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ASYMPTOTIC BEHAVIOR OF NON-AUTONOMOUS STOCHASTIC PARABOLIC EQUATIONS WITH NONLINEAR LAPLACIAN PRINCIPAL PART

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Abstract. We prove the existence and uniqueness of random attractors for the p-Laplace equation driven simultaneously by non-autonomous deterministic and stochastic forcing. The nonlinearity of the equation is allowed to have a polynomial growth rate of any order which may be greater than p. We further establish the upper semicontinuity of random attractors as the intensity of noise approaches zero. In addition, we show the pathwise periodicity of random attractors when all non-autonomous deterministic forcing terms are time periodic.

1. Introduction

In this article, we investigate the existence and upper semicontinuity of random attractors for a class of degenerate parabolic equations driven simultaneously by non-autonomous deterministic and stochastic forcing. More precisely, we consider the p-Laplace equation defined in a bounded domain $\mathcal{O} \subseteq \mathbb{R}^n$:

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f_1(t,x,u) + f_2(t,x,u) + g(t,x) + \alpha u \circ \frac{dW}{dt} \tag{1.1}$$

with homogeneous Dirichlet boundary condition, where $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$, $p > 2$, $\alpha > 0$, and W is a two-sided real-valued Wiener process on a probability space. The symbol $\circ$ means that the stochastic equation is understood in the sense of Stratonovich integration. The functions $f_1, f_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy some growth conditions which will be specified later. In particular, we will assume that f_2 is Lipschitz continuous in its third argument and f_1 has the property:

$$f_1(t,x,s)s \le -\lambda|s|^q + \psi_1(t,x), \quad \text{for all } t,s \in \mathbb{R} \text{ and } x \in \mathcal{O}, \tag{1.2}$$

where $\lambda > 0$, $q > 1$ and $\psi_1 \in L^1_{\rm loc}(\mathbb{R}, L^1(\mathcal{O}))$.

The random attractors of the stochastic p-Laplace equation (1.1) were studied in [22, 23, 24] recently, where the authors successfully established the existence of such attractors under the condition $q \le p$ for the exponents p and q in (1.1)

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and (1.2). In these papers, an abstract framework of the variational approach was introduced and equation (1.1) fits these frameworks only when $q \le p$. To the best of our knowledge, the existence of random attractors for $q > p$ is still open and the first goal of the present paper is to solve this problem. In other words, we will show the stochastic equation (1.1) has a random attractor in $L^2(\mathcal{O})$ for any $p > 2$ and $q > 1$. In addition, if f_1, f_2 and g are periodic in their first argument, then the corresponding random attractor is pathwise periodic. The second goal of this paper is to establish the upper semicontinuity of random attractors when the intensity of noise approaches zero. Such continuity of attractors is new for the p-Laplace equation even when $q \le p$.

If f_1, f_2 and g are independent of $t \in \mathbb{R}$, then (1.1) reduces to an autonomous stochastic equation. The attractors of autonomous stochastic systems have been extensively investigated in [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 18, 19, 20, 21, 22, 25, 26, 29, 30, 31, 33, 34] and the references therein. For non-autonomous random attractors, we refer the reader to [1, 11, 17, 23, 24, 27, 35, 36, 37, 38] for details.

This paper is organized as follows. In the next section, we review an existence result on random attractors for non-autonomous stochastic equations. In Section 3, we establish the well-posedness of equation (1.1) in $L^2(\mathcal{O})$ and define a continuous cocycle for the solution operators. In Section 4, we derive uniform estimates of solutions which are necessary for the existence of random absorbing sets. The asymptotic compactness of solutions for a non-autonomous deterministic equation is also contained in this section. Sections 5 and 6 are devoted to the existence and upper semicontinuity of random attractors for (1.1), respectively.

In the sequel, we use $\|\cdot\|$ and $(\cdot,\cdot)$ for the norm and inner product of $L^2(\mathcal{O})$, respectively. The norm of a general Banach space X is written as $\|\cdot\|_X$, For convenience, we will use c or c_i $(i = 1, 2, \dots)$ to denote a positive number whose value is not of significance.

2. Notation

In this section, we assume (X, d) is a complete separable metric space and $(\Omega, \mathcal{F}, P, \{\theta\}_{t\in\mathbb{R}})$ is a metric dynamical system. Denote by $\mathcal{D}$ a collection of families of nonempty subsets of X. We recall the following definitions from [35].

Definition 2.1. A mapping Φ: $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

(i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X),\ \mathcal{B}(X))$-measurable;
(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;
(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
(iv) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

A continuous cocycle Φ is called T-periodic if $\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot)$ for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 2.2. A cocycle Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.3. $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor of Φ if

(i) $\mathcal{A}$ is measurable and $\mathcal{A}(\tau,\omega)$ is compact for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$.
(ii) $\mathcal{A}$ is invariant, that is, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,
$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega))=\mathcal{A}(\tau+t,\theta_t\omega),\quad \forall\ t\ge 0.$$
(iii) For every $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ and for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,
$$\lim_{t\to\infty} d(\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0.$$

A pullback attractor $\mathcal{A}$ is said to be T-periodic if $\mathcal{A}(\tau+T,\omega)=\mathcal{A}(\tau,\omega)$ for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$.

We borrow the following result from [35]. Similar results can be found in [3, 11, 15, 18, 23, 30].

Proposition 2.4. *Let $\mathcal{D}$ be an inclusion-closed collection of families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega,\mathcal{F},P,\{\theta\}_{t\in\mathbb{R}})$. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ which is characterized by, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*
$$\begin{aligned}\mathcal{A}(\tau,\omega)&=\Omega(K,\tau,\omega)=\cup_{B\in\mathcal{D}}\Omega(B,\tau,\omega)\\&=\{\psi(0,\tau,\omega):\psi \text{ is a } \mathcal{D}\text{-complete solution of } \Phi\}\\&=\{\xi(\tau,\omega):\xi \text{ is a } \mathcal{D}\text{-complete quasi-solution of } \Phi\}.\end{aligned}$$
If, in addition, both Φ and K are T-periodic, then so is the attractor $\mathcal{A}$.

Note that the $\mathcal{F}$-measurability of the attractor $\mathcal{A}$ was proved in [37] which is an improvement of the measurability of $\mathcal{A}$ with respect to the P-completion of $\mathcal{F}$ in [35].

3. Cocycles associated with degenerate equations

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$. Consider the non-autonomous stochastic equation defined in $\mathcal{O}$ for $t>\tau$ with $\tau\in\mathbb{R}$,
$$\frac{\partial u}{\partial t}-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)=f_1(t,x,u)+f_2(t,x,u)+g(t,x)+\alpha u\circ\frac{dW}{dt},\tag{3.1}$$
with boundary condition
$$u(t,x)=0,\quad x\in\partial\mathcal{O}\text{ and } t>\tau,\tag{3.2}$$
and initial condition
$$u(\tau,x)=u_\tau(x),\quad x\in\mathcal{O},\tag{3.3}$$
where $p>2$ and $\alpha>0$ are constants, $g\in L^2_{\rm loc}(\mathbb{R},L^2(\mathcal{O}))$, W is a two-sided real-valued Wiener process on a probability space. Throughout this paper, we assume that $f_1:\mathbb{R}\times\mathcal{O}\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies, for all $t,s\in\mathbb{R}$ and $x\in\mathcal{O}$,
$$f_1(t,x,s)s\le-\lambda|s|^q+\psi_1(t,x),\tag{3.4}$$
$$|f_1(t,x,s)|\le\psi_2(t,x)|s|^{q-1}+\psi_3(t,x),\tag{3.5}$$
$$\frac{\partial f_1}{\partial s}(t,x,s)\le\psi_4(t,x),\tag{3.6}$$
where $\lambda>0$ and $q>1$ are constants, $\psi_1\in L^1_{\rm loc}(\mathbb{R},L^1(\mathcal{O}))$, $\psi_3\in L^{q_1}_{\rm loc}(\mathbb{R},L^{q_1}(\mathcal{O}))$, and $\psi_2,\psi_4\in L^\infty_{\rm loc}(\mathbb{R},L^\infty(\mathcal{O}))$. In this paper, we always use p_1 and q_1 for the

conjugate exponents of p and q, respectively; that is, $\frac{1}{p_1}+\frac{1}{p}=1$ and $\frac{1}{q_1}+\frac{1}{q}=1$. Let $f_2:\mathbb{R}\times\mathcal{O}\times\mathbb{R}\to\mathbb{R}$ be continuous and satisfy, for all $t,s_1,s_2\in\mathbb{R}$ and $x\in\mathcal{O}$,

$$f_2(t,x,0)=0,\quad |f_2(t,x,s_1)-f_2(t,x,s_2)|\le\psi_5(t,x)|s_1-s_2|, \tag{3.7}$$

where $\psi_5\in L^\infty_{\rm loc}(\mathbb{R},L^\infty(\mathcal{O}))$.

In the sequel, we will use the probability space $(\Omega,\mathcal{F},P)$ where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\ \omega(0)=0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the corresponding Wiener measure on $(\Omega,\mathcal{F})$. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be the standard group acting on $(\Omega,\mathcal{F},P)$ given by

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\quad \omega\in\Omega,\ t\in\mathbb{R}. \tag{3.8}$$

In later sections, we will study the dynamics of problem (3.1)-(3.3) over the parametric dynamical system $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$. To that end, we first convert the stochastic equation (3.1) into a non-autonomous deterministic one by using the random variable z given by:

$$z(\omega)=-\int_{-\infty}^0 e^\tau\omega(\tau)d\tau,\quad \omega\in\Omega. \tag{3.9}$$

Then z satisfies

$$dz(\theta_t\omega)+z(\theta_t\omega)dt=dW. \tag{3.10}$$

Moreover, from [2], there exists a θ_t-invariant set $\widetilde{\Omega}\subseteq\Omega$ with $P(\widetilde{\Omega})=1$ such that for every $\omega\in\widetilde{\Omega}$, $z(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0\quad\text{and}\quad \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_t\omega)dt=E(z)=0. \tag{3.11}$$

From now on, we only consider the space $\widetilde{\Omega}$ rather than Ω, and hence write $\widetilde{\Omega}$ as Ω for convenience.

Throughout the rest of this article, we set $V=W^{1,p}_0(\mathcal{O})$. Given $u_\tau\in L^2(\mathcal{O})$, a continuous $L^2(\mathcal{O})$-valued process $\{u(t,\tau,\omega,u_\tau)\}_{t\ge\tau}$ is called a solution of problem (3.1)-(3.3) if $u(\cdot,\tau,\omega,u_\tau)\in L^p_{\rm loc}([\tau,\infty),V)\cap L^q_{\rm loc}([\tau,\infty),L^q(\mathcal{O}))$ for almost all $\omega\in\Omega$, and if for all $\xi\in V\cap L^q(\mathcal{O})$,

$$\begin{aligned}
&(u(t,\tau,\omega,u_\tau),\xi)+\int_\tau^t\int_{\mathcal{O}}|\nabla u(s,\tau,\omega,u_\tau)|^{p-2}\nabla u(s,\tau,\omega,u_\tau)\cdot\nabla\xi\,dxds\\
&=(u_\tau,\xi)+\int_\tau^t\int_{\mathcal{O}} f_1(s,x,u(s,\tau,\omega,u_\tau))\xi\,dx\,ds\\
&\quad+\int_\tau^t\int_{\mathcal{O}} f_2(s,x,u(s,\tau,\omega,u_\tau))\xi\,dx\,ds\\
&\quad+\int_\tau^t\int_{\mathcal{O}} g(s,x)\xi\,dx\,ds+\alpha\int_\tau^t(u(s,\tau,\omega,u_\tau),\xi)\circ dW,
\end{aligned}$$

for all $t\ge\tau$ and for almost all $\omega\in\Omega$. Let u be a solution of (3.1)-(3.3) and set

$$v(t,\tau,\omega,v_\tau)=e^{-\alpha z(\theta_t\omega)}u(t,\tau,\omega,u_\tau)\quad\text{with } v_\tau=e^{-\alpha z(\theta_\tau\omega)}u_\tau. \tag{3.12}$$

By (3.1) and (3.12) we find

$$\begin{aligned}
&\frac{dv}{dt}-e^{\alpha(p-2)z(\theta_t\omega)}\,\mathrm{div}(|\nabla v|^{p-2}\nabla v)\\
&=\alpha z(\theta_t\omega)v+e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v)\\
&\quad+e^{-\alpha z(\theta_t\omega)}f_2(t,x,e^{\alpha z(\theta_t\omega)}v)+e^{-\alpha z(\theta_t\omega)}g(t,x),
\end{aligned} \tag{3.13}$$

with boundary condition

$$v(t,x) = 0, \quad x \in \partial\mathcal{O} \text{ and } t > \tau, \tag{3.14}$$

and initial condition

$$v(\tau,x) = v_\tau(x), \quad x \in \mathcal{O}. \tag{3.15}$$

In the sequel, we will establish the well-posedness of problem (3.13)-(3.15). For that purpose, we need to introduce the definition of weak solutions for the equation.

