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ROBUST STABILITY OF PATTERNED LINEAR SYSTEMS

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Abstract. For a Hurwitz stable matrix $A \in \mathbb{R}^{n\times n}$, we calculate the real structured radius of stability for A with a perturbation $P = B\Delta(t)C$, where A, B, C, $\Delta(t)$ form a patterned quadruple of matrices; i.e., they are polynomials of a common matrix of simple structure $M \in \mathbb{R}^{n\times n}$.

1. Introduction

In the previous decades there has been a considerable of interest in the determination of the radius of stability for perturbed systems. In a few words the radius of stability for a nominal stable system is the norm of the smallest destabilizing perturbation. The concepts of complex and real stability radius for different classes of perturbations was introduced and analyzed in [7, 8]. The problem of the determination of the complex stability radii for several classes of perturbations has been studied in [9], and a formula for the calculation of the real stability radius for time-invariant structured linear perturbations has been given in [11]. In [10] the formulation of the problem of stability radius in great generality is established and fundamental results for the calculation of the different radii are discussed.

The case of the real time-varying perturbations is more difficult and the available results provide formulae for the stability radii which are not practical due to computational issues. Some authors have considered particular classes of perturbed systems for which the computation of the real time-varying stability radius can be efficiently solved. In [12] and in [2] recent results for positive systems with structured perturbations proving that for positive systems the complex stability radius and the positive stability radius coincide. In [1], using a metric similar to Frobenius's norm for matrices, lower estimates for the real structural stability radius is proposed.

The aim of this work is the determination of the real time-varying stability radius for a new class of systems, the patterned linear systems. Patterned systems were introduced in [4, 5, 6] as a generalization of the well known class of circulant systems. The importance and possible applications of this class of systems as well as their observability, controllability and stabilization of patterned systems are studied.

In what follows we define linear patterned systems and adjust the definition of the real time-invariant and time-varying stability radii of such systems.

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Let $M \in \mathbb{R}^{n\times n}$ be an arbitrary matrix, and let m be the degree of its minimal polynomial. For arbitrary polynomials the evaluation at the matrix M can be expressed by the evaluation of a polynomial of degree at most $m-1$. So for M we define the set of matrices:

$$\mathcal{F}(M) := \left\{\delta_0 I + \delta_1 M + \dots + \delta_{m-1}M^{m-1} : (\delta_0, \delta_1, \dots, \delta_{m-1}) \in \mathbb{R}^m\right\}. \tag{1.1}$$

As in [4], we will call a matrix $T \in \mathcal{F}(M)$ an M-patterned matrix. It is easily shown that

$$T, R \in \mathcal{F}(M), \alpha, \beta \in \mathbb{R} \Rightarrow \alpha T + \beta R \in \mathcal{F}(M), \tag{1.2}$$

$$T, R \in \mathcal{F}(M) \Rightarrow TR = RT. \tag{1.3}$$

From (1.2) we have that $\mathcal{F}(M)$ is a real linear space composed by matrices determined by the real m-tuples $(\delta_0, \delta_1, \dots, \delta_{m-1}) \in \mathbb{R}^m$. Note that two m-tuples correspond to the same element of $\mathcal{F}(M)$ if and only if their difference is a real multiple of the vector of coefficients of the minimal polynomial of the matrix M. So if we denote by $\mathcal{V}$ the one-dimensional subspace of $\mathbb{R}^m$ generated by the m-vector whose components are, in order, the coefficients of the minimal polynomial of M, then denoting by $\|.\|$ the 2-norm in the space $\mathbb{R}^m$ in the quotient space $\mathbb{R}^m/\mathcal{V}$ we have the quotient norm

$$\|(\delta)\| := \inf\{\|\delta + b\|, b \in \mathcal{V}\},$$

where (δ) is the equivalence class of δ in the quotient space $\mathbb{R}^m/\mathcal{V}$. Thus we define the measure of the disturbance $\Delta \in \mathcal{F}(M)$ by the norm

$$\|\Delta\| := \|(\delta)\|,$$

where δ is the m-vector that determines the matrix Δ as in (1.1). Furthermore, if $\Delta(.) \in L^\infty(\mathbb{R}_+, \mathcal{F}(M))$ is a time-varying disturbance, then we measure its size by

$$\|\Delta(.)\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}_+} \|\Delta(t)\|.$$

Let $\dot{x} = Ax$ the nominal system, where the matrix $A \in \mathcal{F}(M)$ is Hurwitz stable: $Re\lambda < 0$ for all $\lambda \in \sigma(A)$, $\sigma(A)$ the spectrum of A. Hence the nominal system is asymptotically stable (a.e.).

