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CONTROLLABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEMS WITH NONLOCAL CONDITIONS

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Abstract. In this article, we study the controllability of impulsive functional differential equations with nonlocal conditions. We establish sufficient conditions for controllability, via the measure of noncompactness and Mönch fixed point theorem.

1. Introduction

Consider the impulsive functional differential equation

$$\begin{gathered} x'(t) = A(t)x(t) + f(t, x(t), x_t) + Bu(t), \quad \text{a.e. } t \in [0, a]; \\ \Delta x\big|_{t=t_i} = I_i(x(t_i)), \quad i = 1, 2, \dots k; \\ x(t) = \phi(t), \quad t \in [-\tau, 0); \\ x(0) + M(x) = x_0, \end{gathered} \tag{1.1}$$

where $\Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0)$, $A(t)$ is a family of linear operators which generates an evolution operator

$$U : \Delta = \{(t, s) \in J \times J : 0 \le s \le t \le a\} \to L(X),$$

X is a Banach space, $J = [0, a]$, $L(X)$ is the space of all bounded linear operators in X, $M : PC(J, X) \to X$, B is a bounded linear operator from a Banach space V to X and the control function $u(\cdot)$ is given in $L^2(J, V)$, $0 = t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} = a$, $I_i : X \to X, i = 1, \dots, k$ are impulsive functions, $f : J \times X \times L([-\tau, 0], X) \to X$ is a given function satisfying some assumptions that will be specified later, $\phi \in L([-\tau, 0], X)$ and $L([-\tau, 0], X)$ is the space of X-valued Bochner integrable functions on $[-\tau, 0]$ with the norm $\|\phi\|_{L[-\tau,0]} = \int_{-\tau}^0 \|\phi(t)\| dt$.

Abstract differential systems in infinite-dimensional spaces appear in many branches of science and engineering, such as heat flow in materials with memory, viscoelasticity, and other physical phenomena. Systems with short-term perturbations are often naturally described by impulsive differential equations [18, 25]. Impulsive interruptions are observed in mechanics, radio engineering, communication security, control theory, optimal control, biology, medicine, bio-technologies, electronics, neural networks and economics (see for example [4, 5, 8, 19, 26, 27]). We also refer the

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reader to recent results in impulse theory [6, 7, 24, 28]. The semilinear nonlocal initial problem was first discussed by Byszewski [2, 3]. It was studied extensively under various conditions on A (or $A(t)$) and f by several authors (see [4, 15] and the references therein). Recently, Ji et al [15] studied the impulsive differential equation

$$\begin{gathered} x'(t) = A(t)x(t) + f(t, x(t)) + Bu(t), \quad \text{a.e. } t \in [0, a]; \\ \Delta x\big|_{t=t_i} = I_i(x(t_i)), \quad i = 1, 2, \dots k; \\ x(0) + M(x) = x_0. \end{gathered} \tag{1.2}$$

Time delays are often encountered unavoidably in many practical systems such as automatic control systems, population models, inferred grinding models, the AIDS epidemic, and neural networks; see [9, 10, 17, 11, 22] and the references therein. They describe phenomenon present in real systems where the rate of change of the state depends on not only the current state of the system but also its state at some time in history. Therefore, it is natural and necessary to study (1.2) with time delay, i.e. the (1.1).

To the best of our knowledge there is no paper studying such systems. The purpose of the present paper is to fill this gap. In this paper some sufficient conditions for controllability are established by using the measure of noncompactness and Mönch's fixed point theorem. The main features in the present paper are as follows. First, the (1.1) considers the effect of time delay. Also we relax the assumptions on the functions f, M, and I_i in [15].

The organization of this article is as follows. We shall introduce some preliminaries and some lemmas in Section 2. The main results and their proof are given in Section 3.

2. Preliminaries

For the sake of simplicity, we put $J_0 = [0, t_1]$ and $J_i = (t_i, t_{i+1}]$, $i = 1, \dots, k$. Let $PC(J, X) = \{x : x$ is a map from J into X such that $x(t)$ is continuous at $t \neq t_i$, and left continuous at $t = t_i$, and the right limit $x(t_i^+)$ exists for $i = 1, 2, \dots, k\}$. Evidently, $PC(J, X)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t \in J}\{\|x(t)\|\}, \quad \forall x \in PC(J, X).$$

Notice that the interaction of time delay and impulse give rise to discontinuity. Therefore, we introduce the special complete space $L([-\tau, 0], X)$ to overcome the difficulty arising from time delay. For any function $y \in PC(J, X)$ and any $t \in J$, we denote a function y_t by

$$y_t(\theta) = \begin{cases} y(t+\theta), & t+\theta \geq 0; \\ \phi(t+\theta), & t+\theta < 0 \end{cases} \tag{2.1}$$

for $\theta \in [-\tau, 0]$, where $\phi(t)$ is the same as in (1.1). Then it is easy to see $y_t \in L([-\tau, 0], X)$. Moreover, we have the following Lemma.