Definition 3.1. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in L^2(\mathcal{O})$, a continuous function $v(\cdot,\tau,\omega,v_\tau)\colon [\tau,\infty) \to L^2(\mathcal{O})$ is called a solution of (3.13)-(3.15) if $v(\tau,\tau,\omega,v_\tau) = v_\tau$ and

$$\begin{gathered} v \in L^p_{\rm loc}([\tau,\infty),V) \cap L^q_{\rm loc}([\tau,\infty),L^q(\mathcal{O})),\\ \frac{dv}{dt} \in L^{p_1}_{\rm loc}([\tau,\infty),V^*) + L^{q_1}_{\rm loc}([\tau,\infty),L^{q_1}(\mathcal{O})), \end{gathered}$$

and v satisfies, for every $\xi \in V \cap L^q(\mathcal{O})$,

$$\begin{aligned} &\frac{d}{dt}(v,\xi) - e^{\alpha(p-2)z(\theta_t\omega)} \int_{\mathcal{O}} |\nabla v|^{p-2}\nabla v \cdot \nabla\xi \, dx \\ &= \alpha z(\theta_t\omega)(v,\xi) + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_1(t,x,e^{\alpha z(\theta_t\omega)}v)\xi \, dx \\ &\quad + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_2(t,x,e^{\alpha z(\theta_t\omega)}v)\xi \, dx + e^{-\alpha z(\theta_t\omega)}(g(t,\cdot),\xi) \end{aligned}$$

in the sense of distribution on $[\tau,\infty)$.

We will employ the Galerkin method to prove the existence of solutions for the deterministic equation (3.13). For simplicity, we define an operator $A\colon V \to V^*$ by

$$(A(v_1),v_2)_{(V^*,V)} = \int_{\mathcal{O}} |\nabla v_1|^{p-2}\nabla v_1 \cdot \nabla v_2 dx, \quad \text{for all } v_1,v_2 \in V, \tag{3.16}$$

where $(\cdot,\cdot)_{(V^*,V)}$ is the duality pairing of V^* and V. It follows from [32] that A is hemicontinuous and monotone. Let $\{e_j\}_{j=1}^\infty \subseteq V \cap L^q(\mathcal{O})$ be an orthonormal basis of $L^2(\mathcal{O})$ such that $\operatorname{span}\{e_j : j \in \mathbb{N}\}$ is dense in $V \cap L^q(\mathcal{O})$. Given $n \in \mathbb{N}$, let X_n be the space spanned by $\{e_j : j = 1,\dots,n\}$ and $P_n\colon L^2(\mathcal{O}) \to X_n$ be the projection given by

$$P_n v = \sum_{j=1}^n (v,e_j)e_j, \quad \forall v \in L^2(\mathcal{O}).$$

Note that P_n can be extended to V^* and $(L^q(\mathcal{O}))^*$ by

$$P_n\phi = \sum_{j=1}^n (\phi(e_j))e_j, \quad \text{for } \phi \in V^* \text{ or } \phi \in (L^q(\mathcal{O}))^*.$$

Consider the following system for $v_n \in X_n$ defined for $t > \tau$:

$$\begin{aligned} \frac{dv_n}{dt} + e^{\alpha(p-2)z(\theta_t\omega)}P_nA(v_n) &= \alpha z(\theta_t\omega)v_n + e^{-\alpha z(\theta_t\omega)}P_nf_1(t,\cdot,e^{\alpha z(\theta_t\omega)}v_n) \\ &\quad + e^{-\alpha z(\theta_t\omega)}P_nf_2(t,\cdot,e^{\alpha z(\theta_t\omega)}v_n) + e^{-\alpha z(\theta_t\omega)}P_ng, \end{aligned} \tag{3.17}$$

with initial condition

$$v_n(\tau) = P_nv_\tau. \tag{3.18}$$

Under assumptions (3.4)-(3.7), we find that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in L^2(\mathcal{O})$, problem (3.17)-(3.18) has a unique maximal solution $v_n(\cdot, \tau, \omega) \in C^1([\tau, \tau+T), X_n)$ for some $T > 0$. The uniform estimates given below imply $T = \infty$, and hence the solution is defined for all $t \geq \tau$. In addition, $v_n(t, \tau, \omega, v_\tau)$ is measurable with respect to $\omega \in \Omega$. By examining the limiting behavior of v_n, we prove the following existence and uniqueness of solutions for (3.13)-(3.15).

Lemma 3.2. *Suppose (3.4)-(3.7) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in L^2(\mathcal{O})$, problem (3.13)-(3.15) has a unique solution $v(t, \tau, \omega, v_\tau)$ in the sense of Definition 3.1. This solution is $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O})))$-measurable in ω and continuous in initial data v_τ in $L^2(\mathcal{O})$. Moreover, the solution v satisfies the energy equation:*

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\tau,\omega,v_\tau)\|^2 + 2e^{\alpha(p-2)z(\theta_t\omega)} \int_{\mathcal{O}} |\nabla v|^p dx \\
&= 2\alpha z(\theta_t\omega)\|v\|^2 + 2e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_1(t,x,e^{\alpha z(\theta_t\omega)}v)v\,dx \\
&\quad + 2e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_2(t,x,e^{\alpha z(\theta_t\omega)}v)v dx + 2e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} g(t,x) v dx
\end{aligned}
\tag{3.19}
$$

for almost all $t \geq \tau$.

Proof. The proof consists of several steps. We first derive uniform estimates on the solution v_n of (3.17)-(3.18).

Step 1: Uniform estimates. By (3.17) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_n\|^2 + e^{\alpha(p-2)z(\theta_t\omega)} \int_{\mathcal{O}} |\nabla v_n|^p dx \\
&= \alpha z(\theta_t\omega)\|v_n\|^2 + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n dx \\
&\quad + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_2(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n dx + e^{-\alpha z(\theta_t\omega)}(g(t), v_n).
\end{aligned}
\tag{3.20}
$$

By (3.4) we obtain

$$
\begin{aligned}
&e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n dx \\
&\leq -\lambda e^{\alpha(q-2)z(\theta_t\omega)} \int_{\mathcal{O}} |v_n|^q dx + e^{-2\alpha z(\theta_t\omega)} \int_{\mathcal{O}} \psi_1(t,x) dx.
\end{aligned}
\tag{3.21}
$$

By (3.7) we have

$$
e^{-\alpha z(\theta_t\omega)} \Big| \int_{\mathcal{O}} f_2(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n dx\Big| \leq \|\psi_5(t)\|_{L^\infty(\mathcal{O})}\|v_n\|^2.
\tag{3.22}
$$

It follows from (3.20)-(3.22) that

$$
\begin{aligned}
&\frac{d}{dt}\|v_n\|^2 + 2e^{\alpha(p-2)z(\theta_t\omega)} \int_{\mathcal{O}} |\nabla v_n|^p dx + 2\lambda e^{\alpha(q-2)z(\theta_t\omega)} \int_{\mathcal{O}} |v_n|^q dx \\
&\leq \left(1 + 2\alpha z(\theta_t\omega) + 2\|\psi_5(t)\|_{L^\infty(\mathcal{O})}\right)\|v_n\|^2 + 2e^{-2\alpha z(\theta_t\omega)}\|\psi_1(t)\|_{L^1(\mathcal{O})} \\
&\quad + e^{-2\alpha z(\theta_t\omega)}\|g(t)\|^2.
\end{aligned}
\tag{3.23}
$$

Note that $z(\theta_t\omega)$ is continuous in t for fixed ω. Therefore, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, we obtain from (3.23) that

$$\{v_n\}_{n=1}^\infty \text{ is bounded in } L^\infty(\tau, \tau+T; L^2(\mathcal{O})) \cap L^q(\tau, \tau+T; L^q(\mathcal{O})) \cap L^p(\tau, \tau+T; V). \tag{3.24}$$

By (3.16) and (3.24) we obtain

$$\{A(v_n)\}_{n=1}^\infty \text{ is bounded in } L^{p_1}(\tau, \tau+T; V^*), \tag{3.25}$$

where $\frac{1}{p_1} + \frac{1}{p} = 1$. By (3.5) we have

$$\begin{aligned}
&\int_\tau^{\tau+T} \int_{\mathcal{O}} |f_1(t, x, e^{\alpha z(\theta_t\omega)} v_n)|^{q_1}\, dx\, dt \\
&\le c_1 \int_\tau^{\tau+T} \int_{\mathcal{O}} |v_n|^q\, dx\, dt + \int_\tau^{\tau+T} \int_{\mathcal{O}} |\psi_3(t,x)|^{q_1}\, dx\, dt,
\end{aligned}$$

with $\frac{1}{q_1} + \frac{1}{q} = 1$, which along with (3.24) implies

$$\{f_1(t, x, e^{\alpha z(\theta_t\omega)} v_n)\}_{n=1}^\infty \text{ is bounded in } L^{q_1}(\tau, \tau+T; L^{q_1}(\mathcal{O})). \tag{3.26}$$

By (3.25) and (3.26), we infer from (3.17) that

$$\{\frac{dv_n}{dt}\}_{n=1}^\infty \text{ is bounded in } L^{p_1}(\tau, \tau+T; V^*) + L^{q_1}(\tau, \tau+T; L^{q_1}(\mathcal{O})). \tag{3.27}$$

Next, we prove the existence of solutions for (3.13)-(3.15) based on (3.24)-(3.27).