Let $B, C \in \mathcal{F}(M)$, then we say that the perturbed systems:

$$\dot{x} = [A + B\Delta C]x, \quad \Delta \in \mathcal{F}(M), \tag{1.4}$$

$$\dot{x} = [A + B\Delta(t)C]x, \quad \Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{F}(M)) \tag{1.5}$$

are patterned linear systems, the first with time-invariant patterned linear structured perturbation $P = B\Delta Cx$, and the second with time-varying patterned linear structured perturbation $P = B\Delta(t)Cx$.

Definition 1.1. (i) The real time-invariant stability radius of the matrix $A \in \mathcal{F}(M)$ for M-patterned linear perturbations of structure $(B, C) \in \mathcal{F}(M)^2$ is the number

$$r_{\mathbb{R}}^-(A, B, C) = \inf\{\|\Delta\| : \Delta \in \mathcal{F}(M),\ \sigma(A + B\Delta C) \cap \mathbb{C}_+ \neq \emptyset\},$$

where we set $\inf \emptyset = \infty$, $\sigma(A+B\Delta C)$ denotes the spectrum of the perturbed matrix $A + B\Delta C$, and $\mathbb{C}_+$ denotes the closed right half plane of the complex plane $\mathbb{C}$.

(ii) The real time-varying stability radius of the matrix $A \in \mathcal{F}(M)$ for patterned linear perturbations of structure $(B, C) \in \mathcal{F}(M)^2$ is the number:

$$r_{\mathbb{R},t}^-(A, B, C) = \inf\big\{\|\Delta(\cdot)\|_\infty : \Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{F}(M)) \text{ and } (1.5) \text{ is not asymptotically stable}\big\}.$$

In this work we prove that in the generic case when the matrix $M \in \mathbb{R}^{n\times n}$ is of simple structure; i.e., there exists a basis of $\mathbb{C}^n$ composed by eigenvectors of M, the two radii of stability for patterned systems coincide: $r_{\mathbb{R}}^{-}(A,B,C) = r_{\mathbb{R},t}^{-}(A,B,C)$ and we give a formula for its calculation.

2. Canonical form of linear patterned systems

Let $M \in \mathbb{R}^{n\times n}$ be a matrix of simple structure, and $(A,B,C) \in \mathcal{F}(M)^3$, where A is a stable matrix. By (1.3), the system (1.5) can be written as

$$\dot{x} = [A + BC\Delta(t)]x, \quad \Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{F}(M)).$$

Since M is of simple structure, we can compute a basis of $\mathbb{C}^n$ whose vectors are the eigenvectors of the matrix M. Changing in this basis the vectors z and $\overline{z}$ corresponding to a pair of conjugate eigenvectors to $\frac{1}{2}(z+\overline{z})$ and $\frac{1}{2i}(z-\overline{z})$ and forming the real matrix $U \in \mathbb{R}^{n\times n}$ whose columns are the calculated vectors, then as explained in [3] we have for the matrix $\widetilde{M} = U^{-1}MU$ the block diagonal expression:

$$\widetilde{M} = \begin{bmatrix} m_1 & -n_1 & & & & & & \\ n_1 & m_1 & & & & & & \\ & & \ddots & & & & & \\ & & & m_q & -n_q & & & \\ & & & n_q & m_q & & & \\ & & & & & m_{2q+1} & & \\ & & & & & & \ddots & \\ & & & & & & & m_n \end{bmatrix},$$

where $m_k + in_k, m_k - in_k, k = 1,\dots,q$ are the non-real eigenvalues of M $(n_k \neq 0)$, and $m_{2q+1},\dots,m_n$ are the real eigenvalues of the matrix M. Also for the matrices $\widetilde{A} = U^{-1}AU$, $\widetilde{BC} = U^{-1}BCU$ we have

$$\widetilde{A} = \begin{bmatrix} \mu_1 & -\nu_1 & & & & & & \\ \nu_1 & \mu_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \mu_q & -\nu_q & & & \\ & & & \nu_q & \mu_q & & & \\ & & & & & \mu_{2q+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \mu_n \end{bmatrix},$$