Lemma 2.1. *Suppose $y_n, y_0 \in PC(J, X)$ with $\|y_n - y_0\|_{PC} \to 0$ as $n \to +\infty$. Then for each $t \in J$, we have*

$$\|y_{nt} - y_{0t}\|_{L[-\tau,0]} \to 0, \quad \text{as } n \to +\infty,$$

where $y_{nt}(\theta)$ and $y_{0t}(\theta)$ are defined by (2.1).

Proof. From (2.1), it follows that

$$\|y_{nt} - y_{0t}\|_{L[-\tau,0]} = \begin{cases} \int_0^t |y_n(s) - y_0(s)|ds, & t \le \tau; \\ \int_{t-\tau}^t |y_n(s) - y_0(s)|ds, & t \ge \tau. \end{cases}$$

The conclusion follows. □

The basic space to study (1.1) in this paper is $PC(J, X)$. For a bounded subset Ω of Banach space X, let $\beta(\Omega)$ be the Hausdorff noncompactness measure of Ω, which is defined by $\beta(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net in } X\}$ (see [1, 16]). In this paper, the Hausdorff measure of noncompactness of a bounded set in X, $PC(J, X)$, and $L([-\tau, 0], X)$ are denoted by $\beta(\cdot)$, $\beta_{PC}(\cdot)$, and $\beta_\tau(\cdot)$, respectively. As in [13], we have the following result on the Hausdorff noncompactness measure.

Lemma 2.2. *Suppose E is a Banach space. Let H be a countable set of strongly measurable function $x : J \to E$ such that there exists a $\mu \in L[J, R^+]$ with $\|x(t)\| \le \mu(t)$ a.e. $t \in J$ for all $x \in H$. Then $\beta(H(t)) \in L[J, R^+]$ and*

$$\beta\big(\big\{\int_J x(t)dt : x \in H\big\}\big) \le 2\int_J \beta(H(t))dt,$$

where $\beta(\cdot)$ denotes the Hausdorff noncompactness measure, $J = [0, a]$.

Lemma 2.3 (Mönch fixed point theorem [20])**.** *Suppose E is a Banach space. Let D be a closed and convex subset of E and $u \in D$. Assume that the continuous operator $A : D \to D$ has the following property: $C \subset D$ countable, $C \subset \overline{co}(\{u\} \cup A(C))$ implies C is relatively compact. Then A has a fixed point in D.*

Definition 2.4. A function $x \in PC(J; X)$ is said to be a mild solution of (1.1) if $x(0) + M(x) = x_0$ and

$$x(t) = U(t, 0)x(0) + \int_0^t U(t, s)\big((f(s, x(s), x_s) + Bu(s)\big)ds + \sum_{0<t_i<t} U(t, t_i)I_i(x(t_i)),$$

for all $t \in J$, where x_s is defined by (2.1).

Definition 2.5. Equation (1.1) is said to be nonlocally controllable on J if, for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, V)$ such that the mild solution x of (1.1) satisfies $x(b) + M(x) = x_1$.

A two parameter family of bounded linear operators $U(t, s)$, $0 \le s \le t \le a$ on X is called an evolution system if the following two conditions are satisfied:

(i) $U(s, s) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \le s \le t \le a$;
(ii) $(t, s) \to U(t, s)$ is strongly continuous for $0 \le s \le t \le a$.

Since the evolution system $U(t, s)$ is strongly continuous on the compact set $J \times J$, then there exists $L_U > 0$ such that $\|U(t, s)\| \le L_U$ for any $(t, s) \in J \times J$. More details about evolution systems can be found in [23].

3. Main results

We will use the following hypotheses:

(S1) $A(t)$ is a family of linear operators, $A(t) : \mathscr{D}(A) \to X$, $\mathscr{D}(A)$ not depending on t is a dense subset of X, generating an equicontinuous evolution system $\{U(t, s) : (t, s) \in J \times J\}$, i.e., $(t, s) \to \{U(t, s)x : x \in \Omega\}$ is equicontinuous for $t > 0$ and for all bounded subsets Ω.

