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SUBHARMONIC SOLUTIONS FOR FIRST-ORDER HAMILTONIAN SYSTEMS

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Abstract. In this article, we study the existence of periodic and subharmonic solutions for a class of non-autonomous first-order Hamiltonian systems such that the nonlinearity has a growth at infinity faster than $|x|^\alpha$, $0 \le \alpha < 1$. We also study the minimality of periods for such solutions. Our results are illustrated by specific examples. The proofs are based on the least action principle and a generalized saddle point theorem.

1. Introduction and statement of main results

Consider the non-autonomous first-order Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)) \tag{1.1}$$

where J is the standard symplectic $(2N \times 2N)$-matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $(t, x) \to H(t, x)$ is a continuous function T-periodic ($T > 0$) in t and differentiable with respect to the second variable such that the derivative $H'(t, x) = \frac{\partial H}{\partial x}(t, x)$ is continuous on $\mathbb{R} \times \mathbb{R}^{2N}$. In this article, we are interested first in the existence of periodic solutions and then the existence of subharmonics for the system (1.1). Assuming that $T > 0$ is the minimal period of the time dependence of $H(t, x)$, by subharmonic solution of (1.1) we mean a kT-periodic solution, where k is any integer; when moreover the periodic solution is not T-periodic we call it a true subharmonic.

Using variational methods, there have been various types of results concerning the existence of subharmonic solutions for Hamiltonian systems of first and second order. Most of these works treats the case of second order. Few results have been obtained for the case of the first order, include for example [1, 2, 3, 5, 8, 10, 13, 15]. Many solvability conditions are given, such as the convexity condition (see [2, 5, 13, 15]), the super-quadratic condition (see [3, 6, 8]), the sub-quadratic condition (see

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[10]), the bounded nonlinearity condition (see [12]) and the sub-linear condition (see [1]). In particular, under the assumptions that H satisfies

$$|H'(t,x)| \le f(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T], \tag{1.2}$$

$$H(t,x) \to +\infty \quad \text{as } |x| \to \infty, \text{ uniformly in } t \in [0,T], \tag{1.3}$$

where $f \in L^1(0,T;\mathbb{R}^+)$. It was shown in [12] that (1.1) has subharmonic solutions. In 2007, Daouas and Timoumi [1] generalized these results with a condition of sub-linear growth. Precisely, it was assumed that the nonlinearity satisfies the following assumptions: There exist $\alpha \in [0,1[$, $f \in L^{\frac{2}{1-\alpha}}(0,T;\mathbb{R}^+)$ and $g \in L^2(0,T;\mathbb{R}^+)$ such that

$$|H'(t,x)| \le f(t)|x|^\alpha + g(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T]; \tag{1.4}$$

$$\lim_{|x|\to\infty} \frac{1}{|x|^{2\alpha}} \int_0^T H(t,x) = +\infty; \tag{1.5}$$

There exist a subset C of $[0,T]$ with $\text{meas}(C) > 0$ and $h \in L^1(0,T;\mathbb{R})$ such that

$$\begin{gathered} \lim_{|x|\to\infty} H(t,x) = +\infty, \quad \text{a.e. } t \in C, \\ H(t,x) \ge h(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T]. \end{gathered} \tag{1.6}$$

Under these conditions, subharmonic solutions of (1.1) have been obtained.

In all the results discussed above [1, 12], the nonlinearity is required to grow at infinity at most like $|x|^\alpha$. Consider the Hamiltonian

$$H(t,x) = \gamma(t)\frac{|x|^2}{\ln(2+|x|^2)}, \quad t \in [0,T], \ x \in \mathbb{R}^{2N} \tag{1.7}$$

where

$$\gamma(t) = \begin{cases} \sin(2\pi t/T), & t \in [0,T/2] \\ 0, & t \in [T/2,T], \end{cases}$$

It is easy to verify that H does not satisfy (1.2), (1.3) nor (1.4), (1.5), (1.6). In 2012, Timoumi [14] established the existence of subharmonic solutions for the second-order Hamiltonian system

$$\ddot{u}(t) + V'(t,u(t)) = 0$$

under some conditions which cover a case analogous to (1.7). The goal of the present article is to obtain the existence of periodic and subharmonic solutions for first-order Hamiltonian system (1.1) under conditions covering the case of (1.7) and such that the nonlinearity has a growth at infinity faster than $|x|^\alpha$, $0 \le \alpha < 1$. Our main results are the following theorems.

Theorem 1.1. *Let $\omega \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a non-increasing function with the following properties:*

(a) $\liminf_{s\to\infty} \frac{\omega(s)}{\omega(\sqrt{s})} > 0$,
(b) $\omega(s) \to 0$, $\omega(s)s \to \infty$ *as* $s \to \infty$.

Assume that H satisfies

(H1) *There exist a positive constant a and a function $g \in L^2(0,T;\mathbb{R})$ such that*

$$|H'(t,x)| \le a\omega(|x|)|x| + g(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T];$$

(H2)
$$\frac{1}{[\omega(|x|)|x|]^2}\int_0^T H(t,x)dt \to +\infty \quad \textit{as } |x| \to \infty.$$
Then system (1.1) *possesses at least one T-periodic solution.*

Example 1.2. Consider the Hamiltonian
$$H(t,x) = (\frac{1}{2} + \cos(\frac{2\pi}{T}t))\frac{|x|^2}{\ln(2+|x|^2)}, \quad t \in [0,T],\ x \in \mathbb{R}^{2N},$$
and let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ defined by
$$\omega(s) = \frac{1}{\ln(2+s^2)},\ s \geq 0.$$
A simple computation shows that ω is a continuous non-increasing function satisfying the conditions $(a), (b)$ and the Hamiltonian H satisfies assumptions (H1), (H2) of Theorem 1.1 and does not satisfy the conditions (1.2), (1.3) nor (1.4), (1.5).