$$\widetilde{BC} = \begin{bmatrix} \epsilon_1 & -\kappa_1 & & & & & & \\ \kappa_1 & \epsilon_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \epsilon_q & -\kappa_q & & & \\ & & & \kappa_q & \epsilon_q & & & \\ & & & & & \epsilon_{2q+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \epsilon_n \end{bmatrix},$$

where $\mu_k + i\nu_k$, $\mu_k - i\nu_k$, $k = 1,\dots,q$, $\mu_{2q+1},\dots,\mu_n$ are the eigenvalues of A and $\epsilon_k + i\kappa_k$, $\epsilon_k - i\kappa_k$, $k = 1,\dots,q$, $\epsilon_{2q+1},\dots,\epsilon_n$ are the eigenvalues of the matrix BC. Furthermore, if $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{F}(M))$, then there exists a polynomial $p(\lambda) = \delta_0(.)\lambda + \delta_1(.)\lambda^2 + \cdots + \delta_{n-1}(.)\lambda^{n-1}$, such that $\Delta(.) = p(M)$ and so for $\widetilde{\Delta}(.) = U^{-1}\Delta(.)U$ we have

$$\widetilde{\Delta}(.) = \begin{bmatrix} p\begin{pmatrix} m_1 & -n_1 \\ n_1 & m_1 \end{pmatrix} & & & & & \\ & \ddots & & & & \\ & & p\begin{pmatrix} m_q & -n_q \\ n_q & m_q \end{pmatrix} & & & \\ & & & p(m_{2q+1}) & & \\ & & & & \ddots & \\ & & & & & p(m_n) \end{bmatrix}.$$

Therefore, in the new variables $x = Uy$ the perturbed system (1.5) becomes the decoupled systems:

$$\dot{y}_k = \left(\begin{bmatrix} \mu_k & -\nu_k \\ \nu_k & \mu_k \end{bmatrix} + \begin{bmatrix} \epsilon_k & -\kappa_k \\ \kappa_k & \epsilon_k \end{bmatrix} p\begin{pmatrix} m_k & -n_k \\ n_k & m_k \end{pmatrix} \right) y_k, \quad k = 1,\dots,q; \tag{2.1}$$

$$\dot{y}_k = (\mu_k + \epsilon_k p(m_k))y_k, \quad k = 2q+1,\dots,n\,. \tag{2.2}$$

Setting

$$\begin{gathered} \alpha_{jk} = \Re e(m_k + in_k)^j, \quad \beta_{jk} = \Im m(m_k + in_k)^j, \quad k = 1,\dots,q;\ j = 0,1,\dots,m-1, \\ \boldsymbol{\alpha}_k = (\alpha_{0k}, \alpha_{1k}, \dots, \alpha_{m-1k}), \quad \boldsymbol{\beta}_k = (\beta_{0k}, \beta_{1k}, \dots, \beta_{m-1k}), \\ \boldsymbol{\gamma}_k = (1, m_k, \dots, m_k^{m-1}), \quad k = 1,\dots,q, \\ \boldsymbol{\delta}(.) = (\delta_0(.), \delta_1(.), \dots, \delta_{m-1}(.)), \end{gathered}$$

then, for $k \in \{1,\dots,q\}$, Equation (2.1) becomes

$$\dot{y}_k = \left(\begin{bmatrix} \mu_k & -\nu_k \\ \nu_k & \mu_k \end{bmatrix} + \begin{bmatrix} (\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k)\delta(t) & -(\epsilon_k\boldsymbol{\beta}_k + \kappa_k\alpha_k)\delta(t) \\ (\epsilon_k\boldsymbol{\beta}_k + \kappa_k\alpha_k)\delta(t) & (\epsilon_k\alpha_k - \kappa_k\boldsymbol{\beta}_k)\delta(t) \end{bmatrix} \right) y_k, \tag{2.3}$$

$k = 1,\dots,q$. Also, for $k \in \{2q+1,\dots,n\}$, Equation (2.2) becomes

$$\dot{y}_k = (\mu_k + \epsilon_k\boldsymbol{\gamma}_k\delta(t))y_k. \tag{2.4}$$

Writing (2.3) in polar coordinates, $y_k = (y_k^1, y_k^2) = \rho_k(\cos(\varphi_k), \sin(\varphi_k))$, we have

$$\begin{aligned} \frac{\dot{\rho}_k}{\rho_k} &= \mu_k + (\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k)\delta(t) \\ \dot{\varphi} &= \nu_k + (\epsilon_k\boldsymbol{\beta}_k + \kappa_k\boldsymbol{\alpha}_k)\delta(t). \end{aligned} \tag{2.5}$$