Step 2: Existence of solutions. It follows from (3.24)-(3.26) that there exist $\widetilde{v} \in L^2(\mathcal{O})$, $v \in L^\infty(\tau, \tau+T; L^2(\mathcal{O})) \cap L^p(\tau, \tau+T; V) \cap L^q(\tau, \tau+T; L^q(\mathcal{O}))$, $\chi_1 \in L^{q_1}(\tau, \tau+T; L^{q_1}(\mathcal{O}))$, $\chi_2 \in L^{p_1}(\tau, \tau+T; V^*)$ such that, up to a subsequence,

$$v_n \to v \quad \text{weak-star in } L^\infty(\tau, \tau+T; L^2(\mathcal{O})), \tag{3.28}$$

$$v_n \to v \quad \text{weakly in } L^p(\tau, \tau+T; V) \text{ and } L^q(\tau, \tau+T; L^q(\mathcal{O})), \tag{3.29}$$

$$f_1(t, x, e^{\alpha z(\theta_t\omega)} v_n) \to \chi_1 \quad \text{weakly in } L^{q_1}(\tau, \tau+T; L^{q_1}(\mathcal{O})), \tag{3.30}$$

$$A(v_n) \to \chi_2 \quad \text{weakly in } L^{p_1}(\tau, \tau+T; V^*), \tag{3.31}$$

$$\frac{dv_n}{dt} \to \frac{dv}{dt} \quad \text{weakly in } L^{p_1}(\tau, \tau+T; V^*) + L^{q_1}(\tau, \tau+T; L^{q_1}(\mathcal{O})), \tag{3.32}$$

$$v_n(\tau+T, \tau, \omega) \to \widetilde{v} \quad \text{weakly in } L^2(\mathcal{O}). \tag{3.33}$$

Let $\sigma = \min\{p_1, q_1\}$. By (3.27) we see that $\{\frac{dv_n}{dt}\}$ is bounded in $L^\sigma(\tau, \tau+T; (V \cap L^q(\mathcal{O}))^*)$. Note that $V \hookrightarrow L^2(\mathcal{O}) \hookrightarrow (V \cap L^q(\mathcal{O}))^*$ and the embedding $V \hookrightarrow L^2(\mathcal{O})$ is compact. Therefore, it follows from [28] that, up to a subsequence,

$$v_n \to v \text{ strongly in } L^2(\tau, \tau+T; L^2(\mathcal{O})). \tag{3.34}$$

By (3.7) and (3.34) one can verify that for every $j \in \mathbb{N}$ and $\phi \in C_0^\infty(\tau, \tau+T)$,

$$\begin{aligned}
&\lim_{n\to\infty} \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (P_n f_2(t, \cdot, e^{\alpha z(\theta_t\omega)} v_n), \phi e_j) dt \\
&= \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (f_2(t, \cdot, e^{\alpha z(\theta_t\omega)} v), \phi e_j) dt.
\end{aligned} \tag{3.35}$$

Letting $n \to \infty$ in (3.17), by (3.28)-(3.31) and (3.35), we obtain, for $j \in \mathbb{N}$ and $\phi \in C_0^\infty(\tau, \tau + T)$,

$$
\begin{aligned}
&-\int_\tau^{\tau+T} (v, e_j)\phi' dt + \int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)} (\chi_2, \phi e_j)_{(V^*,V)} dt \\
&= \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v, \phi e_j) dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (\chi_1, \phi e_j)_{(L^{q_1}, L^q)} dt \\
&\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (f_2(t, \cdot, e^{\alpha z(\theta_t\omega)} v), \phi e_j) dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (g, \phi e_j) dt.
\end{aligned} \tag{3.36}
$$

Since $\operatorname{span}\{e_j, j \in \mathbb{N}\}$ is dense in $V \cap L^q(\mathcal{O})$, we find that (3.36) is still valid when e_j is replaced by any element in $V \cap L^q(\mathcal{O})$. Therefore, for every $\xi \in V \cap L^q(\mathcal{O})$, in the sense of distributions, we have

$$
\begin{aligned}
&\frac{d}{dt}(v, \xi) + e^{\alpha(p-2)z(\theta_t\omega)} (\chi_2, \xi)_{(V^*,V)} \\
&= \alpha z(\theta_t\omega)(v, \xi) + e^{-\alpha z(\theta_t\omega)} (\chi_1, \xi)_{(L^{q_1}, L^q)} \\
&\quad + e^{-\alpha z(\theta_t\omega)} (f_2(t, \cdot, e^{\alpha z(\theta_t\omega)} v), \xi) + e^{-\alpha z(\theta_t\omega)} (g, \xi),
\end{aligned} \tag{3.37}
$$

Note that (3.37) implies

$$
\begin{aligned}
\frac{dv}{dt} &= -e^{\alpha(p-2)z(\theta_t\omega)} \chi_2 + \alpha z(\theta_t\omega) v + e^{-\alpha z(\theta_t\omega)} \chi_1 \\
&\quad + e^{-\alpha z(\theta_t\omega)} f_2(t, \cdot, e^{\alpha z(\theta_t\omega)} v) + e^{-\alpha z(\theta_t\omega)} g,
\end{aligned} \tag{3.38}
$$

in $L^{p_1}(\tau, \tau + T; V^*) + L^{q_1}(\tau, \tau + T; L^{q_1}(\mathcal{O}))$. Since $v \in L^p(\tau, \tau + T; V) \cap L^q(\tau, \tau + T; L^q(\mathcal{O}))$ and $\frac{dv}{dt} \in L^{p_1}(\tau, \tau + T; V^*) + L^{q_1}(\tau, \tau + T; L^{q_1}(\mathcal{O}))$, as in [28], we find that $v \in C([\tau, \tau + T], L^2(\mathcal{O}))$ and

$$
\frac{1}{2}\frac{d}{dt}\|v\|^2 = (\frac{dv}{dt}, v)_{(V^* + L^{q_1}, V \cap L^q)} \quad \text{for almost all } t \in (\tau, \tau + T). \tag{3.39}
$$

We now prove $v(\tau) = v_\tau$ and $v(\tau+T) = \widetilde{v}$. Let $\phi \in C^1([\tau, \tau+T])$ and $\xi \in V \cap L^q(\mathcal{O})$. Multiplying (3.17) by $\phi\xi$ and then taking the limit as before, we obtain from (3.18) and (3.33) that

$$
\begin{aligned}
&(\widetilde{v}, \xi)\phi(\tau + T) - (v_\tau, \xi)\phi(\tau) \\
&= \int_\tau^{\tau+T} (v, \xi)\phi' dt - \int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)} (\chi_2, \phi\xi)_{(V^*,V)} dt \\
&\quad + \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v, \phi\xi) dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (\chi_1, \phi\xi)_{(L^{q_1}, L^q)} dt \\
&\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (f_2(t, \cdot, e^{\alpha z(\theta_t\omega)} v), \phi\xi) dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (g, \phi\xi) dt.
\end{aligned} \tag{3.40}
$$

On the other hand, by (3.37) we find that the right-hand side of (3.40) is given by $(v(\tau + T), \xi)\phi(\tau + T) - (v(\tau), \xi)\phi(\tau)$. Therefore we obtain

$$
(v(\tau + T), \xi)\phi(\tau + T) - (v(\tau), \xi)\phi(\tau) = (\widetilde{v}, \xi)\phi(\tau + T) - (v_\tau, \xi)\phi(\tau).
$$

Choose $\psi \in C^1([\tau, \tau + T])$ such that $\psi(\tau) = 1$ and $\psi(\tau + T) = 0$. First letting $\phi = \psi$ and then letting $\phi = 1 - \psi$, from the above, we obtain

$$
v(\tau) = v_\tau \quad \text{and} \quad v(\tau + T) = \widetilde{v}. \tag{3.41}
$$

By (3.33) and (3.41) we obtain

$$v_n(\tau+T,\tau,\omega) \to v(\tau+T) \quad \text{weakly in } L^2(\mathcal{O}), \tag{3.42}$$

which implies that

$$\liminf_{n\to\infty} \|v_n(\tau+T,\tau,\omega)\| \ge \|v(\tau+T)\|. \tag{3.43}$$

Next, we prove $\chi_1 = f_1(t,\cdot,v)$. By (3.34) we see that, up to a subsequence,

$$v_n \to v \quad \text{for almost all } (t,x) \in (\tau,\tau+T)\times\mathcal{O}. \tag{3.44}$$

By (3.44) and the continuity of f_1, we obtain

$$f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n) \to f_1(t,x,e^{\alpha z(\theta_t\omega)}v) \tag{3.45}$$

for almost all $(t,x) \in (\tau,\tau+T)\times\mathcal{O}$. By (3.26) and (3.45), from [28] it follows that

$$f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n) \to f_1(t,x,e^{\alpha z(\theta_t\omega)}v) \quad \text{weakly in } L^{q_1}(\tau,\tau+T;L^{q_1}(\mathcal{O})). \tag{3.46}$$

By (3.30) and (3.46) we have

$$\chi_1 = f_1(t,x,e^{\alpha z(\theta_t\omega)}v). \tag{3.47}$$

We now show that $\chi_2 = A(v)$. By (3.4),

$$e^{-2\alpha z(\theta_t\omega)}\psi_1(t,x) - e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n \ge 0.$$

Therefore, from Fatou's lemma it follows that

$$\begin{aligned}
&\liminf_{n\to\infty}\int_\tau^{\tau+T}\int_{\mathcal{O}}\Big(e^{-2\alpha z(\theta_t\omega)}\psi_1(t,x) - e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n\Big)\,dx\,dt\\
&\ge \int_\tau^{\tau+T}\int_{\mathcal{O}}\liminf_{n\to\infty}\Big(e^{-2\alpha z(\theta_t\omega)}\psi_1(t,x) - e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n\Big)\,dx\,dt
\end{aligned}$$

which along with (3.44)-(3.45) shows that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n\\
&\le \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v)v\,.
\end{aligned} \tag{3.48}$$

By (3.17)-(3.18) we find that

$$\begin{aligned}
&\int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)}(A(v_n),v_n)_{(V^*,V)}dt\\
&= \frac12\|v_n(\tau)\|^2 - \frac12\|v_n(\tau+T)\|^2 + \alpha\int_\tau^{\tau+T} z(\theta_t\omega)\|v_n\|^2dt\\
&\quad + \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)}f_1(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n\,dx\,dt\\
&\quad + \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)}f_2(t,x,e^{\alpha z(\theta_t\omega)}v_n)v_n\,dx\,dt\\
&\quad + \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)}g(t,x)v_n\,dx\,dt.
\end{aligned} \tag{3.49}$$

Letting $n \to \infty$ in (3.49), by (3.34), (3.43) and (3.48) we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)} (A(v_n), v_n)_{(V^*,V)} dt \\
&\le \frac{1}{2}\|v(\tau)\|^2 - \frac{1}{2}\|v(\tau+T)\|^2 + \alpha \int_\tau^{\tau+T} z(\theta_t\omega)\|v\|^2 dt \\
&\quad + \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)} f_1(t,x,e^{\alpha z(\theta_t\omega)}v) v\, dx\, dt \\
&\quad + \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)} f_2(t,x,e^{\alpha z(\theta_t\omega)}v) v\, dx\, dt \\
&\quad + \int_\tau^{\tau+T}\int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)} g(t,x) v\, dx\, dt.
\end{aligned}
\tag{3.50}
$$

On the other hand, by (3.38), (3.39) and (3.47) we see that the right-hand side of (3.50) is given by $\int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)}(\chi_2, v)_{(V^*,V)} dt$. Therefore we obtain

$$
\limsup_{n\to\infty} \int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)} (A(v_n), v_n)_{(V^*,V)} dt \le \int_\tau^{\tau+T} e^{\alpha(p-2)z(\theta_t\omega)} (\chi_2, v)_{(V^*,V)} dt.
\tag{3.51}
$$

Since $A : L^p(\tau, \tau+T; V) \to L^{p_1}(\tau, \tau+T; V^*)$ is hemicontinuous and monotone, by (3.29), (3.31) and (3.51) we infer that

$$
\chi_2 = A(v). \tag{3.52}
$$

It follows from (3.37), (3.41), (3.47) and (3.52) that v is a solution of problem (3.13)-(3.15) in the sense of Definition 3.1. By (3.38), (3.39), (3.47) and (3.52) we find that v satisfies the energy equation (3.19).

Step 3: Uniqueness of solutions. Suppose v_1 and v_2 are two solutions of (3.13)-(3.14) with initial data $v_{1,\tau}$ and $v_{2,\tau}$, respectively. Then $\widetilde{v} = v_1 - v_2$ satisfies

$$
\begin{aligned}
&\frac{d\widetilde{v}}{dt} + e^{\alpha(p-2)z(\theta_t\omega)}(A(v_1) - A(v_2)) \\
&= \alpha z(\theta_t\omega)\widetilde{v} + e^{-\alpha z(\theta_t\omega)} \Big(f_1(t,x,e^{\alpha z(\theta_t\omega)}v_1) - f_1(t,x,e^{\alpha z(\theta_t\omega)}v_2) \Big) \\
&\quad + e^{-\alpha z(\theta_t\omega)} \Big(f_2(t,x,e^{\alpha z(\theta_t\omega)}v_1) - f_2(t,x,e^{\alpha z(\theta_t\omega)}v_2) \Big).
\end{aligned}
$$

By (3.6)-(3.7) and the monotonicity of A, we obtain for $t \in [\tau, \tau+T]$,

$$
\frac{1}{2}\frac{d}{dt}\|\widetilde{v}\|^2 \le \alpha z(\theta_t\omega)\|\widetilde{v}\|^2 + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} \psi_4(t,x)|\widetilde{v}|^2 dx + \int_{\mathcal{O}} \psi_5(t,x)|\widetilde{v}|^2 dx \le c_1\|\widetilde{v}\|^2,
$$

where c_1 is a positive constant depending on τ, T and ω. Therefore, for all $t \in [\tau, \tau+T]$, we have

$$
\|v_1(t,\tau,\omega,v_{1,\tau}) - v_2(t,\tau,\omega,v_{2,\tau})\|^2 \le e^{c_1(t-\tau)}\|v_{1,\tau} - v_{2,\tau}\|^2,
$$

which implies the uniqueness and continuous dependence of solutions on initial data in $L^2(\mathcal{O})$.