(S2) $f : J \times X \times L([-\tau, 0], X) \to X$ satisfies:
  (i) $t \to f(t, x, y)$ is strongly measurable for each $x \in X, y \in L([-\tau, 0], X)$; $(x, y) \to f(t, x, y)$ is continuous for almost all $t \in J$;
  (ii) there exist functions $a_1, b_1, \mu_1 \in L(J; R^+)$ such that

$$\|f(t, x, y)\| \le a_1(t)\|x\| + b_1(t)\|y\|_{L[-\tau,0]} + \mu_1(t),$$

  for all $t \in J$, $x \in X$, $y \in L([-\tau, 0], X)$;
  (iii) there exist $l_1, l_2 \in L^1(J; R^+)$ such that for any bounded subsets $B_1 \subset X, B_2 \subset L([-\tau, 0], X)$,

$$\beta(f(t, B_1, B_2)) \le l_1(t)\beta(B_1) + l_2(t)\beta_\tau(B_2);$$

(S3) $M : PC(J, X) \to X$ is a continuous operator and there exist nonnegative numbers a_2, b_2, l_3 such that

$$\|M(y)\| \le a_2\|y\| + b_2, \quad \forall y \in PC(J, X);$$
$$\beta(M(B_1)) \le l_3\beta_{PC}(B_1), \quad \text{for any bounded } B_1 \subset PC(J, X);$$

(S4) the linear operator $W : L^2(J, V) \to X$ defined by

$$Wu = \int_0^a U(a, s)Bu(s)ds$$

is such that:
  (i) W has an invertible operator W^{-1} which take values in $L^2(J, V)/kerW$ and there exist positive constants L_B and L_W such that $\|B\| \le L_B$ and $\|W^{-1}\| \le L_W$;
  (ii) there is $K_W \in L^1(J, R^+)$ such that, for any bounded set $Q \subset X$,

$$\beta_V((W^{-1}Q)(t)) \le K_W(t)\beta(Q).$$

(S5) $I_i : X \to X (i = 1, \dots, k)$ is a continuous operator and there exist nonnegative numbers c_i, d_i, k_i $(i = 1, 2, \dots, k)$ such that:

$$\|I_i(x)\| \le c_i\|x\| + d_i, \quad \forall x \in X,\ i = 1, 2, \dots, k;$$
$$\beta(I_i(B_1)) \le k_i\beta(B_1), \quad \text{for any bounded } B_1 \subset X,\ i = 1, 2, \dots, k.$$

Theorem 3.1. *Assume that* (S1)–(S5) *are satisfied. In addition, assume that*

$$\begin{aligned} c := L_U\Big[&(1 + L_BL_Wa^{1/2})\Big(a_2 + \int_0^a \big(a_1(s) + \tau b_1(s)\big)ds + \sum_{i=1}^k c_i\Big) \\ &+ L_UL_BL_Wa_2a^{1/2}\Big] < 1, \end{aligned} \tag{3.1}$$

$$\begin{aligned} d := L_U\Big[&\big(l_3 + 2\int_0^a (l_1(s) + \tau l_2(s))ds + \sum_{i=1}^k k_i\big)\big(1 + 2L_BL_U\int_0^a K_W(s)ds\big) \\ &+ 2l_3L_B\int_0^a K_W(s)ds\Big] < 1. \end{aligned} \tag{3.2}$$

Then the impulsive functional differential system (1.1) *is nonlocally controllable on* J.

Proof. From (S4)(i), one can define the control:

$$\begin{aligned} u_x(t) = W^{-1}[&x_1 - M(x) - U(a,0)(x_0 - M(x)) \\ &- \int_0^a U(a,s)f(s,x(s),x_s)ds - \sum_{i=1}^{k} U(a,t_i)I_i(x(t_i))](t), \end{aligned} \tag{3.3}$$

for all $x \in PC(J,X)$. Using this control, define the following operator on $PC(J,X)$ by

$$\begin{aligned} (Gx)(t) =& U(t,0)(x_0 - M(x)) + \int_0^t U(t,s)\big(f(s,x(s),x_s) + Bu_x(s)\big)ds \\ &+ \sum_{0<t_i<t} U(t,t_i)I_i(x(t_i)), \quad \forall x \in PC(J,X). \end{aligned} \tag{3.4}$$

Obviously, $Gx \in PC(J,X)$. We shall show that G has a fixed point, which is then a solution of (1.1). Clearly, if x is a fixed point of G, then $x_1 = M(x) + G(x)(a)$, which implies that the system (1.1) is controllable.

