\mathbf{A}$ is a solution of (1.7) satisfying the condition

$$g^Q(\mathbf{A}) < \frac{1}{3}, \tag{1.8}$$

then $\mathbf{A}$ is also locally stable. If $\mathbf{A}$ is a solution of (1.7) satisfying (1.8), and if we define $\mathbf{H} = \lambda \operatorname{curl} \mathbf{A}$, then $\mathbf{H}$ satisfies a quasilinear system

$$\begin{gathered} -\lambda^2 \operatorname{curl}[F_0(\lambda^2 g^M(\operatorname{curl} \mathbf{H}))M \operatorname{curl} \mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \\ \mathbf{H}_T = \mathcal{H}_T^e \quad \text{on } \partial\Omega \end{gathered} \tag{1.9}$$

where F_0 is defined by the relation (1.5). For the theory of anisotropic superconductors, see Pan [19, 20]. Of course in the special case where M is the identity matrix, (1.7) and (1.9) reduce to (1.1) and (1.3), respectively.

In this paper, we consider the existence and regularity of weak solutions for the following quasilinear system

$$\begin{gathered} -\operatorname{curl}[F(g^M(\operatorname{curl} \mathbf{H}))M \operatorname{curl} \mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \\ \mathbf{H}_T = \mu \mathcal{H}_T^e \quad \text{on } \partial\Omega \end{gathered} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^3$ is a regular bounded domain, $M = M(x)$ is a matrix valued function,

$$g^M(\operatorname{curl} \mathbf{H})(x) = \langle M(x) \operatorname{curl} \mathbf{H}(x), \operatorname{curl} \mathbf{H}(x)\rangle,$$

the function F is defined on a bounded interval $[0, b_f]$ and μ is a real parameter. Throughout this paper, we denote $g^M(\mathbf{a}, \mathbf{b}) = \langle M\mathbf{a}, \mathbf{b}\rangle$ and $g^M(\mathbf{a}) = g^M(\mathbf{a}, \mathbf{a})$. We look for the solution of (1.10) satisfying

$$\|g^M(\operatorname{curl} \mathbf{H})\|_{L^\infty(\Omega)} < b_f. \tag{1.11}$$

The system (1.11) comes from the mathematical theory of anisotropic superconductor, where one wishes to understand the nucleation of instability of the Meissner states when the applied magnetic field increases to a critical magnetic field H_S. The author of [19, 20] considered the existence and regularity of weak solution of (1.10) under the hypotheses that Ω is simply-connected and has no holes, and $\boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}_T^e = 0$ on $\partial\Omega$ where $\boldsymbol{\nu}$ is the normal outer unit vector field on $\partial\Omega$.

We assume that the function F and the matrix valued function $M = M(x)$ satisfy the following conditions: for some $0 < b_f < \infty$, $F \in C^2([0, b_f)) \cap C^0([0, b_f])$ and

$$\begin{gathered} F(u) > 0 \quad \text{for } 0 \le u \le b_f, \\ F'(u) > 0,\ F''(u) > 0 \quad \text{for } 0 < u < b_f, \\ \lim_{u \to b_f - 0} F'(u) = +\infty, \end{gathered} \tag{1.12}$$

and $M \in C(\overline{\Omega}, S_+(3))$ where $S_+(3)$ denotes the set of all positive definite symmetric matrices, that is to say, there exists a constant $\beta(M) > 0$ such that

$$g^M(\xi) = \langle M(x)\xi, \xi\rangle \ge \beta(M)|\xi|^2 \tag{1.13}$$

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

The existence and uniqueness of solutions of (1.9) for small boundary data were given in [18]. He showed that if Ω is smooth and homeomorphic to a ball, and if μ is small, the equation (1.9) has a unique solution $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega}; \mathbb{R}^3)$, and if μ is large, then (1.9) has no solution. The authors of [5] found the optimal bound of boundary data for solvability of (1.9). They assumed the additional assumptions that Ω is simply-connected and has no holes, and boundary data $\mathcal{H}_T^e$ satisfies

$$\boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}_T^e = 0 \quad \text{on} \partial\Omega \tag{1.14}$$

where $\boldsymbol{\nu}$ denotes the unit exterior normal vector field on $\partial\Omega$.

Recently, for the regularity of weak solution of (1.3), [16] succeeded to remove the assumptions that Ω is simply-connected and has no holes, and the condition (1.14). For the quasilinear system (1.3) corresponding to (1.6), see Aramaki [1].

In this paper, we report that for regularity of weak solutions of the system (1.10) we can also remove the assumptions that Ω is simply-connected and has no holes, and condition (1.14). Thus we shall prove the following main theorems on the regularity of weak solutions for the system (1.10) where F satisfies (1.12).

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with $C^{3+\alpha}$ boundary for some $0 < \alpha < 1$. Assume that $M \in C^{1+\alpha}(\overline{\Omega}, S_+(3))$ satisfies* (1.13) *and F is a function satisfying* (1.12)*. Moreover, assume that $\mathcal{H}_T^e \not\equiv 0$ is a given vector fields in $C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$. If $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$ is a weak solution (in the sense of section 3) satisfying* (1.11)*, then $\mathbf{H}_\mu \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$. If furthermore $F \in C^{2+\alpha}([0, b_f))$, then $\mathbf{H}_\mu \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$.*

The proof is given in section 4. We are also interested in the continuity of $\|\mathbf{H}_\mu\|_{C^{2+\alpha}(\overline{\Omega})}$ with respect to μ. For the purpose we must leave the condition (1.14). However, this condition (1.14) is rather natural physically. We note that these topological assumptions were only used to prove the Hölder estimates of weak solution $\mathbf{H}$ of the quasilinear system (1.10). We get the following theorem.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with $C^{3+\alpha}$ boundary for some $0 < \alpha < 1$. Assume that $M \in C^{1+\alpha}(\overline{\Omega}, S_+(3))$ satisfies* (1.13) *and F is a function satisfying* (1.12)*. Moreover, assume that $\mathcal{H}_T^e \not\equiv 0$ is a given vector fields in $C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$ and* (1.14) *holds. Then there exists $\mu^*(\mathcal{H}_T^e) > 0$ such that*

(i) *If $0 < \mu < \mu^*(\mathcal{H}_T^e)$, then* (1.10) *has a unique solution $\mathbf{H}_\mu$ which satisfies* (1.11)*, and $\mathbf{H}_\mu \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$.*
(ii) *The mapping $[0, \mu^*(\mathcal{H}_T^e)) \ni \mu \mapsto \mathbf{H}_\mu \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$ is continuous.*
(iii) *If μ is large, then* (1.10) *has no solution.*

The proof is given in section 6.

2. Preliminaries

2.1. **Properties of the function F.** Let F be the function satisfying (1.12). If we define $\Phi(u) := [F(u)]^2 u$, then $\Phi'(u) > 0$ for $0 < u < b_f$, so $v = \Phi(u)$ is strictly increasing function on $[0, b_f]$. Therefore $v = \Phi(u)$ has an inverse function $u = \Psi(v)$ for $0 \le v \le b_\psi$ where $b_\psi = \Phi(b_f)$. Moreover, we note that since $\Phi''(u) > 0$ for $0 < u < b_f$, and so $\Psi''(v) < 0$ for $0 < v < b_\psi$, $\Phi'(u)$ is strictly increasing on $[0, b_f]$ and $\Psi'(v)$ is strictly decreasing on $[0, b_\psi]$. Define

$$f(v) = \frac{1}{F(\Psi(v))} \quad \text{for } 0 \le v \le b_\psi. \tag{2.1}$$

Then by simple computations, f has the following properties.

(i) $f \in C^2_{\rm loc}([0, b_\psi)) \cap C^0([0, b_\psi])$, $f(v) > 0$ and strictly decreasing on $[0, b_\psi]$.
(ii) We have $1/F(b_f) \le f(v) \le 1/F(0)$ for $0 \le v \le b_f$.
(iii) $f(v) = \sqrt{\Psi(v)/v}$ for $0 < v \le b_\psi$.
(iv) For any l so that $0 < l < b_\psi$, there exists $c(l) > 0$ such that
$$\inf_{0<v<l} [f(v) - 2|f'(v)|v] \ge c(l) := F(0)\Psi'(l).$$
Note that $\lim_{v \to b_\psi - 0} \Psi'(v) = 0$.
(v) Furthermore, if $F \in C^{2+\alpha}_{\rm loc}([0, b_f))$, then $f \in C^{2+\alpha}_{\rm loc}([0, b_\psi))$.

If the function $f \in C^2([0, b_\psi)) \cap C^0([0, b_\psi])$ is first given such that $f(0) > 0$ and $f'(v) < 0$ for $0 < v < b_\psi$ and $f''(v) \le 0$ for $0 < v < b_\psi$, then $\Psi(v) = [f(v)]^2 v$ satisfies $\Psi'(v) > 0$ for $0 < v < b_\psi$, so $u = \Psi(v)$ has the inverse function $v = \Phi(u)$ for $0 \le u \le b_f$ where $b_f = \Psi(b_\psi)$. If we put $F(u) = 1/f(\Phi(u))$, we see that F satisfies (1.12). Thus we can study the semilinear system

$$\begin{gathered} -\operatorname{curl}^2 \mathbf{A} = f(|\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \\ (\operatorname{curl} \mathbf{A})_T = \mathcal{H}_T^e \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

The problem is considered by [2, 3, 21]. In the particular case where $f(t) = 1 - u$, it is an original problem, see [8, 18]. For more general setting, see Pan [19, 20].

2.2. **Local estimates of vector fields.** To prove the regularity of weak solutions, we need some local estimates of vector fields, so we list up them which borrowed from [16]. For the proof, see [16] (cf. also Bolik and Wahl [7] and Wahl [22]).

For $x_0 \in \mathbb{R}^3$ and $R > 0$, define

$$B(x_0, R) = \{x \in \mathbb{R}^3; |x - x_0| < R\}, \quad \overline{B}(x_0, R) = \{x \in \mathbb{R}^3; |x - x_0| \le R\}.$$

For the interior regularity, we will use the following lemma.

Lemma 2.1. *Let* $\mathbf{u} \in H^1(B(x_0,R);\mathbb{R}^3)$.

(i) *If* $\operatorname{curl}\mathbf{u} \in L^q(B(x_0,R);\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u} \in L^q(B(x_0,R))$ *for some* $q > 1$, *then* $\mathbf{u} \in W^{1,q}(B(x_0,3R/4);\mathbb{R}^3)$ *and*

$$\|\mathbf{u}\|_{W^{1,q}(B(x_0,3R/4))} \le C(q,R)\{\|\mathbf{u}\|_{H^1(B(x_0,R))} + \|\operatorname{curl}\mathbf{u}\|_{L^q(B(x_0,R))} + \|\operatorname{div}\mathbf{u}\|_{L^q(B(x_0,R))}\}.$$

(ii) *Let* $k \ge 0$ *be an integer and* $\alpha \in (0,1)$. *If* $\operatorname{curl}\mathbf{u} \in C^{k+\alpha}(\overline{B}(x_0,R);\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u} \in C^{k+\alpha}(\overline{B}(x_0,R))$, *then* $\mathbf{u} \in C^{k+1+\alpha}(\overline{B}(x_0,3R/4);\mathbb{R}^3)$ *and*

$$\|\mathbf{u}\|_{C^{k+1+\alpha}(\overline{B}(x_0,3R/4))} \le C(\alpha,k,R)\{\|\mathbf{u}\|_{H^1(B(x_0,R))} + \|\operatorname{curl}\mathbf{u}\|_{C^{k+\alpha}(\overline{B}(x_0,R))} + \|\operatorname{div}\mathbf{u}\|_{C^{k+\alpha}(\overline{B}(x_0,R))}\}.$$

(iii) *If* $\operatorname{curl}\mathbf{u} \in L^\infty(B(x_0,R);\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u} \in L^\infty(B(x_0,R))$, *then we see that* $\mathbf{u} \in C^\delta(\overline{B}(x_0,3R/4);\mathbb{R}^3)$ *for any* $\delta \in (0,1)$, *and*

$$\|\mathbf{u}\|_{C^\delta(\overline{B}(x_0,3R/4))} \le C(\delta,R)\{\|\mathbf{u}\|_{H^1(B(x_0,R))} + \|\operatorname{curl}\mathbf{u}\|_{L^\infty(B(x_0,R))} + \|\operatorname{div}\mathbf{u}\|_{L^\infty(B(x_0,R))}\}.$$

For the estimates near the boundary, let $x_0 \in \partial\Omega$. Then since Ω is C^2 class, there exist $R = R(\Omega) > 0$ and a function $g \in C^2(\overline{B}(x_0,R))$ such that $B(x_0,R)\cap\Omega$ is contractible, $B(x_0,R)\cap\Omega = \{x \in B(x_0,R); g(x) > 0\}$ and $B(x_0,R)\cap\partial\Omega = \{x \in B(x_0,R); g(x) = 0\}$.