By (3.33), (3.41) and the uniqueness of solutions, we infer that for every $\omega \in \Omega$, the whole sequence $v_n(\tau+T,\tau,\omega) \to v(\tau+T,\tau,\omega)$ weakly in $L^2(\mathcal{O})$. By a similar argument, one can verify that $v_n(t,\tau,\omega) \to v(t,\tau,\omega)$ weakly in $L^2(\mathcal{O})$ for any $t \ge \tau$ and $\omega \in \Omega$. Then the measurability of $v(t,\tau,\omega)$ follows from that of $v_n(t,\tau,\omega)$. This completes the proof. $\square$

From the proof of Lemma 3.2, we see that the solution operator of problem (3.13)-(3.15) is compact in $L^2(\mathcal{O})$ as stated below.

Lemma 3.3. *If* (3.4)-(3.7) *hold, then for every $\omega \in \Omega$ and $t, \tau \in \mathbb{R}$ with $t > \tau$, the solution operator $v(t,\tau,\omega,\cdot)$: $L^2(\mathcal{O}) \to L^2(\mathcal{O})$ of problem* (3.13)-(3.15) *is compact; that is, for every bounded set B in $L^2(\mathcal{O})$, the image $v(t,\tau,\omega,B)$ is precompact in $L^2(\mathcal{O})$.*

Proof. We argue as in [24]. Suppose $\{v_{0,n}\}_{n=1}^{\infty}$ is an arbitrary sequence in B. Choose a positive number T such that $t \in [\tau, \tau+T]$. As in (3.34), one can show that there exists $v \in L^2((\tau,\tau+T), L^2(\mathcal{O}))$ such that, up to a subsequence,

$$v(\cdot,\tau,\omega,v_{0,n}) \to v \quad \text{in } \ L^2((\tau,\tau+T), L^2(\mathcal{O})).$$

Therefore, there exists a subsequence (which is still denoted by $v(\cdot,\tau,\omega,v_{0,n})$) such that

$$v(s,\tau,\omega,v_{0,n}) \to v(s) \quad \text{in } L^2(\mathcal{O}) \text{ for almost all } s \in (\tau,\tau+T). \tag{3.53}$$

Since $t > \tau$, we may choose $s_0 \in (\tau, t)$ such that (3.53) is fulfilled at $s = s_0$. It follows from the continuity of solutions in initial data that

$$v(t,\tau,\omega,v_{0,n}) = v(t,s_0,\omega,v(s_0,\tau,\omega,v_{0,n}) \to v(t,s_0,\omega,v(s_0)),$$

and hence $\{v(t,\tau,\omega,v_{0,n})\}$ has a convergent subsequence in $L^2(\mathcal{O})$. □

We now define a continuous cocycle for the stochastic equation (3.1). Let $u(t,\tau,\omega,u_\tau) = e^{\alpha z(\theta_t\omega)} v(t,\tau,\omega,v_\tau)$ with $u_\tau = e^{\alpha z(\theta_\tau\omega)} v_\tau$, where v is the solution of problem (3.13)-(3.15). It follows from Lemma 3.2 that u is a solution of problem (3.1)-(3.3) which is continuous in $t \in [\tau,\infty)$ as well as in $u_\tau \in L^2(\mathcal{O})$. Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \to L^2(\mathcal{O})$ be a mapping given by, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in L^2(\mathcal{O})$,

$$\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = e^{\alpha z(\theta_t\omega)} v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau), \tag{3.54}$$

where $v_\tau = e^{-\alpha z(\omega)} u_\tau$.

Then Φ is a continuous cocycle on $L^2(\mathcal{O})$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. We will prove the existence of $\mathcal{D}$-pullback attractors for Φ for an appropriate collection of families of subsets of $L^2(\mathcal{O})$. To define such a collection, we first recall the Poincare inequality:

$$\int_{\mathcal{O}} |\nabla v(x)|^p dx \ge \beta \int_{\mathcal{O}} |v(x)|^p dx, \quad \text{for all } v \in W_0^{1,p}(\mathcal{O}), \tag{3.55}$$

where β is a positive constant depending only on $\mathcal{O}$ and p. Note that for any $p \ge 2$,

$$\int_{\mathcal{O}} |v(x)|^2 dx \le \frac{p-2}{p} |\mathcal{O}| + \frac{2}{p} \int_{\mathcal{O}} |v(x)|^p dx, \tag{3.56}$$

where $|\mathcal{O}|$ stands for the Lebesgue measure of $\mathcal{O}$. By (3.55) and (3.56) we have

$$\int_{\mathcal{O}} |\nabla v(x)|^p dx \ge \frac{1}{2}\beta p \|v\|^2 - \frac{1}{2}\beta(p-2)|\mathcal{O}|, \quad \text{for all } \ v \in W_0^{1,p}(\mathcal{O}). \tag{3.57}$$

Given $r \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$h(r,\omega) = \frac{1}{2}\beta p e^{\alpha(p-2)z(\theta_r\omega)} - 2\alpha z(\theta_r\omega). \tag{3.58}$$

By the ergodic theory, we find that

$$\lim_{s\to-\infty}\frac{1}{s}\int_0^s h(r,\omega)dr = E\big(\frac{1}{2}\beta p e^{\alpha(p-2)z(\omega)} - 2\alpha z(\omega)\big) = \frac{1}{2}\beta p E\big(e^{\alpha(p-2)z(\omega)}\big) > 0. \tag{3.59}$$

We now fix a number δ such that

$$0 < \delta < \frac{1}{2}\beta p E\big(e^{\alpha(p-2)z(\omega)}\big). \tag{3.60}$$

It follows from (3.59) and (3.60) that for every $\varepsilon \in \big(0,\ \frac{1}{4}\beta p E\left(e^{\alpha(p-2)z(\omega)}\right) - \frac{1}{2}\delta\big)$, there exists $s_0 = s_0(\omega,\varepsilon) < 0$ such that for all $s \le s_0$,

$$\int_0^s h(r,\omega)dr < (\delta+\varepsilon)s < \delta s. \tag{3.61}$$

Let $D = \{D(\tau,\omega) : \tau\in\mathbb{R},\omega\in\Omega\}$ be a family of bounded nonempty subsets of $L^2(\mathcal{O})$ with the property: for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{s\to-\infty} e^{\int_0^s h(r,\omega)dr - 2\alpha z(\theta_s\omega)}\|D(\tau+s,\theta_s\omega)\|^2 = 0, \tag{3.62}$$

where h is given by (3.58) and $\|S\| = \sup_{u\in S}\|u\|_{L^2(\mathcal{O})}$ for a subset S of $L^2(\mathcal{O})$. Hereafter, we use $\mathcal{D}$ to denote the collection of all families satisfying (3.62):

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau\in\mathbb{R},\omega\in\Omega\} :\ D \text{ satisfies (3.62)}\}. \tag{3.63}$$

From now on, we assume $\psi_5 \in L^\infty(\mathbb{R}, L^\infty(\mathcal{O}))$ and

$$\int_{-\infty}^0 e^{\delta s}\left(\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1(\mathcal{O})}\right)ds < \infty, \quad \forall \tau\in\mathbb{R}, \tag{3.64}$$

where δ is the fixed number satisfying (3.60).

4. Uniform estimates of solutions

We will derive uniform estimates of solutions for problem (3.13)-(3.15) in this section. These estimates will be used to prove the existence of pullback absorbing sets and the asymptotic compactness of the stochastic equation (3.1).

Lemma 4.1. *Suppose* (3.4)-(3.7) *and* (3.64) *hold. Then for every $\sigma\in\mathbb{R}$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D = \{D(\tau,\omega) : \tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T = T(\tau,\omega,D,\sigma) > 0$ such that for all $t \ge T$, the solution v of problem* (3.13)-(3.15) *satisfies*

$$\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\\
&\le c\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_s\omega)}ds\\
&\quad + c\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)}ds + c\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{\frac{4\alpha(p-1)}{2-p}z(\theta_s\omega)}ds\\
&\quad + 2\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})}ds\\
&\quad + \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr}\|g(s+\tau)\|^2 ds,
\end{aligned}$$

where $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ and c is a positive constant independent of τ, ω, D and α.

Proof. We start with the energy identity (3.19). By (3.4), the nonlinearity f_1 satisfies

$$\begin{aligned} &2e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} f_1(t,x,e^{\alpha z(\theta_t\omega)}v)vdx \\ &\le -2\lambda e^{\alpha(q-2)z(\theta_t\omega)}\int_{\mathcal{O}}|v|^q dx + 2\int_{\mathcal{O}} e^{-2\alpha z(\theta_t\omega)}\psi_1(t,x)dx. \end{aligned} \tag{4.1}$$

By the Young inequality and (3.7) we obtain

$$\begin{aligned} &2e^{-\alpha z(\theta_t\omega)}|\int_{\mathcal{O}} f_2(t,x,e^{\alpha z(\theta_t\omega)}v)vdx| \\ &\le 2\|\psi_5(t)\|_{L^\infty(\mathcal{O})}\|v\|^2 \\ &\le c_1 e^{-2\alpha z(\theta_t\omega)} + \frac{1}{2}\beta e^{\alpha(p-2)z(\theta_t\omega)}\int_{\mathcal{O}}|v|^p dx, \end{aligned} \tag{4.2}$$

where c_1 is a positive number independent of τ,ω and α. Similarly, by the Young inequality, we have

$$\begin{aligned} &2e^{-\alpha z(\theta_t\omega)}|\int_{\mathcal{O}} g(t,x)vdx| \\ &\le \|g(t)\|^2 + e^{-2\alpha z(\theta_t\omega)}\|v\|^2 \\ &\le \|g(t)\|^2 + c_2 e^{\frac{4\alpha(p-1)}{2-p}z(\theta_t\omega)} + \frac{1}{2}\beta e^{\alpha(p-2)z(\theta_t\omega)}\int_{\mathcal{O}}|v|^p dx. \end{aligned} \tag{4.3}$$

By (3.55) and (3.57) we obtain

$$\begin{aligned} 2e^{\alpha(p-2)z(\theta_t\omega)}\int_{\mathcal{O}}|\nabla v|^p dx &\ge \beta e^{\alpha(p-2)z(\theta_t\omega)}\int_{\mathcal{O}}|v|^p dx + \frac{1}{2}\beta p e^{\alpha(p-2)z(\theta_t\omega)}\|v\|^2 \\ &\quad - \frac{1}{2}\beta(p-2)|\mathcal{O}|e^{\alpha(p-2)z(\theta_t\omega)}. \end{aligned} \tag{4.4}$$