First we show that G is continuous. To do this, suppose $x_n, x \in PC(J,X)$ and $x_n \to x$ as $n \to +\infty$. Then by (S3) and (S5) we know that

$$\begin{aligned} &\|Gx_n - Gx\|_{PC} \\ &\le L_U\Big(\|M(x_n) - M(x)\| + \int_0^a \|f(s,x_n(s),x_{ns}) - f(s,x(s),x_s)\|ds \\ &\quad + L_B\int_0^a \|u_{x_n}(s) - u_x(s)\|ds + \sum_{i=1}^{k}\|I_i(x_n(t_i)) - I_i(x(t_i))\|\Big) \\ &\le L_U\Big(\|M(x_n) - M(x)\| + \int_0^a \|f(s,x_n(s),x_{ns}) - f(s,x(s),x_s)\|ds \\ &\quad + L_B a^{1/2}\|u_{x_n} - u_x\|_{L^2} + \sum_{i=1}^{k}\|I_i(x_n(t_i) - I_i(x(t_i))\|\Big). \end{aligned} \tag{3.5}$$

Notice that

$$\|x_{ns} - x_s\|_{L[-\tau,0]} \le \tau\|x_n - x\|_{PC}. \tag{3.6}$$

From (3.3), we have

$$\begin{aligned} &\|u_{x_n} - u_x\|_{L^2} \\ &\le L_W\|M(x_n) - M(x)\| + L_W L_U\Big[\|M(x_n) - M(x)\| \\ &\quad + \int_0^a \|f(s,x_n(s),x_{ns}) - f(s,x(s),x_s)\|ds + \sum_{i=1}^{k}\|I_i(x_n(t_i) - I_i(x(t_i))\|\Big]. \end{aligned} \tag{3.7}$$

Then by (3.5)–(3.7), (S2)–(S5), and the Lebesgue dominated convergence theorem, we obtain

$$\|Gx_n - Gx\|_{PC} \to 0 \quad \text{as} \quad n \to +\infty,$$

so G is continuous.

Next, choose a positive number r satisfying

$$\begin{aligned} r > \frac{L_U}{1-c}\Big[&(1+L_U L_B L_W a^{1/2})\Big(\|x_0\|+b_2+\int_0^a b_1(s)ds\cdot\|\phi\|_{L[-\tau,0]} \\ &+\int_0^a \mu_1(s)ds+\sum_{i=1}^k d_i\Big)+L_B L_W a^{1/2}(\|x_1\|+b_2)\Big]. \end{aligned} \tag{3.8}$$

We now show that

$$G: B(0,r)\to B(0,r), \tag{3.9}$$

where $B(0,r)=\{x\in PC(J,X):\|x\|_{PC}\le r\}$. In fact, for each $x\in PC(J,X)$, by (3.3), we have

$$\begin{aligned} \|u_x\|_{L^2} &= \Big(\int_0^a \|u_x(s)\|^2 ds\Big)^{1/2} \\ &\le L_W(\|x_1\|+a_2\|x\|_{PC}+b_2)+L_W L_U\Big[\|x_0\|+a_2\|x\|_{PC}+b_2 \\ &\quad +\int_0^a \big(a_1(s)\|x(s)\|+b_1(s)\|x_s\|_{L[-\tau,0]}+\mu_1(s)\big)ds+\sum_{i=1}^k(c_i\|x(t_i)\|+d_i)\Big] \\ &\le L_W(\|x_1\|+a_2\|x\|_{PC}+b_2)+L_W L_U\Big[\|x_0\|+a_2\|x\|_{PC}+b_2 \\ &\quad +\int_0^a \big(a_1(s)\|x\|_{PC}+b_1(s)(\tau\|x\|_{PC}+\|\phi\|_{L[-\tau,0]})+\mu_1(s)\big)ds \\ &\quad +\sum_{i=1}^k(c_i\|x\|_{PC}+d_i)\Big]. \end{aligned}$$