Lemma 2.2. *Let* $g \in C^1(\overline{B}(x_0,R))$ *such that* $\nabla g\cdot\mathbf{b} > 0$ *for some unit vector* $\mathbf{b}$ *and* $g(x_0) = 0$. *Define* $\boldsymbol{\nu} = \nabla g/|\nabla g|$ *and*

$$B = \{x \in B(x_0,R); g(x) > 0\}, \quad B' = \{x \in B(x_0,3R/4); g(x) > 0\},$$
$$\Sigma = \{x \in B(x_0,R); g(x) = 0\}.$$

Let $\mathbf{u} \in H^1(B;\mathbb{R}^3)$. *Then the following holds.*

(i) *Let* $g \in C^2(\overline{B}(x_0,R))$. *If* $\operatorname{curl}\mathbf{u} \in L^q(B;\mathbb{R}^3)$, $\operatorname{div}\mathbf{u} \in L^q(B)$ *and* $\mathbf{u}_T \in W^{1-1/q,q}(\Sigma;\mathbb{R}^3)$ *for some* $q > 1$, *then* $\mathbf{u} \in W^{1,q}(B';\mathbb{R}^3)$ *and*

$$\|\nabla\mathbf{u}\|_{W^{1,q}(B')} \le C(q,g,R)\big\{\|\mathbf{u}\|_{H^1(B)} + \|\operatorname{curl}\mathbf{u}\|_{L^q(B)} + \|\operatorname{div}\mathbf{u}\|_{L^q(B)} + \|\mathbf{u}_T\|_{W^{1-1/q,q}(\Sigma)}\big\}.$$

(ii) *Let* $k \ge 0$ *be an integer and* $\alpha \in (0,1)$, *and* $g \in C^{k+1+\alpha}(\overline{B}(x_0,R))$. *If* $\operatorname{curl}\mathbf{u} \in C^{k+\alpha}(\overline{B};\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u} \in C^{k+\alpha}(\overline{B})$ *and* $\mathbf{u}\cdot\boldsymbol{\nu} \in C^{k+1+\alpha}(\overline{\Sigma})$, *then* $\mathbf{u} \in C^{k+1+\alpha}(\overline{B'};\mathbb{R}^3)$ *and*

$$\|\nabla\mathbf{u}\|_{C^{k+\alpha}(\overline{B'})} \le C(g,\alpha,k,R)\big\{\|\mathbf{u}\|_{H^1(B)} + \|\operatorname{curl}\mathbf{u}\|_{C^{k+\alpha}(\overline{B})} + \|\operatorname{div}\mathbf{u}\|_{C^{k+\alpha}(\overline{B})} + \|\mathbf{u}\cdot\boldsymbol{\nu}\|_{C^{k+1+\alpha}(\overline{\Sigma})}\big\}.$$

(iii) *Let* $k \ge 0$ *be an integer and* $\alpha \in (0,1)$. *Suppose that* $g \in C^{k+1+\alpha}(\overline{B}(x_0,R))$. *If* $\operatorname{curl}\mathbf{u} \in C^{k+\alpha}(\overline{B};\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u} \in C^{k+\alpha}(\overline{B})$ *and* $\mathbf{u}_T \in C^{k+1+\alpha}(\overline{\Sigma})$, *then* $\mathbf{u} \in C^{k+1+\alpha}(\overline{B'};\mathbb{R}^3)$ *and*

$$\|\nabla\mathbf{u}\|_{C^{k+\alpha}(\overline{B'})} \le C(g,\alpha,k,R)\big\{\|\mathbf{u}\|_{H^1(B)} + \|\operatorname{curl}\mathbf{u}\|_{C^{k+\alpha}(\overline{B})} + \|\operatorname{div}\mathbf{u}\|_{C^{k+\alpha}(\overline{B})} + \|\mathbf{u}_T\|_{C^{k+1+\alpha}(\overline{\Sigma})}\big\}.$$

(iv) *Suppose that* $g \in C^2(\overline{B})$ *and* $\delta \in (0,1)$. *If* $\operatorname{curl}\mathbf{u} \in L^\infty(B;\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u} \in L^\infty(B)$ *and* $\mathbf{u}_T \in C^{0,1}(\overline{\Sigma},\mathbb{R}^3)$, *then* $\mathbf{u} \in C^\delta(\overline{B'};\mathbb{R}^3)$ *and*

$$\|\mathbf{u}\|_{C^\delta(\overline{B'})} \le C(\delta,R)\{\|\mathbf{u}\|_{H^1(B)} + \|\operatorname{curl}\mathbf{u}\|_{L^\infty(B)} + \|\operatorname{div}\mathbf{u}\|_{L^\infty(B)} + \|\mathbf{u}_T\|_{C^{0,1}(\overline{\Sigma})}\}.$$

2.3. **Lifting operator of the boundary values.** We state important properties on "lifting" of the boundary data.

Lemma 2.3. *Let* Ω *be a Lipschitz continuous domain in* $\mathbb{R}^3$ *and* $\mathcal{H}^e_T \in H^{1/2}(\partial\Omega)$. *Then there exists* $\mathcal{H}^e \in H^1(\Omega)$ *such that* $(\mathcal{H}^e)_T = \mathcal{H}^e_T$ *on* $\partial\Omega$ *and* $\operatorname{div}\mathcal{H}^e = 0$ *in* Ω, *and*

$$\|\mathcal{H}^e\|_{H^1(\Omega)} \le C(\Omega)\|\mathcal{H}^e_T\|_{H^{1/2}(\partial\Omega)}.$$

Here $\mathcal{H}^e$ is unique up to an additive function of $V := \{\mathbf{v} \in H^1_0(\Omega,\mathbb{R}^3); \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}$. We note that in [19], he assumed that Ω is C^2 domain and has no holes. But since we follows Girault and Raviart [13], we only assume that Ω is Lipschitz continuous.

Proof of Lemma 2.3. Let $\mathbf{w}$ be any vector field in $H^1(\Omega,\mathbb{R}^3)$ such that $\mathbf{w} = \mathcal{H}^e_T$ on $\partial\Omega$. By the Green formula, we have

$$\int_\Omega \operatorname{div}\mathbf{w}\,dx = \int_{\partial\Omega} \mathbf{w}\cdot\boldsymbol{\nu}\,dS = \int_{\partial\Omega} \mathcal{H}^e_T\cdot\boldsymbol{\nu}\,dS = 0.$$

Thus $\operatorname{div}\mathbf{w} \in L^2_0(\Omega) := \{\mathbf{v} \in L^2(\Omega); \int_\Omega \mathbf{v}\,dx = 0\}$. We consider V to be a Banach space with norm $\|\nabla\mathbf{v}\|_{L^2(\Omega)}$ which is equivalent to $H^1_0(\Omega)$ norm according to the Poincaré inequality. Then since V is a closed subspace of $H^1_0(\Omega,\mathbb{R}^3)$, we can write $H^1_0(\Omega) = V \oplus V^\perp$ in $H^1_0(\Omega)$. Then it follows from [13, Corollary 2.4] that there exists a unique $\mathbf{v} \in V^\perp$ such that $\operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{w}$ in Ω, and $\|\nabla\mathbf{v}\|_{L^2(\Omega)} \le C_1\|\operatorname{div}\mathbf{w}\|_{L^2(\Omega)}$. If we define $\mathbf{u} = \mathbf{w} - \mathbf{v}$, then $\mathbf{u}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega} = \mathcal{H}^e_T$ on $\partial\Omega$, and $\operatorname{div}\mathbf{u} = 0$ in Ω. Thus we obtain

$$\begin{aligned}\|\mathbf{u}\|_{H^1(\Omega)} &\le \|\mathbf{w}\|_{H^1(\Omega)} + \|\mathbf{v}\|_{H^1(\Omega)}\\ &\le \|\mathbf{w}\|_{H^1(\Omega)} + C(\Omega)\|\nabla\mathbf{w}\|_{L^2(\Omega)} \le C\|\mathbf{w}\|_{H^1(\Omega)}.\end{aligned}$$

If we take lower limit of both side and taking the definition of $H^{1/2}(\partial\Omega)$ into consideration, we obtain

$$\inf_{\mathbf{v}\in V}\|\mathbf{u}+\mathbf{v}\|_{H^1(\Omega)} \le C\|\mathcal{H}^e_T\|_{H^{1/2}(\partial\Omega)}.$$

By the standard arguments of variational problem, we see that the left hand side is achieved. If we choose a minimizer $\mathbf{v}$ and define $\mathcal{H}^e = \mathbf{u}+\mathbf{v}$, this $\mathcal{H}^e$ satisfies the conclusion. □

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $C^{3+\alpha}$ *boundary for some* $0 < \alpha < 1$ *and* $\mathcal{H}^e_T \in C^{2+\alpha}(\partial\Omega)$. *Then there exists* $\mathcal{H}^e \in C^{2+\alpha}(\overline{\Omega},\mathbb{R}^3)$ *such that* $\operatorname{div}\mathcal{H}^e = 0$ *in* Ω, $(\mathcal{H}^e)_T = \mathcal{H}^e_T$ *on* $\partial\Omega$, *and*

$$\|\mathcal{H}^e\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\Omega)\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)}.$$

Note that we do not assume that Ω is simply-connected and has no holes.

Proof. It follows from the Gilbarg and Trudinger [12, Lemma 6.38] that there exist an open set $\Omega' \supset \overline{\Omega}$ and $\mathcal{H}^e_1 \in C^{2+\alpha}_0(\Omega')$ such that $\mathcal{H}^e_1|_{\partial\Omega} = \mathcal{H}^e_T$ on $\partial\Omega$, so $(\mathcal{H}^e_1)_T = \mathcal{H}^e_T$ on $\partial\Omega$, and satisfies

$$\|\mathcal{H}^e_1\|_{C^{2+\alpha}(\Omega')} \le C(\alpha,\Omega)\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)}.$$

Choose $\phi \in C^{3+\alpha}(\overline{\Omega})$ satisfying

$$\Delta\phi = \operatorname{div}\mathcal{H}_1^e \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega.$$

Since $\operatorname{div}\mathcal{H}_1^e \in L^q(\Omega)$ for any $q > 1$, we have

$$\|\phi\|_{W^{2,q}(\Omega)} \le C\|\operatorname{div}\mathcal{H}_1^e\|_{L^q(\Omega)} \le C\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\overline{\Omega})}.$$

By the Sobolev imbedding theorem, we have

$$\|\phi\|_{C^{1+(1-3/q)}(\overline{\Omega})} \le C\|\phi\|_{W^{2,q}(\Omega)} \le C'\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}.$$

Define $\mathcal{H}^e = \mathcal{H}_1^e - \nabla\phi \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Then clearly $\operatorname{div}\mathcal{H}^e = 0$ in Ω. Then $\|\mathcal{H}^e\|_{C^0(\overline{\Omega})} \le C\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}$. Thus $\mathcal{H}^e$ satisfies the system

$$\begin{gathered}
\Delta\mathcal{H}^e = -\operatorname{curl}^2\mathcal{H}^e = -\operatorname{curl}^2\mathcal{H}_1^e \in C^\alpha(\overline{\Omega}),\\
(\mathcal{H}^e)_T = (\mathcal{H}_1^e)_T - (\nabla\phi)_T = (\mathcal{H}_1^e)_T = \mathcal{H}_T^e \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3),\\
\operatorname{div}\mathcal{H}^e = 0 \text{ in } C^{1+\alpha}(\partial\Omega).
\end{gathered} \tag{2.3}$$

We note that $\Delta\mathcal{H}^e \in C^\alpha(\overline{\Omega}) \subset L^q(\Omega)$ for any $q > 1$ and the boundary condition of (2.3) satisfies the complementing condition. Thus it follows from Morrey [17, Theorem 6.3.8 and 6.3.9], we obtain $\mathcal{H}^e \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and

$$\begin{aligned}
\|\mathcal{H}^e\|_{C^{2+\alpha}(\overline{\Omega})} &\le C\{\|\mathcal{H}_1^e\|_{C^\alpha(\overline{\Omega})} + \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)} + \|\mathcal{H}^e\|_{C^0(\overline{\Omega})}\}\\
&\le C(\alpha,\Omega)\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}.
\end{aligned}$$

□

3. Weak solutions and an approximation of F

3.1. **Weak solution of** (1.10)**.** In this subsection we give the notion of weak solutions of (1.10). (cf. [5]). Define the function spaces.

$$H^1(\Omega, \mathbb{R}^3, \operatorname{div} 0) = \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^3); \operatorname{div}\mathbf{u} = 0 \text{ a.e. in } \Omega\},$$
$$H^1_{t0}(\Omega, \mathbb{R}^3, \operatorname{div} 0) = \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^3, \operatorname{div} 0); \mathbf{u}_T = 0 \text{ on } \partial\Omega\}.$$

Here we note that $H^1_{t0}(\Omega, \mathbb{R}^3, \operatorname{div} 0)$ is a Hilbert space with the norm

$$\{\|\operatorname{curl}\mathbf{u}\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{L^2(\Omega)}\}^{1/2},$$

which is equivalent to the standard $H^1(\Omega)$ norm. Then we define weak solutions of (1.10).