It follows from (3.19) and (4.1)-(4.4) that

$$\begin{aligned} \frac{d}{dt}\|v\|^2 + h(t,\omega)\|v\|^2 &\le c_3 e^{\alpha(p-2)z(\theta_t\omega)} + c_3 e^{-2\alpha z(\theta_t\omega)} + c_3 e^{\frac{4\alpha(p-1)}{2-p}z(\theta_t\omega)} \\ &\quad + 2e^{-2\alpha z(\theta_t\omega)}\|\psi_1(t)\|_{L^1(\mathcal{O})} + \|g(t)\|^2, \end{aligned} \tag{4.5}$$

where $h(t,\omega)$ is given by (3.58). Multiplying (4.5) by $e^{\int_0^t h(r,\omega)dr}$ and then solving the inequality, we obtain for every $\tau\in\mathbb{R}$, $t\in\mathbb{R}^+$, $\sigma\ge\tau-t$ and $\omega\in\Omega$,

$$\begin{aligned} &\|v(\sigma,\tau-t,\omega,v_{\tau-t})\|^2 \\ &\le e^{\int_\sigma^{\tau-t} h(r,\omega)dr}\|v_{\tau-t}\|^2 + c_3\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\omega)dr}e^{\alpha(p-2)z(\theta_s\omega)}ds \\ &\quad + c_3\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\omega)dr}e^{-2\alpha z(\theta_s\omega)}ds + c_3\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\omega)dr}e^{\frac{4\alpha(p-1)}{2-p}z(\theta_s\omega)}ds \\ &\quad + 2\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|\psi_1(s)\|_{L^1(\mathcal{O})}ds \\ &\quad + \int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\omega)dr}\|g(s)\|^2 ds. \end{aligned} \tag{4.6}$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.6) we obtain

$$
\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\\
&\le e^{\int_\sigma^{\tau-t} h(r,\theta_{-\tau}\omega)dr}\|v_{\tau-t}\|^2 + c_3\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\theta_{-\tau}\omega)dr} e^{\alpha(p-2)z(\theta_{s-\tau}\omega)}ds\\
&\quad + c_3\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\theta_{-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)}ds + c_3\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\theta_{-\tau}\omega)dr} e^{\frac{4\alpha(p-1)}{2-p}z(\theta_{s-\tau}\omega)}ds\\
&\quad + 2\int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\theta_{-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)}\|\psi_1(s)\|_{L^1(\mathcal{O})}ds\\
&\quad + \int_{\tau-t}^{\sigma} e^{\int_\sigma^s h(r,\theta_{-\tau}\omega)dr}\|g(s)\|^2 ds\\
&\le e^{\int_{\sigma-\tau}^{-t} h(r,\omega)dr}\|v_{\tau-t}\|^2 + c_3\int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_s\omega)}ds\\
&\quad + c_3\int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)}ds + c_3\int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{\frac{4\alpha(p-1)}{2-p}z(\theta_s\omega)}ds\\
&\quad + 2\int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})}ds\\
&\quad + \int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr}\|g(s+\tau)\|^2 ds.
\end{aligned}
\tag{4.7}
$$

Next, we estimate every term on the right-hand side of (4.7). First, by (3.61) we find that for all $s \le s_0$,

$$
\begin{aligned}
\int_{-\infty}^{s_0} e^{\int_{\sigma-\tau}^s h(r,\omega)dr}\|g(s+\tau)\|^2 ds &= e^{\int_{\sigma-\tau}^0 h(r,\omega)dr}\int_{-\infty}^{s_0} e^{\int_0^s h(r,\omega)dr}\|g(s+\tau)\|^2 ds\\
&\le e^{\int_{\sigma-\tau}^0 h(r,\omega)dr}\int_{-\infty}^{s_0} e^{\delta s}\|g(s+\tau)\|^2 ds\\
&\le e^{\int_{\sigma-\tau}^0 h(r,\omega)dr}\int_{-\infty}^{0} e^{\delta s}\|g(s+\tau)\|^2 ds < \infty,
\end{aligned}
$$

where we have used (3.64) for the last integral. Therefore, for $t > 0$, we obtain

$$
\int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr}\|g(s+\tau)\|^2 ds \le \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s h(r,\omega)dr}\|g(s+\tau)\|^2 ds < \infty. \tag{4.8}
$$

Note that (3.11) implies that for $\varepsilon > 0$, there exists $s_1 \le s_0$ such that for all $s \le s_1$,

$$
-2\alpha z(\theta_s\omega) < -\varepsilon s. \tag{4.9}
$$

By (3.61), (4.9) and (3.64) we obtain

$$
\begin{aligned}
&\int_{-\infty}^{s_1} e^{\int_{\sigma-\tau}^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})}ds\\
&= e^{\int_{\sigma-\tau}^0 h(r,\omega)dr}\int_{-\infty}^{s_1} e^{\int_0^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})}ds\\
&\le e^{\int_{\sigma-\tau}^0 h(r,\omega)dr}\int_{-\infty}^{s_1} e^{\delta s}\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})}ds < \infty,
\end{aligned}
$$

and hence

$$\begin{aligned} &\int_{-t}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s} h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)} \|\psi_1(s+\tau)\|_{L^1(\mathcal{O})} ds \\ &\le \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s} h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)} \|\psi_1(s+\tau)\|_{L^1(\mathcal{O})} ds < \infty. \end{aligned} \tag{4.10}$$

By a similar argument, one can verify that the following integrals are also convergent:

$$\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s} h(r,\omega)dr} e^{\alpha(p-2)z(\theta_s\omega)} ds < \infty, \tag{4.11}$$

$$\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s} h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)} ds < \infty, \tag{4.12}$$

$$\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s} h(r,\omega)dr} e^{\frac{4\alpha(p-1)}{2-p} z(\theta_s\omega)} ds < \infty. \tag{4.13}$$

For the first term on the right-hand side of (4.7), since $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega) \in \mathcal{D}$, we have

$$e^{\int_0^{-t} h(r,\omega)dr} \|v_{\tau-t}\|^2 \le e^{\int_0^{-t} h(r,\omega)dr - 2\alpha z(\theta_{-t}\omega)} \|D(\tau-t, \theta_{-t}\omega)\|^2 \to 0,$$

as $t \to \infty$. Therefore, there exists $T = T(\tau, \omega, D, \sigma) > 0$ such that for all $t \ge T$,

$$\begin{aligned} e^{\int_{\sigma-\tau}^{-t} h(r,\omega)dr} \|v_{\tau-t}\|^2 &= e^{\int_{\sigma-\tau}^{0} h(r,\omega)dr} e^{\int_0^{-t} h(r,\omega)dr} \|v_{\tau-t}\|^2 \\ &\le \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s} h(r,\omega)dr} e^{\alpha(p-2)z(\theta_s\omega)} ds. \end{aligned} \tag{4.14}$$

Thus, the lemma follows from (4.7)-(4.14) immediately. □

As a consequence of Lemma 4.1, we have the following estimates.

Lemma 4.2. *Suppose* (3.4)-(3.7) *and* (3.64) *hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \ge T$, the solution v of problem* (3.13)-(3.15) *with $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ satisfies, for all $t \ge T$,*

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \le R(\tau, \omega, \alpha), \tag{4.15}$$

where

$$\begin{aligned} &R(\tau, \omega, \alpha) \\ &= c \int_{-\infty}^{0} e^{\int_0^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_s\omega)} ds \\ &\quad + c \int_{-\infty}^{0} e^{\int_0^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)} ds + c \int_{-\infty}^{0} e^{\int_0^s h(r,\omega)dr} e^{\frac{4\alpha(p-1)}{2-p} z(\theta_s\omega)} ds \\ &\quad + 2 \int_{-\infty}^{0} e^{\int_0^s h(r,\omega)dr} e^{-2\alpha z(\theta_s\omega)} \|\psi_1(s+\tau)\|_{L^1(\mathcal{O})} ds \\ &\quad + \int_{-\infty}^{0} e^{\int_0^s h(r,\omega)dr} \|g(s+\tau)\|^2 ds, \end{aligned} \tag{4.16}$$

with c being a positive constant independent of τ, ω, D and α. Moreover, we have

$$\lim_{t\to\infty} e^{\int_0^{-t} h(r,\omega)dr} R(\tau - t, \theta_{-t}\omega, \alpha) = 0. \tag{4.17}$$

Proof. Note that (4.15) follows from Lemma 4.1 by setting $\sigma = \tau$. We only need to show (4.17). By (4.16) we obtain

$$\begin{aligned}
&R(\tau - t, \theta_{-t}\omega, \alpha)\\
&= c\int_{-\infty}^{0} e^{\int_0^s h(r,\theta_{-t}\omega)dr} e^{\alpha(p-2)z(\theta_{s-t}\omega)} ds\\
&\quad + c\int_{-\infty}^{0} e^{\int_0^s h(r,\theta_{-t}\omega)dr} e^{-2\alpha z(\theta_{s-t}\omega)} ds + c\int_{-\infty}^{0} e^{\int_0^s h(r,\theta_{-t}\omega)dr} e^{\frac{4\alpha(p-1)}{2-p} z(\theta_{s-t}\omega)} ds\\
&\quad + 2\int_{-\infty}^{0} e^{\int_0^s h(r,\theta_{-t}\omega)dr} e^{-2\alpha z(\theta_{s-t}\omega)} \|\psi_1(s+\tau-t)\|_{L^1(\mathcal{O})} ds\\
&\quad + \int_{-\infty}^{0} e^{\int_0^s h(r,\theta_{-t}\omega)dr} \|g(s+\tau-t)\|^2 ds\\
&= c\int_{-\infty}^{-t} e^{\int_{-t}^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_{s}\omega)} ds\\
&\quad + c\int_{-\infty}^{-t} e^{\int_{-t}^s h(r,\omega)dr} e^{-2\alpha z(\theta_{s}\omega)} ds + c\int_{-\infty}^{-t} e^{\int_{-t}^s h(r,\omega)dr} e^{\frac{4\alpha(p-1)}{2-p} z(\theta_{s}\omega)} ds\\
&\quad + 2\int_{-\infty}^{-t} e^{\int_{-t}^s h(r,\omega)dr} e^{-2\alpha z(\theta_{s}\omega)} \|\psi_1(s+\tau)\|_{L^1(\mathcal{O})} ds\\
&\quad + \int_{-\infty}^{-t} e^{\int_{-t}^s h(r,\omega)dr} \|g(s+\tau)\|^2 ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&e^{\int_0^{-t} h(r,\omega)dr} R(\tau - t, \theta_{-t}\omega, \alpha)\\
&= c\int_{-\infty}^{-t} e^{\int_0^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_{s}\omega)} ds + c\int_{-\infty}^{-t} e^{\int_0^s h(r,\omega)dr} e^{-2\alpha z(\theta_{s}\omega)} ds\\
&\quad + c\int_{-\infty}^{-t} e^{\int_0^s h(r,\omega)dr} e^{\frac{4\alpha(p-1)}{2-p} z(\theta_{s}\omega)} ds\\
&\quad + 2\int_{-\infty}^{-t} e^{\int_0^s h(r,\omega)dr} e^{-2\alpha z(\theta_{s}\omega)} \|\psi_1(s+\tau)\|_{L^1(\mathcal{O})} ds\\
&\quad + \int_{-\infty}^{-t} e^{\int_0^s h(r,\omega)dr} \|g(s+\tau)\|^2 ds.
\end{aligned} \tag{4.18}$$

By the convergence of the integral

$$\int_{-\infty}^{0} e^{\int_0^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_{s}\omega)} ds < \infty$$

we obtain

$$\lim_{t\to\infty} \int_{-\infty}^{-t} e^{\int_0^s h(r,\omega)dr} e^{\alpha(p-2)z(\theta_{s}\omega)} ds = 0. \tag{4.19}$$

Similarly, we find that all remaining terms on the right-hand side of (4.18) converge to zero as $t \to \infty$. Therefore, from (4.18)-(4.19), the desired limit (4.17) follows. $\square$

The asymptotic compactness of solutions of equation (3.13) is stated as follows.

Lemma 4.3. *Suppose* (3.4)-(3.7) *and* (3.64) *hold. Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *the sequence* $v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})$ *has a convergent subsequence in* $L^2(\mathcal{O})$ *provided* $t_n \to \infty$ *and* $e^{\alpha z(\theta_{-t_n}\omega)} v_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. Thanks to $t_n \to \infty$ and $e^{\alpha z(\theta_{-t_n}\omega)} v_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, it follows from Lemma 4.1 that there exist $c = c(\tau,\omega) > 0$ and $N = N(\tau,\omega,D) > 0$ such that for all $n \geq N$,

$$\|v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\| \leq c(\tau,\omega). \tag{4.20}$$

Note that

$$v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) = v(\tau, \tau - 1, \theta_{-\tau}\omega, v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})). \tag{4.21}$$

Then the precompactness of $\{v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\}$ follows from (4.20)-(4.21) and Lemma 3.3 immediately. $\square$

5. Existence of Random Attractors

In this section, we prove the existence of $\mathcal{D}$-pullback attractor for the stochastic problem (3.1)-(3.3). We first construct a $\mathcal{D}$-pullback absorbing set for the corresponding cocycle based on the uniform estimates derived in the previous section.