Note that (2.3) is stable when $(\epsilon_k, \kappa_k) = (0,0)$ as is (2.4) when $\epsilon_k = 0$ for all disturbance $\delta(t)$. Finally note that from the expressions of the vectors $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}_k$ and the fact that $n_k \neq 0$, $k = 1,\dots,q$ it follows that

$$\begin{gathered} (\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k = 0) \Leftrightarrow (\epsilon_k = 0 \text{ and } \kappa_k = 0), \\ (\epsilon_k\boldsymbol{\gamma}_k = 0) \Leftrightarrow \epsilon_k = 0. \end{gathered}$$

3. Main results

Theorem 3.1. *Let $M \in \mathbb{R}^{n\times n}$ be a matrix of simple structure, and $(A,B,C) \in (\mathcal{F}(M))^3$, where A is a Hurwitz stable matrix and the product BC is not the null matrix, then for the patterned stability radii of the triple (A,B,C) we have: $r^-_{\mathbb{R}}(A,B,C) = r^-_{\mathbb{R},t}(A,B,C)$ and for this numbers we have the expression:*

$$r^-_{\mathbb{R}} = \min\Big\{ \inf_{k=1,\dots,q;\,(\epsilon_k,\kappa_k)\neq(0,0)} \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k\|},\ \inf_{k=2q+1,\dots,n;\ \epsilon_k\neq 0} \frac{-\mu_k}{\|\epsilon_k\gamma_k\|}\Big\}. \tag{3.1}$$

Proof. First we deduce (3.1) for the time-invariant stability radius. Note that from (2.3), (2.4) it follows that the disturbance $\delta \in \mathbb{R}^m$ corresponding to the minimum norm destabilizing perturbation of the system is a disturbance of minimum norm that destabilizes at least one of the subsystems (2.3), (2.4).

For a fixed $k \in \{1,2,\dots,q\}$ when the disturbance $\delta(t) \equiv \delta \in \mathbb{R}^m$ is time-invariant the real part of the eigenvalues of the matrix of the subsystem (2.3) is $\mu_k + (\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k)\delta$ and so the minimum norm destabilizing time-invariant disturbance is the solution $\delta_0 \in \mathbb{R}^m$ of the optimization problem:

$$\min\|\delta\|, \quad \mu_k + (\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k)\delta = 0.$$

For the solution δ_0 of this problem when $(\epsilon_k,\kappa_k) \neq (0,0)$ we have:

$$\delta_0 = \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k\|^2}(\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k)\,, \quad \|\delta_0\| = \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k\|}\,.$$

Thus a minimum norm destabilizing perturbation of (2.3) is the polynomial in M with coefficients

$$\delta_0 = \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k\|^2}(\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k).$$

Similarly we see that for $k \in \{2q+1,\dots,n\}$ the minimum norm time-invariant destabilizing disturbance δ of the subsystem (2.4) is the solution $\delta_0 \in \mathbb{R}^m$ of the optimization problem:

$$\min\|\delta\|, \quad \mu_k + \epsilon_k\boldsymbol{\gamma}_k\delta = 0.$$

Let δ_0 be the solution to this problem when $\epsilon_k \neq 0$. Hence,

$$\delta_0 = \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\gamma}_k\|^2}(\epsilon_k\boldsymbol{\gamma}_k)\,, \quad \|\delta_0\| = \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\gamma}_k\|}\,.$$

Thus a minimum norm destabilizing perturbation of this subsystem is the polynomial in M with coefficients $\delta_0 = \frac{-\mu_k}{\|\epsilon_k\boldsymbol{\gamma}_k\|^2}(\epsilon_k\boldsymbol{\gamma}_k)$. Therefore (3.1) holds also for the time-invariant patterned stability radius.