This together with (3.4) guarantees that

$$\begin{aligned} &\|Gx\|_{PC} \\ &\le L_U\Big[\|x_0\|+\|M(x)\|+\int_0^a \|f(s,x(s),x_s)+Bu_x(s)\|ds+\sum_{i=1}^k\|I_i(x(t_i))\|\Big] \\ &\le L_U\Big[\|x_0\|+a_2\|x\|_{PC}+b_2+\int_0^a \big(a_1(s)\|x(s)\|+b_1(s)\|x_s\|_{L[-\tau,0]}+\mu_1(s)\big)ds \\ &\quad +L_B\int_0^a\|u_x(s)\|ds+\sum_{i=1}^k(c_i\|x(t_i)\|+d_i)\Big] \\ &\le L_U\Big[\|x_0\|+a_2\|x\|_{PC}+b_2+\int_0^a \big(a_1(s)\|x\|_{PC}+b_1(s)(\tau\|x\|_{PC}+\|\phi\|_{L[-\tau,0]}) \\ &\quad +\mu_1(s)\big)ds+L_B a^{1/2}\|u_x\|_{L^2}+\sum_{i=1}^k(c_i\|x\|_{PC}+d_i)\Big] \\ &\le c\|x\|_{PC}+L_U\Big[(1+L_U L_B L_W a^{1/2})\Big(\|x_0\|+b_2+\int_0^a b_1(s)ds\cdot\|\phi\|_{L[-\tau,0]} \\ &\quad +\int_0^a\mu_1(s)ds+\sum_{i=1}^k d_i\Big)+L_B L_W a^{1/2}(\|x_1\|+b_2)\Big]. \end{aligned}$$

From (3.8) we have $\|Gx\|_{PC}\le r$ if $\|x\|_{PC}\le r$; that is, (3.9) holds.

Next we prove that if $D \subset B(0,r)$ is countable and

$$D \subset \overline{co}(\{u_0\} \cup G(D)), \tag{3.10}$$

where $u_0 \in B(0,r)$, then D is relatively compact. Without loss of generality, suppose that $D = \{x_n\}_{n=1}^{\infty}$. First we show $\{Gx_n\}_{n=1}^{\infty}$ is equicontinuous on each J_i, $i = 0, \dots, k$. If this is true then $\overline{co}(\{u_0\} \cup G(D))$ is also equicontinuous on each J_i. To this end, notice that for each $x \in D$, $t', t'' \in J_i$, we have

$$\begin{aligned}
&\|(Gx)(t'') - (Gx)(t')\| \\
&= \|[U(t'',0) - U(t',0)](x_0 - M(x))\| + \|\sum_{j=1}^{i} \big(U(t'',t_j) - U(t',t_j)\big) I_j(x(t_j))\| \\
&\quad + \|\int_0^{t''} U(t'',s)\big(f(s,x(s),x_s) + Bu_x(s)\big)ds \\
&\quad - \int_0^{t'} U(t',s)\big(f(s,x(s),x_s) + Bu_x(s)\big)ds\| \\
&\le \|[U(t'',0) - U(t',0)](x_0 - M(x))\| + \sum_{j=1}^{i} \|\big(U(t'',t_j) - U(t',t_j)\big) I_j(x(t_j))\| \\
&\quad + \int_0^{t'} \|U(t'',s) - U(t',s)\big(f(s,x(s),x_s) + Bu_x(s)\big)\| ds \\
&\quad + \int_{t'}^{t''} \|U(t'',s)\| \cdot \|f(s,x(s),x_s) + Bu_x(s)\| ds
\end{aligned} \tag{3.11}$$

From the equicontinuity property of $U(\cdot,s)$ and the absolute continuity of the Lebesgue integral, we see that the right-hand side of the inequality (3.11) tends to zero independent of $x \in D$ as $|t'' - t'| \to 0, t'', t' \in J_i$. Therefore, $G(D)$ is equicontinuous on every J_i.

Next notice that

$$\|x_{ns} - x_{ms}\|_{L[-\tau,0]} \le \tau \|x_n - x_m\|_{PC}, \quad \|x_n(s) - x_m(s)\| \le \|x_n - x_m\|_{PC}, s \in J,$$

which implies

$$\beta_\tau(\{x_{ns}\}_{n=1}^{\infty}) \le \tau \beta_{PC}(\{x_n\}_{n=1}^{\infty}), \quad \beta(\{x_n(s)\}_{n=1}^{\infty}) \le \beta_{PC}(\{x_n\}_{n=1}^{\infty}), s \in J.$$