Definition 3.1. Let $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ be a given vector field on $\partial\Omega$ which is tangent to $\partial\Omega$. Then $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3, \operatorname{div} 0)$ is called a weak solution of (1.10) if the following conditions are satisfied:

(i) $\|g^M(\operatorname{curl}\mathbf{H})\|_{L^\infty(\Omega)} < b_f$.
(ii) $\mathbf{H}_T = \mu\mathcal{H}_T^e$ on $\partial\Omega$ in the sense of trace in $H^{1/2}(\partial\Omega, \mathbb{R}^3)$.
(iii) For all $\mathbf{B} \in H^1(\Omega, \mathbb{R}^3)$,

$$\begin{aligned}
&\int_\Omega \{F(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}\cdot\operatorname{curl}\mathbf{B} + \mathbf{H}\cdot\mathbf{B}\}\,dx\\
&+ \int_{\partial\Omega} F(g^M(\operatorname{curl}\mathbf{H}))((M\operatorname{curl}\mathbf{H})_T \times \mathbf{B}_T)\cdot\boldsymbol{\nu}\,dS = 0.
\end{aligned} \tag{3.1}$$

If $\mathbf{B} \in H^1(\Omega, \mathbb{R}^3)$, then $\mathbf{B}_T \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Therefore, the surface integral in (3.1) is well defined.

3.2. **An approximation of F.** Let F be the given function as in (1.12) and $\delta > 0$ small enough. Then we can find a function $W_\delta(u) \in C^2([0,\infty))$ (cf. [19]) such that

(i) $W_\delta(0) > 0$ and $W_\delta'(u) = F(u)$ for $0 \le u \le b_f - 2\delta$.

(ii) $W_\delta''(u) \ge 0$ for $u > 0$, and $W_\delta''(u) = 0$ for $u > b_f - \delta$. Thus we can write $W_\delta(u) = c_\delta u + b$ for $u > b_f - \delta$ for some $c_\delta > 0$ and real b.

(iii) If we define $F_\delta = W_\delta'$ and $\Phi_\delta(u) = [F_\delta(u)]^2 u$, then $v = \Phi_\delta(u)$ is strictly increasing in $[0,\infty)$.

(iv) Let $u = \Psi_\delta(v)$ is the inverse function of $v = \Phi_\delta(u)$ defined for $v \ge 0$ and define
$$f_\delta(v) = \frac{1}{F_\delta(\Psi_\delta(v))},$$
then $f_\delta \in C^2_{\rm loc}([0,\infty))$ and there exist $c_1(\delta), c_2(\delta), \varepsilon_2(\delta) > 0$ such that $c_1(\delta) \le f_\delta(v) \le c_2(\delta)$,
$$f_\delta(v) - 2|f_\delta'(v)|v \ge c_1(\delta), \quad \text{for } 0 \le v < \infty,$$
and $f_\delta(v) = 1/c_\delta$ if $v \ge b_\psi - \varepsilon_2(\delta)$.

(v) Furthermore, if $F \in C^{2+\alpha}_{\rm loc}([0, b_f))$, then $f_\delta \in C^{2+\alpha}_{\rm loc}([0,\infty))$.

3.3. **Weak solutions and unique existence of an approximate system.** We set a quasilinear system (called F_δ-system).
$$\begin{gathered} -\operatorname{curl}[F_\delta(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \\ \mathbf{H}_T = \mu\mathcal{H}^e_T \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

Definition 3.2. $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3, \operatorname{div} 0)$ is called a weak solution of (3.2) if $\mathbf{H}_T = \mu\mathcal{H}^e_T$ on $\partial\Omega$ and satisfy
$$\begin{aligned} &\int_\Omega \{F_\delta(g^M(\operatorname{curl}\mathbf{H}))\langle M\operatorname{curl}\mathbf{H}, \operatorname{curl}\mathbf{B}\rangle + \langle \mathbf{H}, \mathbf{B}\rangle\}\,dx \\ &+ \int_{\partial\Omega} F_\delta(g^M(\operatorname{curl}\mathbf{H}))((M\operatorname{curl}\mathbf{H})_T \times \mathbf{B}_T)\cdot\boldsymbol{\nu}\,dS = 0 \end{aligned} \tag{3.3}$$
for all $\mathbf{B} \in H^1(\Omega, \mathbb{R}^3)$.

Since $F_\delta(u)$ is defined for all $u \ge 0$ and constant for large u, and $(M\operatorname{curl}\mathbf{H})_T \in H^{-1/2}(\partial\Omega)$, $\mathbf{B}_T \in H^{1/2}(\partial\Omega)$, the surface integral of (3.3) makes sense.

We shall study the existence of unique weak solution of (3.2).

Proposition 3.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with C^2 boundary, and $M \in C(\overline{\Omega}, S_+(3))$ satisfies that there exists $\beta(M) > 0$ such that*
$$g^M(\xi) = \langle M(x)\xi, \xi\rangle \ge \beta(M)|\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^3. \tag{3.4}$$
If we assume that $\mathcal{H}^e_T \in H^{1/2}(\partial\Omega)$ and F_δ is the function defined in subsection 3.2, *then* (3.2) *has a unique weak solution $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$.*

Proof. From Lemma 2.3, there exists $\mathcal{H}^e \in H^1(\Omega, \mathbb{R}^3)$ such that $\operatorname{div}\mathcal{H}^e = 0$ in Ω and $(\mathcal{H}^e)_T = \mathcal{H}^e_T$ on $\partial\Omega$. We write $\mathbf{H} = \mathcal{H}^e + \mathbf{u}$. Then (3.2) becomes
$$\begin{gathered} -\operatorname{curl}[F_\delta(g^M(\operatorname{curl}(\mathcal{H}^e + \mathbf{u})))M\operatorname{curl}(\mathcal{H}^e + \mathbf{u})] = \mathcal{H}^e + \mathbf{u} \quad \text{in } \Omega, \\ \mathbf{u}_T = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.5}$$

For brevity of notation, we write $Y = H^1_{t0}(\Omega, \mathbb{R}^3, \operatorname{div} 0)$. We define

$$\mathcal{E}[\mathbf{u}] = \mathcal{W}_\delta[\mathcal{H}^e + \mathbf{u}] = \int_\Omega \{W_\delta(g^M(\operatorname{curl}(\mathcal{H}^e + \mathbf{u}))) + |\mathcal{H}^e + \mathbf{u}|^2\}\, dx. \tag{3.6}$$

Then it is clear that $\mathcal{E}$ is well defined on Y and continuous. Put $c = \sqrt{b_f/\beta(M)}$. If $|\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x))| \ge c$, then

$$g^M(\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x)) \ge \beta(M)|\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x))|^2 \ge \beta(M)c^2 = b_f.$$

Therefore, it follows from the properties of W_δ that

$$\begin{aligned} W_\delta(g^M(\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x)))) &= c_\delta g^M(\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x))) + b \\ &\ge c_\delta \beta(M)|\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x))|^2 + b. \end{aligned}$$

Define $\Gamma(\mathcal{H}^e + \mathbf{u}) = \{x \in \Omega; |\operatorname{curl}(\mathcal{H}^e(x) + \mathbf{u}(x))| \ge c\}$. Then we have

$$\begin{aligned} &\int_\Omega W_\delta(g^M(\operatorname{curl}(\mathcal{H}^e + \mathbf{u}))) \, dx \\ &\ge \int_{\Gamma(\mathcal{H}^e+\mathbf{u})} W_\delta(\operatorname{curl}(\mathcal{H}^e + \mathbf{u}))) \, dx \\ &\ge c_\delta \beta(M) \int_{\Gamma(\mathcal{H}^e+\mathbf{u})} |\operatorname{curl}(\mathcal{H}^e + \mathbf{u})|^2 \, dx + b|\Gamma(\mathcal{H}^e + \mathbf{u})| \\ &= c_\delta \beta(M) \int_\Omega |\operatorname{curl}(\mathcal{H}^e + \mathbf{u})|^2 \, dx \\ &\quad - c_\delta \beta(M) \int_{\Omega\setminus\Gamma(\mathcal{H}^e+\mathbf{u})} |\operatorname{curl}(\mathcal{H}^e + \mathbf{u})|^2 \, dx + b|\Gamma(\mathcal{H}^e + \mathbf{u})| \\ &\ge c_\delta \beta(M) \int_\Omega |\operatorname{curl}(\mathcal{H}^e + \mathbf{u})|^2 \, dx - c_\delta \beta(M) c^2 |\Omega \setminus \Gamma(\mathcal{H}^e + \mathbf{u})| + b|\Gamma(\mathcal{H}^e + \mathbf{u})| \\ &\ge c_\delta \beta(M) \int_\Omega |\operatorname{curl}(\mathcal{H}^e + \mathbf{u})|^2 \, dx - c'|\Omega|. \end{aligned}$$

Thus we see that

$$\mathcal{E}_\delta[\mathbf{u}] \ge c_\delta \beta(M) \int_\Omega |\operatorname{curl}(\mathcal{H}^e + \mathbf{u})| \, dx - c'|\Omega| + \int_\Omega |\mathcal{H}^e + \mathbf{u}|^2 \, dx.$$

It follows from Dautray and Lions [10, p.212] that for any vector field $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$, $\|\mathbf{v}\|^2_{H^1(\Omega)}$ is equivalent to

$$\|\operatorname{curl} \mathbf{v}\|^2_{L^2(\Omega)} + \|\operatorname{div} \mathbf{v}\|^2_{L^2(\Omega)} + \|\mathbf{v}\|^2_{L^2(\Omega)} + \|\mathbf{v}_T\|^2_{H^{1/2}(\partial\Omega)}. \tag{3.7}$$

Therefore, we see that $\lim_{\|\mathbf{u}\|_Y \to \infty} \mathcal{E}_\delta[\mathbf{u}] = +\infty$. Since clearly $\mathcal{E}_\delta[\mathbf{u}]$ is strictly convex on Y, $\mathcal{E}_\delta$ has a unique minimizer $\mathbf{u} \in Y$ and $\mathbf{u}$ is a weak solution of (3.5). Then $\mathbf{H} = \mathcal{H}^e + \mathbf{u}$ is a weak solution of (3.2). Since $\mathcal{E}_\delta$ is strictly convex, any critical point of $\mathcal{E}_\delta$ is a global minimizer. Thus $\mathcal{E}_\delta$ has at most one global minimizer, and so (3.2) has exactly one weak solution. □

4. Regularity of the weak solutions of the approximate system

In this section, we shall show the regularity of weak solutions for the approximate system (F_δ-system) (3.2). For brevity of notation, we consider the system (3.2) with $\mu = 1$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{3+\alpha}$ boundary for some $0 < \alpha < 1$, and $M \in C^{1+\alpha}(\overline{\Omega}, S_+(3))$ satisfies (3.4), and let F_δ be as in subsection 3.2. For given $0 \not\equiv \mathcal{H}_T^e \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$, if $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$ is a weak solution of (3.2) (with $\mu = 1$), then $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and*

$$\|\mathbf{H}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\Omega, \|M\|_{C^{1+\alpha}(\overline{\Omega})}, \beta(M), \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}, \alpha, \delta). \tag{4.1}$$

The constant also depends on the behavior of F_δ.

The authors of [16] considered the regularity of F_0-system (1.3). For the purpose they assumed the condition (1.4). However, as we consider the F_δ-system, we need not to assume the condition (1.4).