Lemma 5.1. *Suppose* (3.4)-(3.7) *and* (3.64) *hold. Then the continuous cocycle* Φ *associated with problem* (3.1)-(3.3) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K_\alpha = \{K_\alpha(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *which is given by*

$$K_\alpha(\tau,\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \leq e^{2\alpha z(\omega)} R(\tau,\omega,\alpha)\}, \tag{5.1}$$

where $R(\tau,\omega,\alpha)$ *is the number given by* (4.16)

Proof. Replacing ω by $\theta_{-\tau}\omega$ in (3.12) we obtain

$$u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) = e^{\alpha z(\omega)} v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) \quad \text{with } u_{\tau-t} = e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t}. \tag{5.2}$$

Therefore, for every $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, we have $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$. This along with (4.15) shows that there is $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$, the solution v of problem (3.13)-(3.15) satisfies

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \leq R(\tau,\omega,\alpha), \tag{5.3}$$

where $R(\tau,\omega,\alpha)$ is given by (4.16). By (5.2) and (5.3) we obtain, for all $t \geq T$,

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \leq e^{2\alpha z(\omega)} R(\tau,\omega,\alpha). \tag{5.4}$$

By (3.54) and (5.4) we find that, for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) = u(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_\alpha(\tau,\omega), \tag{5.5}$$

where $K_\alpha(\tau,\omega)$ is given by (5.1). Note that

$$e^{\int_0^s h(r,\omega)dr - 2\alpha z(\theta_s\omega)} \|K_\alpha(\tau + s, \theta_s\omega)\|^2 = e^{\int_0^s h(r,\omega)dr)} R(\tau + s, \theta_s\omega, \alpha). \tag{5.6}$$

It follows from (4.17) and (5.6) that

$$\lim_{s \to -\infty} e^{\int_0^s h(r,\omega)dr - 2\alpha z(\theta_s\omega)} \|K_\alpha(\tau + s, \theta_s\omega)\|^2 = 0,$$

and hence by (3.63) we have $K_\alpha = \{K_\alpha(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. By (4.16), it is evident that $R(\tau,\omega,\alpha)$ is measurable in $\omega \in \Omega$ and so is the set-valued function $K_\alpha(\tau,\omega)$. This along with (5.5) concludes the proof. $\square$

Next, we prove the $\mathcal{D}$-pullback asymptotic compactness of solutions of problem (3.1)-(3.3).

Lemma 5.2. *Suppose* (3.4)-(3.7) *and* (3.64) *hold. Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *the sequence* $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ *is precompact in* $L^2(\mathcal{O})$ *provided* $t_n \to \infty$ *and* $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. Let $v_{0,n} = e^{-\alpha z(\theta_{-t_n}\omega)} u_{0,n}$ for $n \in \mathbb{N}$. Since $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, we have $e^{\alpha z(\theta_{-t_n}\omega)} v_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$. By Lemma 4.3 we find that the sequence $v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})$ has a convergent subsequence in $L^2(\mathcal{O})$, and so is the sequence $u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})$ by (5.2). Therefore, by (3.54), the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ is precompact in $L^2(\mathcal{O})$ as desired. □

We are now in a position to present the existence of $\mathcal{D}$-pullback attractors for (3.1)-(3.3).

Theorem 5.3. *Suppose* (3.4)-(3.7) *and* (3.64) *hold. Then the cocycle* Φ *for problem* (3.1)-(3.3) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_\alpha = \{\mathcal{A}_\alpha(\tau, \omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$ *in* $L^2(\mathcal{O})$. *In addition, for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$
\begin{aligned}
\mathcal{A}_\alpha(\tau, \omega) = \Omega(K, \tau, \omega) &= \cup_{B \in \mathcal{D}} \Omega(B, \tau, \omega) \\
&= \{\psi(0, \tau, \omega) : \psi \textit{ is a } \mathcal{D}\textit{-complete orbit of } \Phi\} \\
&= \{\xi(\tau, \omega) : \xi \textit{ is a } \mathcal{D}\textit{-complete quasi-solution of } \Phi\},
\end{aligned}
$$

where K *is a* $\mathcal{D}$*-pullback absorbing set of* Φ.

Proof. This result follows directly from Lemmas 5.1, 5.2 and Proposition 2.4. □

For the periodicity of the $\mathcal{D}$-pullback attractor $\mathcal{A}_\alpha$, we have the following result.

Theorem 5.4. *Let* (3.4)–(3.7) *hold. Suppose that* $f_1(t, x, s)$ *and* $f_2(t, x, s)$ *are* T*-periodic in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{O}$ *and* $s \in \mathbb{R}$. *If* $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$ *and* $\psi_1 \in L^1_{\rm loc}(\mathbb{R}, L^1(\mathcal{O}))$ *are also* T*-periodic, then problem* (3.1)-(3.3) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_\alpha$ *in* $L^2(\mathcal{O})$ *such that* $\mathcal{A}_\alpha(\tau + T, \omega) = \mathcal{A}_\alpha(\tau, \omega)$ *for all* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$.

Proof. Since $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$ and $\psi_1 \in L^1_{\rm loc}(\mathbb{R}, L^1(\mathcal{O}))$ are both T-periodic, one may check that condition (3.64) is fulfilled in this case. Therefore, by Theorem 5.3, problem (3.1)-(3.3) has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\alpha$ in $L^2(\mathcal{O})$. By the T-periodicity of f_1, f_2 and g, we find that the cocycle Φ is also T-periodic; more precisely, for any $u_0 \in L^2(\mathcal{O})$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\begin{aligned}
\Phi(t, \tau + T, \omega, u_0) &= u(t + \tau + T, \tau + T, \theta_{-\tau - T}\omega, u_0) \\
&= u(t + \tau, \tau, \theta_{-\tau}\omega, u_0) \\
&= \Phi(t, \tau, \omega, u_0).
\end{aligned}
$$

On the other hand, by the T-periodicity of g and ψ_1, we obtain from (4.16) that $R(\tau + T, \omega, \alpha) = R(\tau, \omega, \alpha)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, and hence, by (5.1), the $\mathcal{D}$-pullback absorbing set K_α is T-periodic. Thus, the periodicity of $\mathcal{A}_\alpha$ follows from Proposition 2.4 immediately. □

6. Upper semicontinuity of attractors

In this section, we establish the upper semicontinuity of random attractors of problem (3.1)-(3.3) as $\alpha \to 0$. Hereafter, we assume δ is a fixed number such that

$$0 < \delta < \frac{1}{2}\beta p. \tag{6.1}$$

Note that $\lim_{\alpha\to 0} E(e^{\pm\alpha(p-2)|z(\omega)|}) = 1$ and $\lim_{\alpha\to 0} E(\alpha|z(\omega)|) = 0$. By (6.1) we find that there exists $\alpha_0 \in (0,1)$ such that for all $\alpha \in [0,\alpha_0]$,

$$\delta < \frac{1}{2}\beta p E(e^{\alpha(p-2)z(\omega)}), \quad \delta < \frac{1}{2}\beta p E(e^{-\alpha(p-2)|z(\omega)|}) - 2\alpha E(|z(\omega)|). \tag{6.2}$$

Therefore, condition (3.60) is fulfilled for any $\alpha \in [0,\alpha_0]$. Throughout this section, we always assume $\alpha \in [0,\alpha_0]$ and write the the cocycle associated with (3.1)-(3.3) as Φ_α. By (3.4)-(3.7) and (6.2), it follows from Theorem 5.3 that for every $\alpha \in (0,\alpha_0]$, Φ_α has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\alpha \in \mathcal{D}$ in $L^2(\mathcal{O})$. Moreover, the family $K_\alpha = \{K_\alpha(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ given by (5.1) is a $\mathcal{D}$-pullback absorbing set of Φ_α. Given $r \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$\widetilde{h}(r,\omega) = \frac{1}{2}\beta p e^{-\alpha_0(p-2)|z(\theta_r\omega)|} - 2\alpha_0|z(\theta_r\omega)|. \tag{6.3}$$

As in (4.16), by (6.2) one can verify the following integrals are well defined:

$$\begin{aligned}
L(\tau,\omega) = c\int_{-\infty}^0 & e^{\int_0^s \widetilde{h}(r,\omega)dr} e^{\alpha(p-2)|z(\theta_s\omega)|} ds \\
&+ c\int_{-\infty}^0 e^{\int_0^s \widetilde{h}(r,\omega)dr} e^{2\alpha|z(\theta_s\omega)|} ds \\
&+ c\int_{-\infty}^0 e^{\int_0^s \widetilde{h}(r,\omega)dr} e^{\frac{4\alpha(p-1)}{p-2}|z(\theta_s\omega)|} ds \\
&+ 2\int_{-\infty}^0 e^{\int_0^s \widetilde{h}(r,\omega)dr} e^{2\alpha|z(\theta_s\omega)|}\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})} ds \\
&+ \int_{-\infty}^0 e^{\int_0^s \widetilde{h}(r,\omega)dr}\|g(s+\tau)\|^2 ds,
\end{aligned} \tag{6.4}$$

where c is the same constant as in (4.16). By (3.58) we find that $h(r,\omega) \ge \widetilde{h}(r,\omega)$ for all $r \in \mathbb{R}$, $\omega \in \Omega$ and $\alpha \in [0,\alpha_0]$. Therefore, by (4.16) we have $R(\tau,\omega,\alpha) \le L(\tau,\omega)$ for all $\alpha \in [0,\alpha_0]$, which along with (5.1) implies that for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\cup_{0<\alpha\le\alpha_0} \mathcal{A}_\alpha(\tau,\omega) \subseteq \cup_{0<\alpha\le\alpha_0} K_\alpha(\tau,\omega) \subseteq K(\tau,\omega), \tag{6.5}$$

where $K(\tau,\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \le e^{2\alpha_0|z(\omega)|}L(\tau,\omega)\}$.

We now consider (3.1) with $\alpha = 0$. In this case we have

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f_1(t,x,u) + f_2(t,x,u) + g(t,x), \quad t > \tau,\ x \in \mathcal{O}, \tag{6.6}$$

which is supplemented with

$$u|_{\partial\mathcal{O}} = 0, \quad u(\tau,\cdot) = u_\tau. \tag{6.7}$$

We will use Φ_0 to denote the cocycle generated by (6.6)-(6.7) in $L^2(\mathcal{O})$. Let $\mathcal{D}_0$ be the collection given by

$$\mathcal{D}_0 = \{D = \{D(\tau) \subseteq L^2(\mathcal{O}) : \tau \in \mathbb{R}\} : \lim_{s\to-\infty} e^{\frac{1}{2}\beta ps}\|D(\tau+s)\|^2 = 0,\ \forall \tau \in \mathbb{R}\}.$$

It is evident that the results in the previous sections also hold true for $\alpha = 0$. Particularly, Φ_0 has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau), \tau \in \mathbb{R}\} \in \mathcal{D}_0$ in $L^2(\mathcal{O})$ and has a $\mathcal{D}_0$-pullback absorbing set $K_0 = \{K_0(\tau) : \tau \in \mathbb{R}\}$ with $K_0(\tau)$ given by

$$K_0(\tau) = \{u \in L^2(\mathcal{O}) : \ \|u\|^2 \le R_0(\tau)\}, \tag{6.8}$$

where $R_0(\tau)$ is defined by

$$R_0(\tau) = \frac{6c}{\beta p} + \int_{-\infty}^0 e^{\frac{1}{2}\beta p s}(\|g(s+\tau)\|^2 + 2\|\psi_1(s+\tau)\|_{L^1(\mathcal{O})})ds, \tag{6.9}$$

and c is as in (4.16). By (4.16), (5.1) and (6.8)-(6.9) we obtain for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\limsup_{\alpha \to 0} \|K_\alpha(\tau, \omega)\| \le \|K_0(\tau)\|. \tag{6.10}$$

Next, we establish the convergence of solutions of problem (3.1)-(3.3) as $\alpha \to 0$, for which the following condition is needed: there exists $\psi_6 \in L^\infty_{\rm loc}(\mathbb{R}, L^\infty(\mathcal{O}))$ such that for all $t, s \in \mathbb{R}$ and $x \in \mathcal{O}$,

$$|\frac{\partial f_1}{\partial s}(t, x, s)| \le \psi_6(t, x)\left(1 + |s|^{q-2}\right). \tag{6.11}$$

Lemma 6.1. *Let (3.4)-(3.7) and (6.11) hold. If u_α and u are the solutions of (3.1)-(3.3) and (6.6)-(6.7) with initial data $u_{\alpha,\tau}$ and u_τ, respectively, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in [0, 1]$, there exists $\alpha_1 = \alpha_1(\tau, \omega, T, \varepsilon) > 0$ such that for all $\alpha \le \alpha_1$ and $t \in [\tau, \tau + T]$,*

$$\begin{aligned} &\|u_\alpha(t, \tau, \omega, u_{\alpha,\tau}) - u(t, \tau, u_\tau)\|^2 \\ &\le c_1 e^{c_2(t-\tau)}\|u_{\alpha,\tau} - u_\tau\|^2 \\ &\quad + c_1\varepsilon e^{c_2(t-\tau)}\Big(1 + \|u_\tau\|^2 + \|u_{\alpha,\tau}\|^2 + \int_\tau^t (\|\psi_1(s)\|_{L^1(\mathcal{O})} + \|g(s)\|^2)ds\Big), \end{aligned} \tag{6.12}$$

where c_1 and c_2 are positive constants independent of ε and α.