Then from (S2), (S3), (S4) and (S5), for each $t \in J$, we have

$$\begin{aligned}
&\beta_V(\{u_{x_n}(t)\}_{n=1}^{\infty}) \\
&\le K_W(t)\beta\Big(\{M(x_n) + U(a,0)(x_0 - M(x_n)) + \int_0^a U(a,s)f(s,x_n(s),x_{ns})ds \\
&\quad + \sum_{i=1}^{k} U(a,t_i)I_i(x_n(t_i))\}_{n=1}^{\infty}\Big) \\
&\le K_W(t)\Big(l_3(1+L_U)\beta_{PC}(\{x_n\}_{n=1}^{\infty}) + 2L_U \int_0^a \big[l_1(s)\beta(\{x_n(s)\}_{n=1}^{\infty}) \\
&\quad + l_2(s)\beta_\tau(\{x_{ns}\}_{n=1}^{\infty})\big]ds + L_U \sum_{i=1}^{k} k_i \beta(\{x_n(t_i)\}_{n=1}^{\infty})\Big)
\end{aligned}$$

$$\le K_W(t)\Big(l_3(1+L_U)+2L_U\int_0^a \big[l_1(s)+\tau l_2(s)\big]ds+L_U\sum_{i=1}^k k_i\Big)\beta_{PC}(\{x_n\}_{n=1}^\infty),$$

and

$$
\begin{aligned}
&\beta(\{(Gx_n)(t)\}_{n=1}^\infty)\\
&\le \beta\Big(\{U(t,0)(x_0-M(x_n))\}_{n=1}^\infty\Big)\\
&\quad+\beta\Big(\{\int_0^t U(t,s)\big(f(s,x_n(s),x_{ns})+Bu_{x_n}(s)\big)ds\}_{n=1}^\infty\Big)\\
&\quad+\beta\Big(\{\sum_{0<t_i<t} U(t,t_i)I_i(x_n(t_i))\}_{n=1}^\infty\Big)\\
&\le L_U l_3\beta_{PC}(\{x_n\}_{n=1}^\infty)+2L_U\int_0^a \big[l_1(s)\beta(\{x_n(s)\}_{n=1}^\infty)+l_2(s)\beta_\tau(\{x_{ns}\}_{n=1}^\infty)\big]ds\\
&\quad+2L_UL_B\int_0^a \beta_V(\{u_{x_n}(s)\}_{n=1}^\infty)ds+L_U\sum_{i=1}^k k_i\beta(\{x_n(t_i)\}_{n=1}^\infty)\\
&\le L_U\Big[l_3+2\int_0^a \big[l_1(s)+\tau l_2(s)\big]ds+2L_B\Big(l_3(1+L_U)+2L_U\int_0^a \big[l_1(s)+\tau l_2(s)\big]ds\\
&\quad+L_U\sum_{i=1}^k k_i\Big)\int_0^a K_W(s)ds+\sum_{i=1}^k k_i\Big]\beta_{PC}(\{x_n\}_{n=1}^\infty)\\
&\le L_U\Big[\big(l_3+2\int_0^a (l_1(s)+\tau l_2(s))ds+\sum_{i=1}^k k_i\big)\big(1+2L_BL_U\int_0^a K_W(s)ds\big)\\
&\quad+2l_3L_B\int_0^a K_W(s)ds\Big]\beta_{PC}(\{x_n\}_{n=1}^\infty)\\
&= d\cdot\beta_{PC}(\{x_n\}_{n=1}^\infty).
\end{aligned}
\tag{3.12}
$$

Note since $\{Gx_n\}_{n=1}^\infty$ is equicontinuous on each J_i, $i=0,\dots,k$ we have (from a well known result on measures of noncompactness)

$$\beta_{PC}(\{Gx_n\}_{n=1}^\infty)=\sup_{0\le i\le k}\sup_{t\in J_i}\beta(\{(Gx_n)(t)\}_{n=1}^\infty).$$

This together with (3.2), (3.10) and (3.12) guarantees that

$$\beta_{PC}(\{x_n\}_{n=1}^\infty)\le\beta_{PC}(\{Gx_n\}_{n=1}^\infty)\le d\cdot\beta_{PC}(\{x_n\}_{n=1}^\infty),$$

which implies that $D=\{x_n\}_{n=1}^\infty$ is relatively compact.

From Mönch's fixed point theorem, G has a fixed point in $B(0,r)$ and immediately the system (1.1) is nonlocally controllable on J. □

Remark 3.2. Note that (1.1) with no effect of time delay was considered in [15]. The assumptions on f, M, and I_i in [15] are relaxed in this paper. For example M is not necessarily compact here, and the the assumptions (S2), (S3), and (S5) in our paper are weaker than assumptions (H2), (H3), and (H5) in [15].

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