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary and let $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega)$. If $\mathbf{H}$ is a weak solution of (3.2) with $\mu = 1$, then we have*

$$\|\mathbf{H}\|_{H^1(\Omega)} \le C(\Omega, \beta(M), \|M\|_{C^0(\overline{\Omega})}, W_\delta, \|\mathcal{H}_T^e\|_{H^{1/2}(\partial\Omega)}).$$

Proof. Let $\mathcal{H}^e$ be a lifting of $\mathcal{H}_T^e$. Then the weak solution of (3.2) is of the form $\mathbf{H} = \mathcal{H}^e + \mathbf{u}$ where $\mathbf{u}$ is the minimizer of (3.6). Therefore $\mathcal{E}[\mathbf{u}] \le \mathcal{E}[\mathbf{0}]$. Since W_δ is strictly increasing and $W_\delta(0) > 0$, using (3.4), we see that

$$\begin{aligned} &\int_\Omega \{W_\delta(0)\beta(M)|\operatorname{curl}(\mathcal{H}^e + \mathbf{u})|^2 + |\mathcal{H}^e + \mathbf{u}|^2\}\,dx \\ &\le \mathcal{E}[0] = \int_\Omega \{W_\delta(g^M(\operatorname{curl}\mathcal{H}^e)) + |\mathcal{H}^e|^2\}\,dx. \end{aligned}$$

Since $\mathbf{u} \in H^1_{t0}(\Omega, \mathbb{R}^3.\operatorname{div} 0)$, it follows from [10, p.212] that $\|\mathbf{u}\|_{H^1(\Omega)}$ is equivalent to $\|\operatorname{curl}\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^2(\Omega)}$. Therefore from the above estimate, we have the estimate

$$\|\mathbf{u}\|_{H^1(\Omega)} \le C(\Omega, \beta(M), \|M\|_{C^0(\overline{\Omega})}, W_\delta, \|\mathcal{H}^e\|_{H^1(\Omega)}).$$

Thus we have

$$\|\mathbf{H}\|_{H^1(\Omega)} \le C_1(\Omega, \beta(M), \|M\|_{C^0(\overline{\Omega})}, W_\delta, \|\mathcal{H}^e\|_{H^1(\Omega)}).$$

Taking Lemma 2.3 into consideration, we complete the proof. □

Remark 4.3. If $\mathcal{H}_T^e \in C^{0,1}(\partial\Omega, \mathbb{R}^3)$, it follows from Lemmas 2.1 and 2.2 that $\mathbf{H} \in C^\delta(\overline{\Omega})$ for any $0 < \delta < 1$.

Along the idea of [16] we shall show that the regularity of weak solutions of the approximate F_δ-system. We only consider the boundary regularity. For the proof of Theorem 4.1, it suffices to prove the next proposition. For the purpose, let $x_0 \in \partial\Omega$ and $0 < \alpha < 1$. Since Ω is C^2 class, we can choose $R(\Omega) > 0$ such that $B(x_0, R(\Omega))$ is contractible. For $0 < R < R(\Omega)$, let $g \in C^{2+\alpha}(\overline{B}(x_0, R))$ such that $g(x_0) = 0$. Define for $r \in (0, R]$,

$$\begin{aligned} \Omega[r] &= \{x \in B(x_0, r); g(x) > 0\}, \quad \overline{\Omega}[r] = \{x \in \overline{B}(x_0, r); g(x) \ge 0\}, \\ \Sigma[r] &= \{x \in B(x_0, r); g(x) = 0\}, \quad \overline{\Sigma}[r] = \{x \in \overline{B}(x_0, r); g(x) = 0\}. \end{aligned}$$

Proposition 4.4. *Let $\mathbf{H} \in H^1(\Omega[R], \mathbb{R}^3)$ be a weak solution of the following F_δ-system*

$$\begin{gathered} -\operatorname{curl}[F_\delta(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}] = \mathbf{H} \quad \text{in } \Omega[R], \\ \mathbf{H}_T = \mathcal{H}_T^e \quad \text{on } \Sigma[R]. \end{gathered} \tag{4.2}$$

If $\mathcal{H}_T^e \in C^{2+\alpha}(\overline{\Sigma}[R], \mathbb{R}^3)$, then $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega}[R/8], \mathbb{R}^3)$, and the following estimate holds.

$$\|\mathbf{H}\|_{C^{2+\alpha}(\overline{\Omega}[R/8])} \le C(g, R, M, \alpha, \delta, \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\overline{\Sigma}[R])}, \|\mathbf{H}\|_{H^1(\Omega[R])}).$$

We note that if $\mathbf{H}$ is the weak solution of (4.2), it follows from Lemma 4.2 that $\|\mathbf{H}\|_{H^1(\Omega[R])}$ is controlled by $C(\Omega, M, \|\mathcal{H}_T^e\|_{H^{1/2}(\partial\Omega)})$.

Since we treat the approximate system, it is not necessary to assume the boundedness of $\operatorname{curl} \mathbf{H}$ as in [16] in which the authors proved the regularity for the system associated with F_0. Though the proof look like the proof of [16], we have to modify it for our general setting. Therefore we give a complete proof despite the redundancy.

Proof. **Step 1.** We can find a vector field $\mathbf{B}$ such that

$$\begin{gathered} \operatorname{curl} \mathbf{B} = \mathbf{H},\ \operatorname{div} \mathbf{B} = 0 \quad \text{in } \Omega[3R/4], \\ \boldsymbol{\nu} \cdot \mathbf{B} = 0 \quad \text{on } \Sigma[3R/4]. \end{gathered} \tag{4.3}$$

In fact, according to the contractibility of $\Omega[7R/8]$, we can choose a C^2 contractible domain Ω^* such that $\Omega[3R/4] \subset \Omega^* \subset \Omega[7R/8]$. From the contractibility of Ω^* and the fact that $\operatorname{div} \mathbf{H} = 0$ in $\Omega[R]$, we can see from [5] that there exists $\mathbf{B} \in H^2(\Omega^*, \mathbb{R}^3)$ such that

$$\begin{gathered} \operatorname{curl} \mathbf{B} = \mathbf{H},\ \operatorname{div} \mathbf{B} = 0 \quad \text{in } \Omega^*, \\ \boldsymbol{\nu} \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega^*. \end{gathered}$$

By the Sobolev imbedding theorem, we see that $\mathbf{B} \in C^\tau(\overline{\Omega^*}, \mathbb{R}^3)$ for any $0 < \tau < 1/2$. Since $\mathbf{H}$ is a weak solution, for any $\mathbf{v} \in Y = H^1_{t0}(\Omega^*, \mathbb{R}^3, \operatorname{div} 0)$,

$$\int_{\Omega^*} F_\delta(g^M(\operatorname{curl} \mathbf{H}))M \operatorname{curl} \mathbf{H} + \mathbf{B}) \cdot \operatorname{curl} \mathbf{v}\, dx = 0.$$

If we put $\mathbf{w} = F_\delta(g^M(\operatorname{curl} \mathbf{H}))M \operatorname{curl} \mathbf{H} + \mathbf{B}$, since $F_\delta(u) = c_\delta$ for $u \ge b_f$, we see that $\mathbf{w} \in L^2(\Omega^*, \mathbb{R}^3)$, and $\mathbf{w} \perp \operatorname{curl} H^1_{t0}(\Omega^*, \mathbb{R}^3, \operatorname{div} 0)$ in $L^2(\Omega^*, \mathbb{R}^3)$. Since it follows from [10, p. 226] that

$$(\operatorname{curl} Y)^\perp = Z = \{\mathbf{z} \in L^2(\Omega^*, \mathbb{R}^3); \operatorname{curl} \mathbf{z} = 0 \text{ in } \Omega^*\}.$$

Since Ω^* is contractible, we can write $Z = \{\nabla\phi; \phi \in H^1(\Omega^*)\}$. Therefore, there exists $\varphi \in H^1(\Omega^*)$ such that

$$F_\delta(g^M(\operatorname{curl} \mathbf{H}))M \operatorname{curl} \mathbf{H} + \mathbf{B} = \nabla\varphi \quad \text{in } \Omega^*. \tag{4.4}$$

Applying $Q = M^{-1}$,

$$F_\delta(g^M(\operatorname{curl} \mathbf{H})) \operatorname{curl} \mathbf{H} = Q(\nabla\varphi - \mathbf{B}). \tag{4.5}$$

From (4.4) and (4.5), we see that

$$F_\delta(g^M(\operatorname{curl} \mathbf{H}))^2 g^M(\operatorname{curl} \mathbf{H}) = g^Q(\nabla\varphi - \mathbf{B}).$$

Putting $u(x) = g^M(\operatorname{curl} \mathbf{H}(x)), v(x) = g^Q(\nabla\varphi(x) - \mathbf{B}(x))$, from the properties (iii) and (iv) in subsection 3.2, $\Phi_\delta(u(x)) = v(x)$. Therefore,

$$g^M(\operatorname{curl} \mathbf{H}(x)) = u(x) = \Psi_\delta(g^Q(\nabla\varphi(x) - \mathbf{B}(x)) = \Psi_\delta(v(x)),$$

and

$$f_\delta(v) = \frac{1}{F_\delta(\Psi_\delta(v))}.$$

Define

$$\mathcal{A}(x, \mathbf{p}) = f_\delta(g^Q(\mathbf{p} - \mathbf{B}))Q(\mathbf{p} - \mathbf{B}).$$

From (4.5), we can see that

$$\begin{aligned} \operatorname{curl} \mathbf{H} &= \frac{1}{F_\delta(g^M(\operatorname{curl} \mathbf{H}))} Q(\nabla\varphi - \mathbf{B}) \\ &= \frac{1}{F_\delta(u)} Q(\nabla\varphi - \mathbf{B}) \\ &= f_\delta(g^Q(\nabla\varphi - \mathbf{B}))Q(\nabla\varphi - \mathbf{B}) \\ &= \mathcal{A}(x, \nabla\varphi). \end{aligned} \tag{4.6}$$

If we write $\mathbf{H} = \mathcal{H}^e + \mathbf{u}$ where $\mathcal{H}^e \in C^{2+\alpha}(\Omega[3R/4]; \mathbb{R}^3)$ as in Lemma 2.4, then $\mathbf{u} \in H^1_{t0}(\Omega[3R/4], \mathbb{R}^3, \operatorname{div} 0)$, and

$$\begin{aligned} \boldsymbol{\nu} \cdot Q(\nabla\varphi - \mathbf{B}) &= F_\delta(g^M(\operatorname{curl} \mathbf{H}))\boldsymbol{\nu} \cdot \operatorname{curl} \mathbf{H} \\ &= F_\delta(g^M(\operatorname{curl} \mathbf{H}))\boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}^e_T \quad \text{on } \Sigma[3R/4]. \end{aligned}$$

Here we used the fact that $\boldsymbol{\nu} \cdot \operatorname{curl} \mathbf{H}$ depends only on the tangent trace $\mathbf{H}_T$ of $\mathbf{H}$. Thus we can see

$$\begin{aligned} \boldsymbol{\nu} \cdot \mathcal{A}(x, \nabla\varphi) &= f_\delta(g^Q(\nabla\varphi - \mathbf{B}))\boldsymbol{\nu} \cdot Q(\nabla\varphi - \mathbf{B}) \\ &= f_\delta(g^Q(\nabla\varphi - \mathbf{B}))F_\delta(g^M(\operatorname{curl} \mathbf{H}))\boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}^e_T \\ &= \boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}^e_T \quad \text{on } \Sigma[3R/4]. \end{aligned}$$

Hence taking (4.6) into consideration, we can see that φ is a weak solution of the co-normal derivative problem

$$\begin{gathered} \operatorname{div}[\mathcal{A}(x, \nabla\varphi)] = 0 \quad \text{in } \Omega[3R/4], \\ \boldsymbol{\nu} \cdot \mathcal{A}(x, \nabla\varphi) = \boldsymbol{\nu} \cdot \operatorname{curl} \mathcal{H}^e_T \quad \text{on } \Sigma[3R/4]. \end{gathered} \tag{4.7}$$

Step 2. $W^{1,p}$ regularity of φ. Let φ be a weak solution of (4.7). Since $\mathcal{A}(x, \nabla\varphi) = f_\delta(g^Q(\nabla\varphi - \mathbf{B}))Q(\nabla\varphi - \mathbf{B})$, if $g^Q(\nabla\varphi - \mathbf{B}) > b_\psi - \varepsilon_2$, then $f_\delta(g^Q(\nabla\varphi - \mathbf{B}) = 1/c_\delta$. Thus we can write

$$\mathcal{A}_i(x, \nabla\varphi) = \frac{1}{c_\delta} \sum_{j=1}^3 q_{ij}(\frac{\partial\varphi}{\partial x_j} - B_j)$$

where $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3), \mathbf{B} = (B_1, B_2, B_3)$ and $Q = (q_{ij})$. Define $\mathbf{f} = (f^1, f^2, f^3)$ such that

$$f^i(x) = \mathcal{A}_i(x, \nabla\varphi(x)) - \frac{1}{c_\delta} \sum_{j=1}^3 q_{ij}(x) \frac{\partial\varphi}{\partial x_j} - (\operatorname{curl} \mathcal{H}^e)_i$$

where $\operatorname{curl} \mathcal{H}^e = ((\operatorname{curl} \mathcal{H}^e)_1, (\operatorname{curl} \mathcal{H}^e)_2, (\operatorname{curl} \mathcal{H}^e)_3)$. Then we can write $\mathcal{A}(x, \nabla\varphi) = \frac{1}{c_\delta} Q\nabla\varphi + \mathbf{f}$. Therefore, (4.7) becomes the system

$$\begin{gathered} \operatorname{div}\Big(\frac{1}{c_\delta} Q\nabla\varphi + \mathbf{f}\Big) = 0 \quad \text{in } \Omega[3R/4], \\ \boldsymbol{\nu} \cdot \Big(\frac{1}{c_\delta} Q\nabla\varphi + \mathbf{f}\Big) = 0 \quad \text{in } \Sigma[3R/4]. \end{gathered} \tag{4.8}$$