The inequality $r^-_{\mathbb{R}}(A,B,C) \geq r^-_{\mathbb{R},\,t}(A,B,C)$ follows from the definitions of the stability radii. So for the equality of this radii it is sufficient to show that for the time-varying perturbations $\Delta(t)$ with coefficients satisfying $\|\delta(t)\| < r^-_{\mathbb{R}}(A,B,C)$ the corresponding subsystems (2.4) and (2.5) are all asymptotically stable. To see this we note that for such disturbance $\delta(t)$ and any fixed $k \in \{2q+1,\dots,n\}$ there exists $\epsilon > 0$ such that: $\mu_k + \epsilon_k\boldsymbol{\gamma}_k\delta(t) < -\epsilon$, for all $t \in [0,\infty)$ and for fixed $k \in \{1,\dots,q\}$ there exists $\epsilon > 0$ such that: $\mu_k + (\epsilon_k\boldsymbol{\alpha}_k - \kappa_k\boldsymbol{\beta}_k)\delta(t) < -\epsilon$ for all $t \in [0,\infty)$. $\square$

4. Example

In this section using Maple we present a numerical example, showing the calculation of the stability radius $r^-_{\mathbb{R},t}(A,B,C)$ for a triple of patterned matrices of sixth order. Let

$$M := \begin{bmatrix} -0.3 & 0.2 & 0.2 & 0 & -0.1 & -0.4 \\ -0.2 & -0.4 & 0.1 & 0.3 & 0.1 & 0.4 \\ 0 & -0.3 & -0.5 & -0.3 & 0 & -0.3 \\ -0.1 & -0.1 & -0.1 & -0.5 & -0.1 & -0.1 \\ 0.1 & 0.4 & 0.4 & 0.9 & 0.2 & 0.7 \\ 0 & -0.3 & -0.6 & -0.9 & -0.6 & -0.8 \end{bmatrix}$$

$$A := \begin{bmatrix} -0.2604 & 0.1812 & 0.1794 & -0.0027 & -0.0927 & -0.3639 \\ -0.1812 & -0.3498 & 0.0936 & 0.2757 & 0.0927 & 0.3639 \\ -0.0009 & -0.2739 & -0.4443 & -0.2748 & -0.0018 & -0.273 \\ -0.0909 & -0.0909 & -0.0909 & -0.4425 & -0.0909 & -0.0909 \\ 0.0918 & 0.3648 & 0.3657 & 0.819 & 0.1944 & 0.6351 \\ 0.0009 & -0.2739 & -0.546 & -0.8172 & -0.5442 & -0.7137 \end{bmatrix}$$

$$B := \begin{bmatrix} 0.09662 & -0.00244 & 0.00222 & -0.00054 & -0.00164 & -0.00518 \\ -0.00244 & 0.09564 & 0.00182 & 0.00444 & 0.00164 & 0.00518 \\ -0.00018 & -0.00408 & 0.09364 & -0.00426 & -0.00036 & -0.0039 \\ -0.00128 & -0.00128 & -0.00128 & 0.094 & -0.00128 & -0.00128 \\ 0.00146 & 0.00536 & 0.00554 & 0.0117 & 0.10308 & 0.00872 \\ 0.00018 & -0.00408 & -0.0078 & -0.01134 & -0.00744 & 0.09046 \end{bmatrix}$$

$$C := \begin{bmatrix} -0.20564 & -0.01632 & -0.02287 & -0.02506 & -0.01126 & -0.01094 \\ 0.01632 & -0.16676 & 0.02911 & 0.0313 & 0.01126 & 0.01094 \\ -0.01818 & -0.02442 & -0.21405 & -0.03116 & -0.00688 & -0.00656 \\ -0.00438 & -0.00438 & -0.00438 & -0.2075 & -0.00438 & -0.00438 \\ -0.0226 & 0.0288 & 0.0175 & 0.01937 & -0.19 & 0.010632 \\ -0.01818 & -0.02442 & -0.0128 & -0.01249 & -0.00625 & -0.20719 \end{bmatrix}$$

Easily we can verify that

$$\begin{gathered} A = 0.01I + 0.9M - 0.01M^2, \\ B = 0.1I + 0.011M - 0.002M^2, \\ C = -0.2I + 0.003M - 0.002M^3 + 0.2M^5, \end{gathered}$$

thus the matrices A, B, C are M-patterned matrices. The eigenvalues of the matrix M are all different: $-0.5+0.3i, -0.5-0.3i, -0.2+0.3i, -0.2-0.3i, -0.4, -0.5$, so the matrix M is of simple structure. Using Maple and the procedure explained in section 2 we calculate the invertible matrix U, which corresponds to the canonical form of the patterned system

$$U := \begin{bmatrix} 0 & -1 & 0 & -1 & -1 & 2 \\ 1 & 1 & 0 & 1 & 1 & -2 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & 1 & -2 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{4.1}$$