Proof. Let v_α be the solution of (3.13)-(3.15) and $\xi = v_\alpha - u$. By (3.13) and (6.6) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \int_{\mathcal{O}} \Big(e^{\alpha(p-2)z(\theta_t\omega)}|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u\Big)\nabla\xi\, dx \\ &= \alpha z(\theta_t\omega)\|\xi\|^2 + \alpha z(\theta_t\omega)(u, \xi) \\ &\quad + \int_{\mathcal{O}} \big(e^{-\alpha z(\theta_t\omega)} f_1\big(t, x, e^{\alpha z(\theta_t\omega)} v_\alpha\big) - f_1(t, x, u)\big)\xi\, dx \\ &\quad + \int_{\mathcal{O}} \big(e^{-\alpha z(\theta_t\omega)} f_2(t, x, e^{\alpha z(\theta_t\omega)} v_\alpha) - f_2(t, x, u)\big)\xi\, dx \\ &\quad + \big(e^{-\alpha z(\theta_t\omega)} - 1\big)\int_{\mathcal{O}} g(t, x)\xi\, dx. \end{aligned} \tag{6.13}$$

For the second term on the left-hand side of (6.13) we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \Big(e^{\alpha(p-2)z(\theta_t\omega)} |\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u \Big) \nabla\xi \, dx \\
&= \int_{\mathcal{O}} e^{\alpha(p-2)z(\theta_t\omega)} \big(|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u \big) \nabla\xi \, dx \\
&\quad + \int_{\mathcal{O}} \big(e^{\alpha(p-2)z(\theta_t\omega)} - 1 \big) |\nabla u|^{p-2}\nabla u \cdot \nabla\xi \, dx.
\end{aligned}
\tag{6.14}
$$

From [16], we know that there is a positive number γ such that

$$
\big(|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u \big) \cdot (\nabla v_\alpha - \nabla u) \ge \gamma |\nabla v_\alpha - \nabla u|^p. \tag{6.15}
$$

On the other hand, since $z(\theta_t\omega)$ is continuous in $t \in \mathbb{R}$, by Young's inequality we find that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in [0,1)$, there exists $\alpha_2 = \alpha_2(\tau,\omega,T,\varepsilon) > 0$ such that for all $\alpha \in [0,\alpha_2]$ and $t \in [\tau, \tau+T]$,

$$
\begin{aligned}
&| \int_{\mathcal{O}} \big(e^{\alpha(p-2)z(\theta_t\omega)} - 1 \big) |\nabla u|^{p-2}\nabla u \cdot \nabla\xi \, dx | \\
&\le \frac{1}{2}\gamma e^{\alpha(p-2)z(\theta_t\omega)} \int_{\mathcal{O}} |\nabla\xi|^p dx + \varepsilon \int_{\mathcal{O}} |\nabla u|^p dx.
\end{aligned}
\tag{6.16}
$$

It follows from (6.14)-(6.16) that for all $\alpha \in [0,\alpha_2]$ and $t \in [\tau, \tau+T]$,

$$
\begin{aligned}
&\int_{\mathcal{O}} \Big(e^{\alpha(p-2)z(\theta_t\omega)} |\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u \Big) \nabla\xi \, dx \\
&\ge \frac{1}{2}\gamma e^{\alpha(p-2)z(\theta_t\omega)} \int_{\mathcal{O}} |\nabla\xi|^p dx - \varepsilon \int_{\mathcal{O}} |\nabla u|^p dx.
\end{aligned}
\tag{6.17}
$$

For the nonlinearity f_1 in (6.13), by (3.5)-(3.6) and (6.11), we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \Big(e^{-\alpha z(\theta_t\omega)} f_1\big(t,x,e^{\alpha z(\theta_t\omega)} v_\alpha\big) - f_1(t,x,u) \Big) \xi \, dx \\
&= \int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)} \big(f_1(t,x,e^{\alpha z(\theta_t\omega)} v_\alpha) - f_1(t,x,e^{\alpha z(\theta_t\omega)} u) \big) \xi \, dx \\
&\quad + \int_{\mathcal{O}} \big(e^{-\alpha z(\theta_t\omega)} - 1 \big) f_1(t,x,e^{\alpha z(\theta_t\omega)} u) \xi \, dx \\
&\quad + \int_{\mathcal{O}} \big(f_1(t,x,e^{\alpha z(\theta_t\omega)} u) - f_1(t,x,u) \big) \xi \, dx \\
&= \int_{\mathcal{O}} \xi^2 \frac{\partial f_1}{\partial s}(t,x,s) dx + \big(e^{-\alpha z(\theta_t\omega)} - 1 \big) \int_{\mathcal{O}} f_1(t,x,e^{\alpha z(\theta_t\omega)} u) \xi \, dx \\
&\quad + \big(e^{\alpha z(\theta_t\omega)} - 1 \big) \int_{\mathcal{O}} \xi u \frac{\partial f_1}{\partial s}(t,x,s) dx \\
&\le \int_{\mathcal{O}} \xi^2 \psi_4(t,x) dx + c|e^{-\alpha z(\theta_t\omega)} - 1| \int_{\mathcal{O}} \big(e^{\alpha(q-1)z(\theta_t\omega)} |u|^{q-1}|\xi| + \psi_3(t,x)|\xi| \big) dx \\
&\quad + c|e^{\alpha z(\theta_t\omega)} - 1| \int_{\mathcal{O}} \psi_6(t,x) \big(e^{\alpha(q-2)z(\theta_t\omega)} |u|^{q-1}|\xi| + |u|^{q-1}|\xi| + |u||\xi| \big) dx,
\end{aligned}
$$

from which we find that there is $\alpha_3 = \alpha_3(\tau, \omega, T, \varepsilon) \le \alpha_2$ such that for all $\alpha \in [0, \alpha_3]$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&\int_{\mathcal{O}} \Big(e^{-\alpha z(\theta_t \omega)} f_1\big(t, x, e^{\alpha z(\theta_t \omega)} v_\alpha\big) - f_1(t, x, u) \Big) \xi \, dx \\
&\le c\|\xi\|^2 + c\varepsilon + c\varepsilon \int_{\mathcal{O}} (|u|^q + |v_\alpha|^q) dx.
\end{aligned} \tag{6.18}
$$

Similarly, by (3.7), one can verify that there exists $\alpha_4 = \alpha_4(\tau, \omega, T, \varepsilon) \le \alpha_3$ such that for all $\alpha \in [0, \alpha_4]$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&\int_{\mathcal{O}} \Big(e^{-\alpha z(\theta_t \omega)} f_2\big(t, x, e^{\alpha z(\theta_t \omega)} v_\alpha\big) - f_2(t, x, u) \Big) \xi \, dx \\
&\le c\|\xi\|^2 + c\varepsilon + c\varepsilon \int_{\mathcal{O}} |u|^q dx,
\end{aligned} \tag{6.19}
$$

$$
\big(e^{-\alpha z(\theta_t \omega)} - 1\big) \int_{\mathcal{O}} g(t, x) \xi \, dx \le \varepsilon \|\xi\|^2 + \varepsilon \|g(t)\|^2. \tag{6.20}
$$

By (6.13), (6.17) and (6.18)-(6.20) we obtain for all $\alpha \in [0, \alpha_4]$ and $t \in [\tau, \tau + T]$,

$$
\frac{d}{dt}\|\xi\|^2 \le c_1 \|\xi\|^2 + c_2 \varepsilon \left(1 + \|\nabla u\|_p^p + \|u\|_q^q + \|v_\alpha\|_q^q + \|g(t)\|^2\right). \tag{6.21}
$$

It follows from (6.21) that for all $\alpha \in [0, \alpha_4]$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
\|\xi(t)\|^2 &\le e^{c_1(t-\tau)} \|\xi(\tau)\|^2 + c_2 \varepsilon e^{c_1(t-\tau)} \\
&\quad \times \int_\tau^t \Big(1 + \|v_\alpha(s, \tau, \omega, v_{\alpha,\tau})\|_q^q + \|u\|_q^q + \|\nabla u\|_p^p + \|g(s)\|^2\Big) ds.
\end{aligned} \tag{6.22}
$$

By (3.4) and (3.7) we obtain from (3.19) that, for all $\alpha \in [0, 1]$,

$$
\begin{aligned}
&\frac{d}{dt}\|v_\alpha(t)\|^2 + 2e^{\alpha(p-2)z(\theta_t \omega)} \|\nabla v_\alpha(t)\|_p^p + 2\lambda e^{\alpha(q-2)z(\theta_t \omega)} \|v_\alpha(t)\|_q^q \\
&\le c_3 \|v_\alpha\|^2 + c_4 (\|\psi_1(t)\|_{L^1(\mathcal{O})} + \|g(t)\|^2).
\end{aligned}
$$

Solving this inequality, we obtain that for all $\alpha \in [0, 1]$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&\|v_\alpha(t, \tau, \omega, v_{\alpha,\tau})\|^2 + 2 \int_\tau^t e^{c_3(t-s)} e^{\alpha(p-2)z(\theta_s \omega)} \|\nabla v_\alpha(s)\|_p^p ds \\
&+ 2\lambda \int_\tau^t e^{c_3(t-s)} e^{\alpha(q-2)z(\theta_s \omega)} \|v_\alpha(s)\|_q^q ds \\
&\le e^{c_3(t-\tau)} \|v_{\alpha,\tau}\|^2 + c_4 \int_\tau^t e^{c_3(t-s)} (\|\psi_1(s)\|_{L^1(\mathcal{O})} + \|g(s)\|^2) ds.
\end{aligned} \tag{6.23}
$$

By (6.23) we have, for all $\alpha \in [0, 1]$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&\|v_\alpha(t, \tau, \omega, v_{\alpha,\tau})\|^2 + \int_\tau^t \big(\|\nabla v_\alpha(s, \tau, \omega, v_{\alpha,\tau})\|_p^p + \|v_\alpha(s, \tau, \omega, v_{\alpha,\tau})\|_q^q\big) ds \\
&\le c_5 e^{c_3(t-\tau)} \|v_{\alpha,\tau}\|^2 + c_5 e^{c_3(t-\tau)} \int_\tau^t (\|\psi_1(s)\|_{L^1(\mathcal{O})} + \|g(s)\|^2) ds.
\end{aligned} \tag{6.24}
$$