Since $\mathbf{B} \in C^\tau(\overline{\Omega}[3R/4], \mathbb{R}^3)$ for any $0 < \tau < 1/2$, in particular, $\mathbf{B}$ is bounded on $\overline{\Omega}[3R/4]$. On $\{x \in \Omega[3R/4]; g^Q(\nabla\varphi - \mathbf{b}) > b_\psi - \varepsilon_2\}$,

$$f^i(x) = -\frac{1}{c_\delta}\sum_{j=1}^{3} q_{ij}(x)B_j(x) - (\operatorname{curl}\mathcal{H}^e)_i(x).$$

If $g^Q(\nabla\varphi - \mathbf{B}) \le b_\psi - \varepsilon_2$, then $\beta(Q)|\nabla\varphi - \mathbf{B}|^2 \le b_\psi - \varepsilon_2$. Therefore, $|\nabla\varphi|$ is bounded, so $\mathcal{A}(x, \nabla\varphi) = f_\delta(g^Q(\nabla\varphi - \mathbf{B}))Q(\nabla\varphi - \mathbf{B})$ is bounded. Hence we see that $\mathbf{f} \in L^\infty(\Omega[3R/4], \mathbb{R}^3)$, and

$$\|\mathbf{f}\|_{L^\infty(\Omega[3R/4])} \le C(1 + \frac{1}{c_\delta}\|\mathbf{B}\|_{C^0(\overline{\Omega}[3R/4])})$$

where $C = C(\Omega[3R/4], \|Q\|_{C^0(\Omega[3R/4])}, \|\mathcal{H}^e\|_{C^1(\overline{\Omega}[3R/4])})$. Thus we see that $\mathbf{f} \in L^p(\Omega[3R/4], \mathbb{R}^3)$ for any $1 < p < \infty$. By the classical L^p Schauder theory, it follows that (4.8) has a weak solution in $W^{1,p}(\Omega[3R/4])$. The system

$$\begin{gathered} \operatorname{div}(Q\nabla\varphi) = 0 \quad \text{in } \Omega[3R/4], \\ \boldsymbol{\nu} \cdot Q\nabla\varphi = 0 \quad \text{on } \Sigma[3R/4] \end{gathered}$$

has only constant solution. Therefore the weak solution of (4.8) is unique up to an additive constant. Thus we see that $\varphi \in W^{1,p}(\Omega[3R/4])$ for any $1 < p < \infty$, and there exists a constant $C_1 = C_1(\Omega, \|Q\|_{C^1(\overline{\Omega}[3R/4])}, \beta(Q), p)$ such that

$$\|\nabla\varphi\|_{L^p(\Omega[3R/4])} \le C_1 c_\delta \|\mathbf{f}\|_{L^p(\Omega[3R/4])} \le C_2(c_\delta + \|\mathbf{B}\|_{L^p(\Omega[3R/4])}).$$

By the Sobolev imbedding theorem, $\varphi \in C^\tau(\overline{\Omega}[3R/4])$ for any $0 < \tau < 1/2$. We can choose φ so that $\int_{\Omega[3R/4]} \varphi\, dx = 0$. Hence we obtain

$$\begin{aligned} \|\varphi\|_{C^\tau(\overline{\Omega}[3R/4])} &\le C\|\varphi\|_{W^{1,p}(\Omega[3R/4])} \\ &\le C(c_\delta, \|\mathbf{B}\|_{L^p(\Omega[3R/4])}) \\ &\le C(c_\delta, \|\mathbf{H}\|_{H^1(\Omega[3R/4])}) \end{aligned}$$

Let $\mathbf{I} = Q^t\boldsymbol{\nu}$ where Q^t is the transpose matrix of Q. Then the boundary condition of (4.7) is written in the form

$$f_\delta(g^Q(\nabla\varphi - \mathbf{B}))(\nabla\varphi - \mathbf{B}) \cdot \mathbf{I} = \boldsymbol{\nu} \cdot \operatorname{curl}\mathcal{H}^e_T$$

on $\Sigma[3R/4]$. Since $\mathbf{I} = Q^t\boldsymbol{\nu} \in C^1(\Sigma[3R/4], \mathbb{R}^3)$, $\mathbf{I} \cdot \mathbf{B} \in C^\tau(\Sigma[3R/4], \mathbb{R}^3)$ for any $0 < \tau < 1/2$. If we define $\gamma = \mathbf{I}\cdot\boldsymbol{\nu}$, then $\gamma = Q^t\boldsymbol{\nu}\cdot\boldsymbol{\nu} = \boldsymbol{\nu}\cdot Q\boldsymbol{\nu} \ge \beta(Q) > 0$. Therefore, we can write $\mathbf{I} = \gamma(\boldsymbol{\nu} + \mathbf{t})$, where $\mathbf{t}$ is tangent vector. Then the boundary condition of (4.7) is rewritten in the form

$$\frac{\partial\varphi}{\partial\boldsymbol{\nu}} + \mathbf{t} \cdot \nabla\varphi = \frac{\mathbf{I} \cdot \mathbf{B}}{\gamma} + \frac{1}{\gamma f_\delta(g^Q(\nabla\varphi - \mathbf{B}))}\boldsymbol{\nu} \cdot \operatorname{curl}\mathcal{H}^e_T. \tag{4.9}$$

However, in this stage we do not have the C^α regularity of the right hand side of (4.9). According to this reason, we shall use the arguments of [16] for the system (4.7). In order to do so, we remember $\mathcal{A}(x, \mathbf{p}) = f_\delta(g^Q(\mathbf{p} - \mathbf{B}))Q(\mathbf{p} - \mathbf{B})$ where $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{B} \in C^\tau(\overline{\Omega^*}, \mathbb{R}^3)$ for any $0 < \tau < 1/2$. Then simple calculation leads to

$$\frac{\partial\mathcal{A}_i}{\partial p_j} = f_\delta(g^Q(\mathbf{p} - \mathbf{B}))q_{ij}(x) + 2f'_\delta(g^Q(\mathbf{p} - \mathbf{B}))\sum_{k,m=1}^{3} q_{ik}q_{jm}(p_k - B_k)(p_m - B_m).$$

Therefore, using the Schwarz inequality and the property (iv) of F_δ,

$$\begin{aligned}\sum_{i,j=1}^{3}\frac{\partial\mathcal{A}_i}{\partial p_j}\xi_i\xi_j &= f_\delta(g^Q(\mathbf{p}-\mathbf{B}))g^Q(\xi)+2f_\delta(g^Q(\mathbf{p}-\mathbf{B}))g^Q(\mathbf{p}-\mathbf{B},\xi)^2\\ &\geq f_\delta(g^Q(\mathbf{p}-\mathbf{B}))g^Q(\xi)-2|f_\delta(g^Q(\mathbf{p}-\mathbf{B}))|g^Q(\mathbf{p}-\mathbf{B})g^Q(\xi)\\ &\geq c_1(\delta)g^Q(\xi)\\ &\geq c_1(\delta)\beta(Q)|\xi|^2.\end{aligned}$$

Since $c_1(\delta)\leq f_\delta(v)\leq 1/F(0)$ from the property (iv) of F_δ, we see that

$$2|f_\delta'(v)|v\leq f_\delta(v)-c_1(\delta)\leq f_\delta(v)\leq\frac{1}{F(0)}.$$

Therefore,

$$\begin{aligned}\sum_{i,j=1}^{3}\frac{\partial\mathcal{A}_i}{\partial p_j}\xi_i\xi_j &\leq f_\delta(g^Q(\mathbf{p}-\mathbf{B}))g^Q(\xi)+2|f_\delta'(g^Q(\mathbf{p}-\mathbf{B}))|g^Q(\mathbf{p}-\mathbf{B})g^Q(\xi)\\ &=\{f_\delta(g^Q(\mathbf{p}-\mathbf{B}))+2|f_\delta'(g^Q(\mathbf{p}-\mathbf{B}))|g^Q(\mathbf{p}-\mathbf{B})\}g^Q(\xi)\\ &\leq\frac{2}{F(0)}g^Q(\xi)\\ &\leq\frac{1}{F(0)}\|M\|_{C^0(\overline{\Omega})}|\xi|^2.\end{aligned}$$

Thus there exist $\lambda,\Lambda>0$ such that the eigenvalues of the matrix $\left(\frac{\partial\mathcal{A}_i}{\partial p_j}\right)$ is contained in the interval $[\lambda,\Lambda]$. Next, we estimate $|\mathcal{A}_i(x,\mathbf{p})-\mathcal{A}_i(y,\mathbf{p})|$. We have

$$\begin{aligned}&|\mathcal{A}_i(x,\mathbf{p})-\mathcal{A}_i(y,\mathbf{p})|\\ &=|\{f_\delta(g^{Q(y)}(\mathbf{p}-\mathbf{B}(y))\\ &\quad+(f_\delta(g^{Q(x)}(\mathbf{p}-\mathbf{B}(x))-f_\delta(g^{Q(y)}(\mathbf{p}-\mathbf{B}(y))\}Q(x)(\mathbf{p}-\mathbf{B}(x))\\ &\quad-f_\delta(g^{Q(y)}(\mathbf{p}-\mathbf{B}(y))Q(y)(\mathbf{p}-\mathbf{B}(y))|\\ &\quad\leq|f_\delta(g^{Q(y)}(\mathbf{p}-\mathbf{B}(y))\{Q(x)(\mathbf{p}-\mathbf{B}(x)-Q(y)(\mathbf{p}-\mathbf{B}(y))\}|\\ &\quad+|\{f_\delta(g^{Q(x)}(\mathbf{p}-\mathbf{B}(x)))-f_\delta(g^{Q(y)}(\mathbf{p}-\mathbf{B}(y))\}Q(x)(\mathbf{p}-\mathbf{B}(x))|.\end{aligned}$$

Since $Q\in C^{1+\alpha}$, we have $|Q(x)\mathbf{p}-Q(y)\mathbf{p}|\leq C|x-y||\mathbf{p}|$. Moreover, since $\mathbf{B}\in C^\tau(\overline{\Omega}[3R/4],\mathbb{R}^3)$, we have $|Q(x)\mathbf{B}(x)-Q(y)\mathbf{B}(y)|\leq C|x-y|^\tau$. Therefore, we have for some $0<\theta<1$,

$$\begin{aligned}&|f_\delta(g^{Q(x)}(\mathbf{p}-\mathbf{B}(x))-f_\delta(g^{Q(y)}(\mathbf{p}-\mathbf{B}(y))|\\ &\leq|x-y||f_\delta'(g^Q(y)(\mathbf{p}-\mathbf{B}(y)+\theta(g^{Q(x)}(\mathbf{p}-\mathbf{B}(x))-g^{Q(y)}(\mathbf{p}-\mathbf{B}(x)))|\\ &\leq C|x-y|.\end{aligned}$$

If we note that $|f_\delta|\leq 1/F(0)$, we have for some $m>0$

$$|\mathcal{A}_i(x,\mathbf{p})-\mathcal{A}_i(y,\mathbf{p})|\leq m|x-y|^\tau(1+|\mathbf{p}|).$$

Since $\frac{\partial\mathcal{A}_i}{\partial p_j}$ is continuous with respect to $\mathbf{p}$ and $|\varphi|$ is bounded, we can apply Lieberman [15, Theorem 5.1 and the remark]. Hence $\varphi\in C^1(\overline{\Omega}[3R/4])$ and

$$|\nabla\varphi(x)-\nabla\varphi(y)|\leq C(\tau,m,\Lambda,\lambda,g,\|\varphi\|_{C^0(\overline{\Omega}([3R/4])})|x-y|^\tau.$$

That is to say, $\varphi \in C^{1+\tau}(\overline{\Omega}[3R/4])$.