Then we have for the matrices $\widetilde{A} = U^{-1}AU$ and $\widetilde{BC} = U^{-1}BCU$:

$$\widetilde{A} = \begin{bmatrix} -0.4416 & 0.273 & 0 & 0 & 0 & 0 \\ -0.273 & -0.4416 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1695 & 0.2712 & 0 & 0 \\ 0 & 0 & -0.2712 & -0.1695 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3516 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4425 \end{bmatrix},$$

$$\widetilde{BC} = \begin{bmatrix} -0.0178544936 & -0.0001506648 & 0 & 0 & 0 & 0 \\ 0.0001506648 & -0.0178544936 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.01967052992 & -0.00074185824 & 0 & 0 \\ 0 & 0 & 0.00074185824 & -0.01967052992 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0193532736 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.019505 \end{bmatrix}.$$

In this case the parameters needed to apply the formula given by Theorem 3.1 we have the values:

$$\begin{aligned} m_1 &= -0.5, & n_1 &= -0.3, \\ m_2 &= -0.2, & n_2 &= -0.3, \\ m_5 &= -0.4, & m_6 &= -0.5 \\ \mu_1 &= -0.4416, & \nu_1 &= -0.273, \\ \mu_2 &= -0.1695, & \nu_2 &= -0.2712, \\ \mu_5 &= -0.3516, & \mu_6 &= -0.4425 \\ \epsilon_1 &= -0.0178544936, & \kappa_1 &= 0.0001506648, \\ \epsilon_2 &= -0.01967052992, & \kappa_2 &= 0.00074185824, \\ \epsilon_5 &= -0.0193532736, & \epsilon_6 &= -0.019505 \end{aligned}$$

$$\begin{aligned} \alpha_1 &= (1, -0.5, 0.16, 0.01, -0.0644, 0.061), \\ \beta_1 &= (0, -0.3, 0.3, -0.198, 0.096, -0.02868), \\ \alpha_2 &= (1, -0.2, -0.05, 0.046, -0.0119, -0.00122), \\ \beta_2 &= (0, -0.3, 0.12, -0.009, -0.0120, 0.00597), \\ \gamma_5 &= (1, -0.4, 0.16, -0.064, 0.0256, -0.01024), \\ \gamma_6 &= (1, -0.5, 0.25, -0.125, 0.0625, -0.03125), \end{aligned}$$

Simple computations show that

$$-\frac{\mu_1}{\|\epsilon_1\alpha_1 - \kappa_1\beta_1\|} = 21.8038, \quad -\frac{\mu_2}{\|\epsilon_2\alpha_2 - \kappa_2\beta_2\|} = 8.41345,$$
$$-\frac{\mu_5}{\|\epsilon_5\gamma_5\|} = 16.65090014, \quad -\frac{\mu_6}{\|\epsilon_6\gamma_6\|} = 19.64947599\,.$$

From this and the formula of Theorem 3.1 we conclude that

$$r^-_{\mathbb{R},t}(A, B, C) = 8.41345 \tag{4.2}$$

and a minimum norm destabilizing perturbation is the polynomial in M with coefficients

$$\delta = \frac{-\mu_2}{\|\epsilon_2\alpha_2 - \kappa_2\beta_2\|^2}(\epsilon_2\alpha_2 - \kappa_2\beta_2).$$

Finally we have

$$\delta = (-8.21476, 1.7359, 0.37356, -0.37509, 0.101473, 0.00817242)$$

and the disturbance Δ corresponding to the minimum norm destabilizing perturbation is

$$\Delta = \begin{bmatrix} -8.72494 & 0.307791 & 0.4566 & 0.232859 & 0.022962 & -0.450774 \\ -0.307791 & -9.01559 & -0.131673 & 0.0920681 & -0.022962 & 0.450774 \\ 0.0963806 & -0.228547 & -8.78754 & -0.188449 & 0.136478 & -0.337257 \\ -0.113516 & -0.113516 & -0.113516 & -8.93635 & -0.113516 & -0.113516 \\ 0.017135 & ,0.342063 & 0.301965 & 0.999442 & -8.14832 & 0.92451 \\ 0.0963806 & -0.228547 & -0.662185 & -1.13592 & -0.810993 & -9.41008 \end{bmatrix}.$$

Conclusions. We have established that for the class of linear patterned systems the time-invariant and the time-varying stability radii are equal and we have obtained a simple computable formula for this radius in terms of the nominal matrix and the matrices that give the structure of the linear patterned perturbation. The results in this paper give a complete solution to the problem of calculation of the real stability radii of patterned linear systems.

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