Note that (6.24) is also valid for $\alpha = 0$, and hence we have, for all $t \in [\tau, \tau + T]$,

$$\begin{aligned} &\int_\tau^t \left(\|\nabla u(s,\tau,u_\tau)\|_p^p + \|u(s,\tau,u_\tau)\|_q^q\right) ds \\ &\le c_5 e^{c_3(t-\tau)}\|u_\tau\|^2 + c_5 e^{c_3(t-\tau)} \int_\tau^t (\|\psi_1(s)\|_{L^1(\mathcal{O})} + \|g(s)\|^2) ds. \end{aligned} \tag{6.25}$$

By (6.22) and (6.24)-(6.25) we obtain for all $\alpha \in [0, \alpha_4]$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned} &\|v_\alpha(t,\tau,\omega,v_{\alpha,\tau}) - u(t,\tau,\omega)\|^2 \\ &\le e^{c_1(t-\tau)}\|v_{\alpha,\tau} - u_\tau\|^2 \\ &\quad + c_6 \varepsilon e^{c_7(t-\tau)}\Big(1 + \|u_\tau\|^2 + \|v_{\alpha,\tau}\|^2 + \int_\tau^t (\|\psi_1(s)\|_{L^1(\mathcal{O})} + \|g(s)\|^2) ds\Big). \end{aligned} \tag{6.26}$$

Note that

$$\|u_\alpha(t,\tau,\omega,u_{\alpha,\tau}) - v_\alpha(t,\tau,\omega,v_{\alpha,\tau})\| = |e^{\alpha z(\theta_t\omega)} - 1|\|v_\alpha(t,\tau,\omega,v_{\alpha,\tau})\|. \tag{6.27}$$

Then (6.24) and (6.26)-(6.27) conclude the proof. □

Lemma 6.2. *Suppose* (3.4)-(3.7), (3.64) *and* (6.11) *hold. Let $\tau \in \mathbb{R}$ and $\omega \in \Omega$ be fixed. If $\alpha_n \to 0$ and $u_n \in \mathcal{A}_{\alpha_n}(\tau,\omega)$, then the sequence $\{u_n\}_{n=1}^\infty$ has a convergent subsequence in $L^2(\mathcal{O})$.*

Proof. By $u_n \in \mathcal{A}_{\alpha_n}(\tau,\omega)$ and the invariance of $\mathcal{A}_{\alpha_n}$ we find that there exists $\widetilde{u}_n \in \mathcal{A}_{\alpha_n}(\tau - 1, \theta_{-1}\omega)$ such that

$$u_n = u_{\alpha_n}(\tau, \tau-1, \theta_{-\tau}\omega, \widetilde{u}_n). \tag{6.28}$$

Let v_{α_n} be the solution of (3.13) with α replaced by α_n. Then we have

$$u_{\alpha_n}(\tau,\tau-1,\theta_{-\tau}\omega,\widetilde{u}_n) = e^{\alpha_n z(\omega)} v_{\alpha_n}(\tau,\tau-1,\theta_{-\tau}\omega,\widetilde{v}_n) \quad \text{with } \widetilde{v}_n = e^{-\alpha_n z(\theta_{-1}\omega)}\widetilde{u}_n. \tag{6.29}$$

Since $\alpha_n \to 0$, by (6.5), there exist $c = c(\tau,\omega) > 0$ and $N = N(\tau,\omega) \ge 1$ such that $\|\widetilde{u}_n\| \le c$ for all $n \ge N$, which along with (6.29) shows that $\{\widetilde{v}_n\}_{n=1}^\infty$ is bounded in $L^2(\mathcal{O})$. Therefore, as in (3.53), one can prove that there exists $\bar{v} \in L^2((\tau-1,\tau), L^2(\mathcal{O}))$ such that, up to a subsequence,

$$v_{\alpha_n}(s,\tau-1,\theta_{-\tau}\omega,\widetilde{v}_n) \to \bar{v}(s) \quad \text{in } L^2(\mathcal{O}) \text{ for almost all } s \in (\tau-1,\tau). \tag{6.30}$$

By (3.12) and (6.30) we obtain

$$u_{\alpha_n}(s,\tau-1,\theta_{-\tau}\omega,\widetilde{u}_n) \to \bar{v}(s) \quad \text{in } L^2(\mathcal{O}) \text{ for almost all } s \in (\tau-1,\tau). \tag{6.31}$$

Since $\alpha_n \to 0$, by (6.31) and Lemma 6.1 we obtain, for almost all $s \in (\tau-1,\tau)$,

$$u_{\alpha_n}(\tau,s,\theta_{-\tau}\omega,u_{\alpha_n}(s,\tau-1,\theta_{-\tau}\omega,\widetilde{u}_n)) \to u(\tau,s,\theta_{-\tau}\omega,\bar{v}(s)) \quad \text{in } L^2(\mathcal{O}) \tag{6.32}$$

where u is the solution of (6.6). By the cocycle property, we find that the left-hand side of (6.32) is the same as $u_{\alpha_n}(\tau,\tau-1,\theta_{-\tau}\omega,\widetilde{u}_n)$. Thus, by (6.28) and (6.32) we obtain, for almost all $s \in (\tau-1,\tau)$,

$$u_n = u_{\alpha_n}(\tau,\tau-1,\theta_{-\tau}\omega,\widetilde{u}_n) \to u(\tau,s,\theta_{-\tau}\omega,\bar{v}(s)) \quad \text{in } L^2(\mathcal{O}),$$

which completes the proof. □

We are now ready to present the upper semicontinuity of random attractors as the intensity of noise approaches zero.

Theorem 6.3. *Suppose* (3.4)-(3.7), (3.64) *and* (6.11) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\alpha \to 0} \operatorname{dist}_{L^2(\mathcal{O})}(\mathcal{A}_\alpha(\tau, \omega), \mathcal{A}_0(\tau)) = 0. \tag{6.33}$$

Proof. Given a sequence $\alpha_n \to 0$ and $u_{0,n} \to u_0$ in $L^2(\mathcal{O})$, it follows from Lemma 6.1 that, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi_{\alpha_n}(t, \tau, \omega, u_{0,n}) \to \Phi_0(t, \tau, u_0) \quad \text{in } L^2(\mathcal{O}). \tag{6.34}$$

By (6.10), (6.34) and Lemma 6.2, we find that all conditions of Theorem 3.2 in [36] are satisfied, and thus (6.33) follows immediately. □

Remark 6.4. In this paper, we discuss the random attractors of parabolic equation of type (3.1) driven by linear multiplicative noise and non-autonomous deterministic forcing g. It is interesting to consider the case when g is replaced by an additive white noise. The attractors for such a system have been investigated recently in [23]. Since the main objective of this paper is to deal with the nonlinearity f_1 with polynomial growth of any order, we do not consider the additive noise here for the sake of simplicity, and leave this case for future investigation.

References

[1] A. Adili, B. Wang; Random attractors for stochastic FitzHugh-Nagumo systems driven by deterministic non-autonomous forcing, *Continuous and Discrete Dynamical Systems Series B*, **18** (2013), 643-666.

[2] L. Arnold; *Random Dynamical Systems*, Springer-Verlag, 1998.

[3] P. W. Bates, H. Lisei, K. Lu; Attractors for stochastic lattice dynamical systems, *Stoch. Dyn.*, **6** (2006), 1-21.

[4] P. W. Bates, K. Lu, B. Wang; Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **246** (2009), 845-869.

[5] W. J. Beyn, B. Gess, P. Lescot, M. Röckner; The global random attractor for a class of stochastic porous media equations, *Comm. Partial Differential Equations*, **36** (2011), 446-469.

[6] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss, J. Valero; Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete Continuous Dynamical Systems*, **21** (2008), 415-443.

[7] T. Caraballo, J. Real, I. D. Chueshov; Pullback attractors for stochastic heat equations in materials with memory, *Discrete Continuous Dynamical Systems B*, **9** (2008), 525-539.

[8] T. Caraballo, J. A. Langa; On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, **10** (2003), 491-513.

[9] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss, J. Valero; Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions, *Discrete Contin. Dyn. Syst. Ser. B*, **14** (2010), 439-455.

[10] T. Caraballo, M. J. Garrido-Atienza, T. Taniguchi; The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.*, **74** (2011), 3671-3684.

[11] T. Caraballo, J. A. Langa, V.S. Melnik, J. Valero; Pullback attractors for nonautonomous and stochastic multivalued dynamical systems, *Set-Valued Analysis*, **11** (2003), 153-201.

[12] I. Chueshov, M. Scheutzow; On the structure of attractors and invariant measures for a class of monotone random systems, *Dynamical Systems*, **19** (2004), 127-144.

[13] I. Chueshow; *Monotone Random Systems - Theory and Applications*, Lecture Notes in Mathematics 1779, Springer, Berlin, 2001.

[14] H. Crauel, A. Debussche, F. Flandoli; Random attractors, *J. Dyn. Diff. Eqns.*, **9** (1997), 307-341.

[15] H. Crauel, F. Flandoli; Attractors for random dynamical systems, *Probab. Th. Re. Fields*, **100** (1994), 365-393.

[16] E. DiBennedeto; *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.
[17] J. Duan, B. Schmalfuss; The 3D quasigeostrophic fluid dynamics under random forcing on boundary, *Comm. Math. Sci.*, **1** (2003), 133-151.
[18] F. Flandoli, B. Schmalfuss; Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Stoch. Rep.*, **59** (1996), 21-45.
[19] M. J. Garrido-Atienza, B. Schmalfuss, Ergodicity of the infinite dimensional fractional Brownian motion, *J. Dynam. Differential Equations*, **23** (2011), 671-681.
[20] M. J. Garrido-Atienza, A. Ogrowsky, B. Schmalfuss; Random differential equations with random delays, *Stoch. Dyn.*, **11** (2011), 369-388.
[21] M. J. Garrido-Atienza, B. Maslowski, B. Schmalfuss; Random attractors for stochastic equations driven by a fractional Brownian motion, *International J. Bifurcation and Chaos*, **20** (2010), 2761-2782.
[22] B. Gess, W. Liu, M. Rockner; Random attractors for a class of stochastic partial differential equations driven by general additive noise, *J. Differential Equations*, **251** (2011), 1225-1253.
[23] B. Gess; Random attractors for degenerate stochastic partial differential equations, *J. Dynamics and Differential Equations*, **25** (2013), 121-157.
[24] B. Gess; Random attractors for singular stochastic evolution equations, *J. Differential Equations*, **255** (2013), 524-559.
[25] J. Huang, W. Shen; Pullback attractors for nonautonomous and random parabolic equations on non-smooth domains, *Discrete and Continuous Dynamical Systems*, **24** (2009), 855-882.
[26] P. E. Kloeden, J. A. Langa; Flattening, squeezing and the existence of random attractors, *Proc. Royal Soc. London Serie A.*, **463** (2007), 163-181.
[27] P.E. Kloeden, M. Rasmussen, *Nonautonomous Dynamical Systems*, Mathematical Surveys and Monographs, Vol. 176, Amer. Math. Soc., Providence, 2011.
[28] J. L. Lions; *Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires*, Dunod, Paris, 1969.
[29] Y. Lv, W. Wang, Limiting dynamics for stochastic wave equations, *J. Differential Equations*, **244** (2008), 1-23.
[30] B. Schmalfuss; Backward cocycles and attractors of stochastic differential equations, *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*, 185-192, Dresden, 1992.
[31] Z. Shen, S. Zhou, W. Shen; One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation, *J. Differential Equations*, **248** (2010), 1432-1457.
[32] R. E. Showalter; *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Providence, 1997.
[33] B. Wang; Random Attractors for the Stochastic Benjamin-Bona-Mahony Equation on Unbounded Domains, *J. Differential Equations*, **246** (2009), 2506-2537.
[34] B. Wang; Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Transactions of American Mathematical Society*, **363** (2011), 3639-3663.
[35] B. Wang; Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012),1544-1583.
[36] B. Wang; Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, arXiv:1205.4658v1, 2012.
[37] B. Wang; Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete and Continuous Dynamical Systems Series A*, **34** (2014), 269-300.
[38] B. Wang; Existence, stability and bifurcation of random complete and periodic solutions of stochastic parabolic equations, arXiv:1304.4884v1, 2013.

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