Step 3. Improvement of regularity of $\mathbf{B}$ and φ. Since $\nabla\varphi \in L^p(\Omega[3R/4],\mathbb{R}^3)$ for any $1<p<\infty$ and $\mathbf{B} \in C^\tau(\overline{\Omega}[3R/4],\mathbb{R}^3)$, we see that $\nabla\varphi - \mathbf{B} \in L^p(\Omega[3R/4],\mathbb{R}^3)$. Since Q is continuous on $\overline{\Omega}$, $Q(\nabla\varphi-\mathbf{B}) \in L^p(\Omega[3R/4],\mathbb{R}^3)$. If $g^Q(\nabla\varphi-\mathbf{B}) \geq b_\psi - \varepsilon_2$, we have $f_\delta(g^Q(\nabla\varphi-\mathbf{B})) = 1/c_\delta$. Therefore $\mathcal{A}(x,\varphi) \in L^p(\Omega[3R/4],\mathbb{R}^3)$, so $\operatorname{curl}\mathbf{H} = \mathcal{A}(x,\varphi) \in L^p(\Omega[3R/4],\mathbb{R}^3)$. Since $\operatorname{div}\mathbf{H} = 0$ in $\Omega[3R/4]$ and $\mathbf{H}_T = \mathcal{H}_T^e \in C^{2+\alpha}(\Sigma[3R/4],\mathbb{R}^3)$, it follows from Lemma 2.2 (i) (cf. [16]) that $\mathbf{H} \in W^{1,p}(\Omega[R/2],\mathbb{R}^3)$ for any $1<p<\infty$. By the Sobolev imbedding theorem, we see that $\mathbf{H} \in C^\tau(\overline{\Omega}[R/2],\mathbb{R}^3)$ for any $0<\tau<1$. From these arguments, we see that $\operatorname{curl}\mathbf{B} = \mathbf{H} \in C^\tau(\overline{\Omega}[R/2],\mathbb{R}^3)$ for any $0<\tau<1$ and $\operatorname{div}\mathbf{B} = 0$ in $\Omega[R/2]$ and $\mathbf{B}\cdot\boldsymbol{\nu} = 0$ on $\Sigma[R/2]$. Using Lemma 2.2 (ii), we see $\mathbf{B} \in C^{1+\tau}(\overline{\Omega}[3R/8],\mathbb{R}^3)$ for any $0<\tau<1$, and

$$
\begin{aligned}
\|\mathbf{B}\|_{C^{1+\tau}(\overline{\Omega}[3R/8])} &\leq C(R,\tau)\{\|\operatorname{curl}\mathbf{B}\|_{C^\tau(\overline{\Omega}[R/2])} + \|\operatorname{div}\mathbf{B}\|_{C^\tau(\overline{\Omega}[R/2])} \\
&\quad + \|\mathbf{B}\|_{H^1(\Omega[R/2])} + \|\mathbf{B}\cdot\boldsymbol{\nu}\|_{C^{1+\tau}(\Sigma[R/2])}\} \\
&= C(R,\tau)\{\|\mathbf{H}\|_{C^\tau(\overline{\Omega}[R/2])} + \|\mathbf{B}\|_{H^1(\Omega[R/2])}\}.
\end{aligned}
$$

In particular, we have $\mathbf{B} \in C^{1+\tau}(\overline{\Omega}[R/2],\mathbb{R}^3)$. Thus we can return to the arguments of Step 1 with $\tau=\alpha$. So we have $\varphi \in C^{1+\alpha}(\overline{\Omega}[R/2])$, and

$$
\begin{aligned}
\|\varphi\|_{C^{1+\alpha}(\overline{\Omega}[R/2])} &\leq C(g,\lambda,M,\alpha,\|\mathbf{B}\|_{C^\alpha(\overline{\Omega}[3R/4])},\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\Sigma[3R/r]))}) \\
&\leq C(g,\lambda,M,\alpha,\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\Sigma[R])},\|\mathbf{H}\|_{H^1(\Omega[R])}).
\end{aligned}
$$

Step 4. $C^{2+\alpha}$ regularity of φ. We use the arguments [16, Step 5 in the Proof of Theorem 4.1]. We rewrite the co-normal derivative problem

$$
\begin{gathered}
\operatorname{div}[\mathcal{A}(x,\nabla\varphi) = 0 \quad \text{in } \Omega[R/2], \\
\boldsymbol{\nu}\cdot\mathcal{A}(x,\nabla\varphi) = \boldsymbol{\nu}\cdot\operatorname{curl}\mathcal{H}_T^e \quad \text{on } \Sigma[R/2]
\end{gathered}
$$

into the form of a linear system with nonlinear boundary condition

$$
\begin{gathered}
\sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j} + f(x) = 0 \quad \text{in } \Omega[R/2], \\
h(x,\nabla\varphi) = 0 \quad \text{on } \Sigma[R/2],
\end{gathered}
$$

where

$$
\begin{aligned}
a_{ij} &= f_\delta(g^Q(\nabla\varphi-\mathbf{B}))q_{ij} \\
&\quad + 2f_\delta'(g^Q(\nabla\varphi-\mathbf{B}))\sum_{l,m=1}^{3} q_{il}q_{jm}(\frac{\partial\varphi}{\partial x_m}-B_m)(\frac{\partial\varphi}{\partial x_l}-B_l),
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
f = \sum_{i,j=1}^{3}\Big\{&-f_\delta(g^Q(\nabla\varphi-\mathbf{B})q_{ij}\frac{\partial B_j}{\partial x_i} + f_\delta(g^Q(\nabla\varphi-\mathbf{B}))\frac{\partial q_{ij}}{\partial x_i}(\frac{\partial\varphi}{\partial x_j}-B_j) \\
&+ f_\delta'(g^Q(\nabla\varphi-\mathbf{B}))q_{ij}\sum_{l,m=1}^{3}\frac{\partial q_{lm}}{\partial x_i}(\frac{\partial\varphi}{\partial x_l}-B_l)(\frac{\partial\varphi}{\partial x_m}-B_m)(\frac{\partial\varphi}{\partial x_j}-B_j) \\
&- 2f_\delta'(g^Q(\nabla\varphi-\mathbf{B}))\sum_{l,m=1}^{3} q_{ij}q_{lm}\frac{\partial B_l}{\partial x_i}(\frac{\partial\varphi}{\partial x_m}-B_m)(\frac{\partial\varphi}{\partial x_j}-B_j)\Big\},
\end{aligned}
$$

$$\begin{aligned}h(x,\nabla\varphi) &= \boldsymbol{\nu}\cdot\mathcal{A}(x,\nabla\varphi) - \boldsymbol{\nu}\cdot\operatorname{curl}\mathcal{H}_T^e\\ &= \boldsymbol{\nu}\cdot f_\delta(g^Q(\nabla\varphi-\mathbf{B}))Q(\nabla\varphi-\mathbf{B}) - \boldsymbol{\nu}\cdot\operatorname{curl}\mathcal{H}_T^e.\end{aligned}$$

Here we note that $a_{ij}, f \in C^\alpha(\overline{\Omega}[R/2]), h(x,\mathbf{p}) \in C^{1+\alpha}(\overline{\Sigma}[R/2]\times\mathbb{R}^3)$. By the Schwarz inequality, we have

$$\begin{aligned}\sum_{i,j=1}^3 a_{ij}(x)\xi_i\xi_j &= f_\delta(g^Q(\nabla\varphi-\mathbf{B}))g^Q(\xi) + 2f_\delta'(g^Q(\nabla\varphi-\mathbf{B}))g^Q(\nabla\varphi-\mathbf{B},\xi)^2\\ &\geq \big\{f_\delta(g^Q(\nabla\varphi-\mathbf{B})) - 2|f_\delta'(g^Q(\nabla\varphi-\mathbf{B}))|g^Q(\nabla\varphi-\mathbf{B})\big\}g^Q(\xi)\\ &\geq c_1(\delta)g^Q(\xi)\\ &\geq c_1(\delta)\beta(Q)|\xi|^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\sum_{i=1}^3\frac{\partial h}{\partial p_i}\nu_i &= \sum_{i,j,l=1}^3 f_\delta(g^Q(\mathbf{p}-\mathbf{B}))g^Q(\boldsymbol{\nu}) + 2f_\delta'(g^Q(\mathbf{p}-\mathbf{B}))g^Q(\mathbf{p}-\mathbf{B},\boldsymbol{\nu})^2\\ &\geq f_\delta(g^Q(\mathbf{p}-\mathbf{B})g^Q(\boldsymbol{\nu}) - 2|f_\delta'(g^Q(\mathbf{p}-\mathbf{B}))|g^Q(\mathbf{p}-\mathbf{B},\boldsymbol{\nu})^2\\ &\geq \{f_\delta(g^Q(\mathbf{p}-\mathbf{B})) - 2|f_\delta'(g^Q(\mathbf{p}-\mathbf{B}))|\}g^Q(\boldsymbol{\nu})\\ &\geq c_1(\delta)\beta(Q) > 0.\end{aligned}$$

Thus it follows from Lieberman [14, Lemma 4.2] that $\varphi\in C^{2+\alpha}(\overline{\Omega}[R/4])$, and

$$\|\varphi\|_{C^{2+\alpha}(\overline{\Omega}[R/4])} \leq C(\Omega,\alpha,\|a_{ij}\|_{C^\alpha(\overline{\Omega}[R/2])},\|f\|_{C^\alpha(\overline{\Omega}[R/2])},\|h\|_{C^{1+\alpha}(\overline{\Sigma}[R/2]\times\mathbb{R}^3)}).$$

Step 5. Regularity of $\mathbf{H}$. We again borrow the arguments of [16]. By the facts that $\mathbf{B},\nabla\varphi\in C^{1+\alpha}(\overline{\Omega}[R/4],\mathbb{R}^3)$, we can see that

$$\mathbf{J} := f_\delta(g^Q(\nabla\varphi-\mathbf{B}))(\nabla\varphi-\mathbf{B}) \in C^{1+\alpha}(\overline{\Omega}[R/4],\mathbb{R}^3)$$

and $\|\mathbf{J}\|_{C^{1+\alpha}(\overline{\Omega}[R/4])}$ is controlled by $\|\varphi\|_{C^{2+\alpha}(\overline{\Omega}[R/4])}$ and $\|\mathbf{B}\|_{C^{1+\alpha}(\overline{\Omega}[R/4])}$, so by $\|\mathbf{H}\|_{C^\alpha(\overline{\Omega}[R])}$. Thus $\operatorname{curl}\mathbf{H} = Q\mathbf{J}\in C^{1+\alpha}(\overline{\Omega}[R/4],\mathbb{R}^3)$, $\operatorname{div}\mathbf{H}=0$ in $\Omega[R/4]$ and $\mathbf{H}_T=\mathcal{H}_T^e$ on $\Sigma[R/4]$. Since $\mathbf{H}\in H^1(\Omega[R/4])$, it follows from Lemma 2.2 (iii) that $\mathbf{H}\in C^{2+\alpha}(\overline{\Omega}[R/8],\mathbb{R}^3)$ and satisfies

$$\begin{aligned}\|\mathbf{H}\|_{C^{2+\alpha}(\overline{\Omega}[R/8])} &\leq C(g,\alpha)\{\|\mathbf{J}\|_{C^{1+\alpha}(\overline{\Omega}[R/4])} + \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\Sigma[R/4])} + \|\mathbf{H}\|_{H^1(\Omega[R/4])})\\ &\leq C(g,R,M,\alpha,\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\Sigma[R])},\|\mathbf{H}\|_{H^1(\Omega[R])}).\end{aligned}$$

This completes the proof. □

Corollary 4.5 ([19])**.** *In addition to the condition of Theorem 4.1, if furthermore* $F_\delta\in C^{2+\alpha}_{\rm loc}([0,b_f))$ *and* $M\in C^{2+\alpha}(\overline{\Omega},S_+(3))$*, then* $\mathbf{H}\in C^{3+\alpha}_{\rm loc}(\Omega,\mathbb{R}^3)$.

Proof. Under the hypotheses, $Q\in C^{2+\alpha}(\overline{\Omega},S_+(3)), f_\delta\in C^{2+\alpha}_{\rm loc}([0,\infty))$. Therefore we have $a_{ij},h\in C^{1+\alpha}(\overline{\Omega})$. Repeating the proof of Theorem 4.1, we see that $\varphi\in C^{3+\alpha}(\overline{\Omega}[R/4])$ and $\mathbf{B}\in C^{2+\alpha}(\overline{\Omega}[R/4],\mathbb{R}^3)$. Therefore $\operatorname{curl}\mathbf{H}\in C^{2+\alpha}(\overline{\Omega}[R/4])$. Let $\eta\in C^{3+\alpha}$ be a cut-off function. Then $\mathbf{H},\operatorname{curl}(\eta\mathbf{H})\in C^{2+\alpha}(\overline{\Omega}[R/4])$, $\operatorname{div}(\eta\mathbf{H})\in C^{2+\alpha}(\overline{\Omega}[R/4])$ and $(\eta\mathbf{H})_T=0$ on $\Sigma[R/4]$. Thus it follows from Lemma 2.2 (iii) that $\eta\mathbf{H}\in C^{3+\alpha}(\overline{\Omega}[R/8],\mathbb{R}^3)$, so $\mathbf{H}\in C^{3+\alpha}_{\rm loc}(\Omega[R/8],\mathbb{R}^3)$. Thus we see that $\mathbf{H}\in C^{2+\alpha}(\overline{\Omega},\mathbb{R}^3)\cap C^{3+\alpha}_{\rm loc}(\Omega,\mathbb{R}^3)$. □

Proof of Theorem 1.1. If $\mathbf{H}_\mu$ is a weak solution of (1.10) satisfying (1.11), choose $\delta > 0$ small so that $\|g^M(\operatorname{curl}\mathbf{H})\|_{L^\infty(\overline{\Omega})} < b_f - 2\delta$. Then $\mathbf{H}_\mu$ is also a weak solution of (3.2). Therefore the conclusion follows from Theorem 4.1 and Corollary 4.5. □

5. Continuity in the parameter of the weak solutions of approximate system

In this section, we consider the continuity of the following F_δ-system with respect to a parameter μ,

$$\begin{gathered} -\operatorname{curl}[F_\delta(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \\ \mathbf{H}_T = \mu\mathcal{H}_T^e \quad \text{on } \partial\Omega. \end{gathered} \tag{5.1}$$

When $\mu = 0$, it is trivial that (5.1) has only one solution $\mathbf{H} = 0$.

Lemma 5.1. *Let $\Omega, M, \mathcal{H}_T^e, \delta, F_\delta$ be as in Theorem 4.1, and let $\mathbf{H}_\mu$ be a unique solution of* (5.1)*. In addition, we assume that $\boldsymbol{\nu}\cdot\operatorname{curl}\mathcal{H}_T^e = 0$ on $\partial\Omega$. Then*

$$[0,\infty) \ni \mu \mapsto \mathbf{H}_\mu \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$$

is continuous. In particular, $\lim_{\mu\to 0}\|\mathbf{H}_\mu\|_{C^{2+\alpha}(\overline{\Omega})} = 0$.

Proof. Suppose the conclusion were false. Then there exist $\mu_0 \geq 0, \varepsilon_0 > 0$ and a sequence $\{\mu_k\}$ converging to μ_0 as $k \to \infty$ such that

$$\|\mathbf{H}_{\mu_k} - \mathbf{H}_{\mu_0}\|_{C^{2+\alpha}(\overline{\Omega})} \geq \varepsilon_0$$

for all k. By Theorem 4.1, $\{\mathbf{H}_{\mu_k}\}$ is bounded in $C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\mathbf{H}_{\mu_k} \to \widetilde{\mathbf{H}}$ in $C^{2+\tau}(\overline{\Omega}, \mathbb{R}^3)$ for any $\tau \in (0,\alpha)$. Therefore, $\widetilde{\mathbf{H}}$ is a solution of (5.1) with $\mu = \mu_0$. By the uniqueness of solution, $\widetilde{\mathbf{H}} = \mathbf{H}_{\mu_0}$. That is $\|\mathbf{H}_{\mu_k} - \mathbf{H}_{\mu_0}\|_{C^{2+\tau}(\overline{\Omega})} \to 0$. We can write $\Omega = \cup_{l=1}^N \Omega_l$ where Ω_l is of the form of $B(x_0, R)$ or $\Omega[R]$ which is simply-connected and without holes. For every k, there exists l_k such that $\|\mathbf{H}_{\mu_k} - \mathbf{H}_{\mu_0}\|_{C^{2+\alpha}(\overline{\Omega}_{l_k})} \geq \varepsilon_0$. Thus there exist a subsequence (still denoted by $\{\mathbf{H}_{\mu_k}\}$) and l_0 such that

$$\|\mathbf{H}_{\mu_k} - \mathbf{H}_{\mu_0}\|_{C^{2+\alpha}(\overline{\Omega_{l_0}})} \geq \varepsilon_0.$$

We consider only the case where Ω_{l_0} is of the form $\Omega[R]$. Let $\mathbf{B}_k = (B_{k,1}, B_{k,2}, B_{k,3})$ be in $C^{2+\alpha}(\overline{\Omega}[R])$ and be the solution of

$$\begin{gathered} \operatorname{curl}\mathbf{B}_k = \mathbf{H}_{\mu_k} \quad \text{in } \Omega[R], \\ \operatorname{div}\mathbf{B}_k = 0 \quad \text{in } \Omega[R], \\ \mathbf{B}_k\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Sigma[R]. \end{gathered}$$

Since $\Omega[R]$ is simply-connected and without holes, it follows from Lemma 2.2 that $\mathbf{B}_k \to \mathbf{B}_0$ in $C^{2+\tau}(\overline{\Omega}[R], \mathbb{R}^3)$ as $k \to \infty$. Here we note that $\mathbf{B}_0$ is a solution of $\operatorname{curl}\mathbf{B}_0 = \mathbf{H}_{\mu_0}$, $\operatorname{div}\mathbf{B}_0 = 0$ in $\Omega[R]$ and $\mathbf{B}_0\cdot\boldsymbol{\nu} = 0$ on $\Sigma[R]$. Next, there exists $\varphi_k \in C^{2+\alpha}(\overline{\Omega}[R])$ such that

$$\nabla\varphi_k = F_\delta(g^M(\operatorname{curl}\mathbf{H}_{\mu_k}))M\operatorname{curl}\mathbf{H}_{\mu_k} + \mathbf{B}_k$$

in $\Omega[R]$ and $\int_{\Omega[R]}\varphi_k\,dx = 0$. Then we have φ_k is bounded in $C^{2+\alpha}(\overline{\Omega}[R])$, and $\|\varphi_k - \varphi_0\|_{C^{2+\tau}(\overline{\Omega}[R])} \to 0$. Thus φ_k is a solution of the system

$$-\sum_{i,j=1}^3 a_{k,ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j} = h_k(x) \quad \text{in } \Omega[R],$$

$$\frac{\partial\varphi}{\partial\boldsymbol{\nu}}+\mathbf{t}\cdot\nabla\varphi=\frac{1}{\gamma}\mathbf{I}\cdot\mathbf{B}_k\quad\text{on }\Sigma[R]$$

where $a_{k,ij}$ is defined by (4.10) with $\varphi=\varphi_k$ and $\mathbf{B}=\mathbf{B}_k$. If $\psi_k=\varphi_k-\varphi_0$, then ψ_k satisfies

$$-\sum_{i,j=1}^{3}a_{k,ij}(x)\frac{\partial^2\psi_k}{\partial x_i\partial x_j}=(h_k(x)-h_0(x))+\widehat{h}_k\quad\text{in }\Omega[R],$$
$$\frac{\partial\psi_k}{\partial\boldsymbol{\nu}}+\mathbf{t}\cdot\nabla\psi_k=\frac{1}{\gamma}\mathbf{I}\cdot(\mathbf{B}_k-\mathbf{B}_0)\quad\text{on }\Sigma[R],$$

where

$$\widehat{h}_k=\sum_{i,j=1}^{3}(a_{k,ij}-a_{0,ij})\frac{\partial^2\psi_k}{\partial x_i\partial x_j}.$$

Since $v_k:=g^Q(\nabla\varphi_k-\mathbf{B}_k)\to v_0:=g^Q(\nabla\varphi_0-\mathbf{B}_0)$ in $C^{1+\tau}(\overline{\Omega}[R])$, $a_{k,ij}\to a_{0,ij},h_k\to h_0$ in $C^1(\overline{\Omega}[R])$ and $\|\varphi_k\|_{C^{2+\alpha}(\overline{\Omega}[R])}$ is uniformly bounded, so we see that $\|\psi_k\|_{C^{2+\alpha}(\overline{\Omega}[R])}$ is uniformly bounded. Thus we have

$$\|\widehat{h}_k\|_{C^\alpha(\overline{\Omega}[R])}\le\sum_{i,j=1}^{3}\|a_{k,ij}-a_{0,ij}\|_{C^\alpha(\overline{\Omega}[R])}\|\psi_k\|_{C^{2+\alpha}(\overline{\Omega}[R])}\to 0$$

as $k\to\infty$. By the Fiorenza Schauder estimate [11] (cf. [12, Theorem 6.30]),

$$\begin{aligned}&\|\psi_k\|_{C^{2+\alpha}(\overline{\Omega}[R])}\\&\le C\{\|h_k-h_0\|_{C^\alpha(\overline{\Omega}[R])}+\|\widehat{h}_k\|_{C^\alpha(\overline{\Omega}[R])}+\frac{1}{\gamma}\|\mathbf{I}\cdot(\mathbf{B}_k-\mathbf{B}_0)\|_{C^{1+\alpha}(\overline{\Sigma}[R])}\}\to 0\end{aligned}$$

where C depends on $\alpha,\lambda,\Lambda,\Omega[R]$. Therefore, $\varphi_k\to\varphi_0$ in $C^{2+\alpha}(\overline{\Omega}[R])$, and so

$$\operatorname{curl}\mathbf{H}_{\mu_k}=f_\delta(v_k)Q(\nabla\varphi_k-\mathbf{B}_k)\to f_\delta(v_0)Q(\nabla\varphi_0-\mathbf{B}_0)=\operatorname{curl}\mathbf{H}_{\mu_0}$$

in $C^{1+\alpha}(\overline{\Omega}[R])$. We also have $\operatorname{div}\mathbf{H}_{\mu_k}=0$ in $\Omega[R]$. Moreover, we see that $(\mathbf{H}_{\mu_k})_T=\mu_k\mathcal{H}_T^e$ on $\Sigma[R]$. Thus we see that $\mathbf{H}_{\mu_k}\to\mathbf{H}_{\mu_0}$ in $C^{2+\alpha}(\overline{\Omega_{l_0}},\mathbb{R}^3)$. This leads to a contradiction. □

6. Regularity of weak solutions of the F-system (1.10)

In this section we shall prove Theorem 1.2.

Lemma 6.1. *Let $\Omega,F,M,\mathcal{H}_T^e$ be as in Theorem 1.2. Then there exist $0<\mu_1<\mu_2$ depending on $\Omega,\|M\|_{C^{1+\alpha}(\overline{\Omega})},\beta(M),\mathcal{H}_T^e,\alpha,F$ such that*

(i) *If $0\le\mu<\mu_1$, (1.10) has a solution $\mathbf{H}_\mu\in C^{2+\alpha}(\overline{\Omega},\mathbb{R}^3)$ satisfying*
$$\|g^M(\operatorname{curl}\mathbf{H}_\mu)\|_{C^0(\overline{\Omega})}<b_f.$$
Such solution is unique.

(ii) *If $\mu>\mu_2$, (1.10) has no $C^{2+\alpha}$ solution.*

Proof. We choose $\delta>0$ small enough and define F_δ as in subsection 3.2. From Lemma 5.1, if $\mu>0$ is small, then (5.1) has a solution $\mathbf{H}_\mu$ and $\|\mathbf{H}_\mu\|_{C^{2+\alpha}(\overline{\Omega})}$ is small. Therefore, $\|g^M(\operatorname{curl}\mathbf{H}_\mu)\|_{C^0(\overline{\Omega})}<b_f-2\delta$, so we see that $F_\delta(g^M(\operatorname{curl}\mathbf{H}_\mu))=F(g^M(\operatorname{curl}\mathbf{H}_\mu))$. Thus $\mathbf{H}_\mu$ is a solution of (1.10). Hence μ_1 exists. It is clear that from Proposition 3.3, a solution satisfying $\|g^M(\operatorname{curl}\mathbf{H}_\mu)\|_{C^0(\overline{\Omega})}<b_f$ is unique.

Next, we show that (1.10) has no $C^{2+\alpha}$ solution for large $\mu > 0$. Consider the following functional

$$\mathcal{T}[\mathbf{H}] = \int_\Omega \{g^M(\operatorname{curl}\mathbf{H}) + |\mathbf{H}|^2\}\, dx.$$

Define

$$c(\mathcal{H}_T^e) = \inf\{\mathcal{T}[\mathbf{H}]; \mathbf{H} \in H^1(\Omega, \mathbb{R}^3, \operatorname{div} 0), \mathbf{H}_T = \mathcal{H}_T^e \text{ on } \partial\Omega\}. \tag{6.1}$$

By a standard arguments, it is clear that (6.1) has a minimizer. By the hypothesis $\mathcal{H}_T^e \not\equiv 0$, we see that $c(\mathcal{H}_T^e) > 0$. We can also see that $c(\mu\mathcal{H}_T^e) = \mu^2 c(\mathcal{H}_T^e)$. If (1.10) has a solution $\mathbf{H}$, then it follows from the definition of F that $g^M(\operatorname{curl}\mathbf{H}) \le b_f$. Therefore,

$$\begin{aligned}
\min\{1, F(0)\}\mu^2 c(\mathcal{H}_T^e) &\le \int_\Omega \{F(g^M(\operatorname{curl}\mathbf{H}))g^M(\operatorname{curl}\mathbf{H}) + |\mathbf{H}|^2\}\, dx \\
&= \int_\Omega \{\operatorname{curl}[F(g^M(\operatorname{curl}\mathbf{H}))M \operatorname{curl}\mathbf{H}] + \mathbf{H}\} \cdot \mathbf{H}\, dx \\
&\quad + \int_{\partial\Omega} (\boldsymbol{\nu} \times \mathbf{H}) \cdot F(g^M(\operatorname{curl}\mathbf{H}))M \operatorname{curl}\mathbf{H}\, dS.
\end{aligned}$$

Here using the facts that $\mathbf{H}$ is a solution of (1.10),

$$\beta(M)|\operatorname{curl}\mathbf{H}|^2 \le g^M(\operatorname{curl}\mathbf{H}) \le b_f$$

and $\|\boldsymbol{\nu} \times \mathbf{H}\|_{C^0(\partial\Omega)} = \mu\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}$, we can see that

$$\min\{1, F(0)\}\mu^2 c(\mathcal{H}_T^e) \le \mu F(b_f)\|M\|_{C^0(\overline{\Omega})}\Big(\frac{b_f}{\beta(M)}\Big)^{1/2}\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}|\partial\Omega|.$$

Thus μ_2 exists, and

$$\mu_2 \le \frac{1}{\min\{1, F(0)\}\mu^2 c(\mathcal{H}_T^e)}\mu F(b_f)\|M\|_{C^0(\overline{\Omega})}\Big(\frac{b_f}{\beta(M)}\Big)^{1/2}\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}|\partial\Omega|.$$

This completes the proof. □

We define an optimal bound for the existence of solutions for (1.10).

$$\begin{aligned}
\mu^*(\mathcal{H}_T^e) = \sup\big\{&b > 0; (1.10) \text{ has a unique } C^{2+\alpha} \text{ solution } \mathbf{H}_\mu \\
&\text{for any } \mu \in (0, b), \text{ and } \sup_{0<\mu\le b}\|g^M(\operatorname{curl}\mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} < b_f\big\}.
\end{aligned} \tag{6.2}$$

Theorem 6.2. *Let $\Omega, F, M, \mathcal{H}_T^e$ be as in Theorem 1.2. Then the following holds.*

- (i) $0 < \mu^*(\mathcal{H}_T^e) < \infty$.
- (ii) $[0, \mu^*(\mathcal{H}_T^e)) \ni \mu \mapsto \mathbf{H}_\mu \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$ *is continuous.*
- (iii) $\lim_{\mu\to\mu^*(\mathcal{H}_T^e)} \|g^M(\operatorname{curl}\mathbf{H})\|_{C^0(\overline{\Omega})} = b_f$.
- (iv) *For any $b \in (0, \mu^*(\mathcal{H}_T^e))$, we have*

$$\sup_{0\le\mu\le b}\|\mathbf{H}_\mu\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\Omega, \|M\|_{C^{1+\alpha}(\overline{\Omega})}, \beta(M), \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}, \alpha, b).$$

The constant also depends on the behavior of F.

Proof. For brevity of notation, we write $\mu^*(\mathcal{H}_T^e) = \mu^*$.

(i) From Lemma 6.1, $\mu^* < \infty$. We show $\mu^* > 0$. From Lemma 6.1, there exists $b > 0$ such that (1.10) has a $C^{2+\alpha}$ solution $\mathbf{H}_\mu$ for any $\mu \in [0, b]$, and $\|g^M(\operatorname{curl}\mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} \le b_f - 2\delta$ for some $\delta > 0$. Then $\mathbf{H}_\mu$ is also the solution of (5.1). Thus $\mu^* > 0$.

(ii) Let $b \in (0, \mu^*)$. Then

$$\sup_{0<\mu\le b} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} < b_f.$$

If we choose $\delta > 0$ small enough, we have

$$\sup_{0<\mu\le b} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} \le b_f - 2\delta.$$

For $\mu \in [0, b]$, $\mathbf{H}_\mu$ is the solution of (5.1), and so from Lemma 5.1, $\mu \mapsto \mathbf{H}_\mu$ is continuous from $[0, b]$ to $C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Since $b \in (0, \mu^*)$ is arbitrary, we see that (ii) holds.

(iii) Suppose the conclusion of (iii) were false. Then there exists $0 < L < b_f$ such that

$$\sup_{0\le\mu<\mu^*} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} \le L.$$

Choose $\delta > 0$ so that $L < b_f - 4\delta$. Then for any $0 \le \mu < \mu^*$, $\mathbf{H}_\mu$ is also a solution of (5.1). Let $\mathbf{H}^\delta_\mu$ be a solution of (5.1). Then $\mathbf{H}^\delta_\mu = \mathbf{H}_\mu$ for $0 \le \mu \le \mu^*$. We claim that

$$\sup_{0<\mu<\infty} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} \ge b_f - 2\delta.$$

In fact, if $\sup_{0<\mu<\infty} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} < b_f - 2\delta$, $\mathbf{H}^\delta_\mu$ is a solution of (1.10) for any $\mu \in (0, \infty)$ satisfying

$$\|g^M(\operatorname{curl} \mathbf{H}^\delta_\mu)\|_{C^0(\overline{\Omega})} < b_f.$$

Therefore, $\mu^*(\mathcal{H}^e_T) = \infty$. This is a contradiction. Thus there exists $\mu_0 > \mu^*$ such that $\|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} < b_f - 2\delta$ if $0 < \mu < \mu_0$ and $\|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} = b_f - 2\delta$ if $\mu = \mu_0$. Then for any $0 < \mu \le \mu_0$, (1.10) has a solution $\mathbf{H}_\mu = \mathbf{H}^\delta_\mu$. By Lemma 5.1, since $[0, \mu_0] \ni \mu \mapsto \mathbf{H}_\mu = \mathbf{H}^\delta_\mu \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$ is continuous, we have

$$\sup_{0<\mu<\mu_0} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} < b_f - 2\delta < b_f.$$

This is a contradiction to the definition of μ^*.

(iv) For any $0 < b < \mu^*$, using the conclusion of (ii) and the definition of μ^*, we have

$$\sup_{0\le\mu\le b} \|g^M(\operatorname{curl} \mathbf{H}_\mu)\|_{C^0(\overline{\Omega})} \le L(b) < b_f.$$

If we choose $\delta > 0$ small enough so that $L(b) < b_f - 2\delta$, $\mathbf{H}_\mu$ is a solution of (1.10). Thus (iv) follows from Theorem 4.1. □

Now the proof of Theorem 1.2 follows from Lemma 6.1 and Theorem 6.2.

Theorem 6.3. *Let Ω, M, F be as in Theorem 1.2. Then there exists $\mu_* > 0$ such that for any $\mathcal{H}^e_T \in C^{2+\alpha}(\partial\Omega)$ with $\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)} = 1$, we have $\mu^*(\mathcal{H}^e_T) \ge \mu_*$.*

Proof. Suppose the conclusion were false. Then there exists $\{\mathcal{H}^e_{j,T}\}$ satisfying $\mu^*_j := \mu^*(\mathcal{H}^e_{j,T}) \to 0$ as $j \to \infty$. By Theorem 6.2, for any $\varepsilon > 0$ small and any j, there exist $\mu_j \in (0, \mu^*_j)$ and the solution $\mathbf{H}_{\mu_j}$ with the boundary condition $\mathbf{H}_{\mu_j,T} = \mu_j \mathcal{H}^e_{j,T}$ such that

$$b_f - 2\varepsilon \le \|g^M(\operatorname{curl} \mathbf{H}_{\mu_j})\|_{C^0(\overline{\Omega})} < b_f - \varepsilon.$$

By Theorem 6.2 (iv), since we may assume that $\mu_j \le b$, we have

$$\|\mathbf{H}_{\mu_j}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\Omega, \|M\|_{C^{1+\alpha}(\overline{\Omega})}, \beta(M), \alpha, b).$$

Therefore $\{\mathbf{H}_{\mu_j}\}$ is uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$. Thus passing to a subsequence, we may assume that $\mathbf{H}_{\mu_j} \to \mathbf{H}_0$ in $C^{2+\tau}(\overline{\Omega}, \mathbb{R}^3)$ as $j \to \infty$ for any $0 < \tau < \alpha$. Here $\mathbf{H}_0$ is a solution of (1.10) with the boundary condition $(\mathbf{H}_0)_T = 0$. Therefore $\mathbf{H}_0 = 0$. However

$$\|g^M(\operatorname{curl}\mathbf{H}_0)\|_{C^0(\overline{\Omega})} = \lim_{j\to\infty} \|g^M(\operatorname{curl}\mathbf{H}_{\mu_j})\|_{C^0(\overline{\Omega})} \ge b_f - 2\varepsilon.$$

This is a contradiction. □

Now we consider the semilinear problem

$$\begin{gathered} -\operatorname{curl}^2 \mathbf{A} = f(g^Q(\mathbf{A}))Q\mathbf{A} \quad \text{in } \Omega, \\ (\operatorname{curl}\mathbf{A})_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega \end{gathered} \tag{6.3}$$

satisfying the condition $\|g^Q(\mathbf{A})\|_{L^\infty(\Omega)} < b_\psi$ where the function f is defined in (2.1). The corresponding quasilinear problem becomes

$$\begin{gathered} -\operatorname{curl}[F(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \\ \mathbf{H}_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega \end{gathered} \tag{6.4}$$

satisfying the condition $\|g^M(\operatorname{curl}\mathbf{H})\|_{L^\infty(\Omega)} < b_f$.

Finally, we can prove as in [16, Remark 4.4] that (6.2) and (6.4) is equivalent without topological restriction for Ω and (1.14) in the general setting.

Proposition 6.4. *Problem* (6.2) *with the condition* $\|g^Q(\mathbf{A})\|_{L^\infty(\Omega)} < b_\psi$ *is equivalent to problem* (6.4) *with the condition* $\|g^M(\operatorname{curl}\mathbf{H})\|_{L^\infty(\Omega)} < b_f$.

Proof. Let $\mathbf{A}$ be a solution of (6.2) satisfying $\|g^Q(\mathbf{A})\|_{L^\infty(\Omega)} < b_\psi$. If we define $\mathbf{H} = \operatorname{curl}\mathbf{A}$, then $-\operatorname{curl}\mathbf{H} = f(g^Q(\mathbf{A}))Q\mathbf{A}$. Here we note that

$$\begin{aligned} F(g^M(\operatorname{curl}\mathbf{H})) &= F(g^M(\operatorname{curl}^2\mathbf{A})) \\ &= F(g^M(-f(g^Q(\mathbf{A}))Q\mathbf{A})) \\ &= F(f(g^Q(\mathbf{A})))^2\langle MQ\mathbf{A}, Q\mathbf{A}\rangle \\ &= F(f(g^Q(\mathbf{A})))^2 g^Q(\mathbf{A})) \\ &= F(\Psi(g^Q(\mathbf{A}))) \\ &= \frac{1}{f(g^Q(\mathbf{A}))}. \end{aligned}$$

Therefore, $-\mathbf{A} = -F(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}$. Thus

$$\mathbf{H} = \operatorname{curl}\mathbf{A} = -\operatorname{curl}[F(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}],$$

and $\mathbf{H}_T = (\operatorname{curl}\mathbf{A})_T = \mathcal{H}^e_T$ on $\partial\Omega$.

Conversely, let $\mathbf{H}$ be a solution of (6.4) satisfying $\|g^M(\operatorname{curl}\mathbf{H})\|_{L^\infty(\Omega)} < b_f$. Define $\mathbf{A} = -F(g^M(\operatorname{curl}\mathbf{H}))M\operatorname{curl}\mathbf{H}$. Then $-\operatorname{curl}\mathbf{H} = f(g^Q(\mathbf{A}))Q\mathbf{A}$. From (6.4), $\operatorname{curl}\mathbf{A} = \mathbf{H}$. Therefore, $-\operatorname{curl}^2\mathbf{A} = f(g^Q(\mathbf{A}))Q\mathbf{A}$, and $(\operatorname{curl}\mathbf{A})_T = \mathbf{H}_T = \mathcal{H}^e_T$ on $\partial\Omega$. In both case, since $-\operatorname{curl}\mathbf{H} = f(g^Q(\mathbf{A}))Q\mathbf{A}$, so

$$g^M(\operatorname{curl}\mathbf{H}) = f(g^Q(\mathbf{A}))^2 g^Q(\mathbf{A}) = \Phi(g^Q(\mathbf{A})),$$

and $\Psi(g^M(\operatorname{curl}\mathbf{H})) = g^Q(\mathbf{A})$. Therefore, $\|g^Q(\mathbf{A})\|_{L^\infty(\Omega)} < b_\psi$ is equivalent to

$$\|g^M(\operatorname{curl}\mathbf{H})\|_{L^\infty(\Omega)} < \Phi(b_\psi) = b_f.$$

□

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