Theorem 1.3. *Assume that H satisfies* (H1), (H2) *and*

(H3) *There exist a subset C of $[0,T]$ with* $\operatorname{meas}(C) > 0$ *and a T-periodic function $h \in L^1(0,T;\mathbb{R})$ such that*
  - (i) $\lim_{|x|\to\infty} H(t,x) = +\infty$, *a.e.* $t \in C$,
  - (ii) $H(t,x) \geq h(t)$, *for all* $x \in \mathbb{R}^{2N}$, *a.e.* $t \in [0,T]$.

Then, for all integer $k \geq 1$, Equation (1.1) *possesses a kT-periodic solution x_k such that*
$$\lim_{k\to\infty} \|x_k\|_\infty = +\infty, \tag{1.8}$$
where $\|x\|_\infty = \sup_{t\in\mathbb{R}} |x(t)|$.

Corollary 1.4. *Assume that H satisfies* (H1) *and*

(H4) *There exist a subset C of $[0,T]$ with* $\operatorname{meas}(C) > 0$ *and a T-periodic function $h \in L^1(0,T;\mathbb{R})$ such that*
  - (i) $\lim_{|x|\to\infty} \frac{H(t,x)}{[\omega(|x|)|x|]^2} = +\infty$, *a.e.* $t \in C$,
  - (ii) $H(t,x) \geq h(t)$ *for all* $x \in \mathbb{R}^{2N}$, *a.e.* $t \in [0,T]$.

Then the conclusion of Theorem 1.3 holds.

Example 1.5. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as in example 1.2. It is easy to see that the Hamiltonian H defined in (1.7) satisfies assumptions (H1), (H2) and (H3) of Theorem 1.3 with $C =]0, T/4[$ and does not satisfy the conditions (1.2), (1.3) nor (1.4), (1.5), (1.6).

Concerning the minimality of periods, we have the following result.

Theorem 1.6. *Assume that H satisfies* (H1) *and*

(H5) $\frac{H'(t,x).x}{[\omega(|x|)|x|]^2} \to +\infty$ *as* $|x| \to \infty$ *uniformly in* $t \in [0,T]$.

Then, for all integer $k \geq 1$, (1.1) *possesses a kT-periodic solution x_k such that* $\lim_{k\to\infty} \|x_k\|_\infty = +\infty$.

Moreover, if the following assumption holds

(H1') If $u(t)$ is a periodic function with minimal period rT, r rational, and $H'(t,u(t))$ is a periodic function with minimal period rT, then r is necessarily an integer, then, for any sufficiently large prime number k, kT is the minimal period of x_k.

Here, $x.y$ denotes the usual inner product of $x, y \in \mathbb{R}^{2N}$.

Example 1.7. The Hamiltonian

$$H(t,x) = (\frac{3}{2} + \cos(\frac{2\pi}{T}t))\frac{|x|^2}{\ln(2+|x|^2)},$$

satisfies (H1), (H5) and (H1') with $\omega(s) = \frac{1}{\ln(2+s^2)}$.

Remark 1.8. Let $x(t)$ be a periodic solution of (1.1). By replacing t by $-t$ in (1.1), we obtain

$$J\dot{x}(-t) + H'(-t, x(-t)) = 0.$$

So, it is clear that the function $y(t) = x(-t)$ is a periodic solution of the system

$$J\dot{y}(t) - H'(-t, y(t)) = 0.$$

Moreover, $-H(-t,x)$ satisfies (H2), (H3), (H4), (H5) whenever $H(t,x)$ satisfies respectively the following assumptions:

(H2')
$$\frac{1}{[\omega(|x|)|x|]^2}\int_0^T H(t,x)dt \to -\infty \quad \text{as } |x| \to \infty;$$

(H3') There exist a subset C of $[0,T]$ with $\text{meas}(C) > 0$ and a T-periodic function $h \in L^1(0,T;\mathbb{R})$ such that
- (i) $\lim_{|x|\to\infty} H(t,x) = -\infty$, a.e. $t \in C$,
- (ii) $H(t,x) \le h(t)$ for all $x \in \mathbb{R}^{2N}$, a.e. $t \in [0,T]$;

(H4') There exist a subset C of $[0,T]$ with $\text{meas}(C) > 0$ and a T-periodic function $h \in L^1(0,T;\mathbb{R})$ such that
- (i) $\lim_{|x|\to\infty} \frac{H(t,x)}{[\omega(|x|)|x|]^2} = -\infty$, a.e. $t \in C$,
- (ii) $H(t,x) \le h(t)$ for all $x \in \mathbb{R}^{2N}$, a.e. $t \in [0,T]$;

(H5') $\frac{H'(t,x).x}{[\omega(|x|)|x|]^2} \to -\infty$ as $|x| \to \infty$, uniformly in $t \in [0,T]$.

Consequently, we can replace (H2)–(H5) in the above results by (H2')–(H5') and obtain the same results.

2. Preliminaries

Let $L^2(S^1, \mathbb{R}^{2N})$ denote the set of $2N$-tuples of T-periodic functions which are square integrable on $S^1 = \mathbb{R}/T\mathbb{Z}$. If $x \in L^2(S^1, \mathbb{R}^{2N})$, then x has a Fourier expansion

$$x(t) \simeq \sum_{m\in\mathbb{Z}} \exp\big(2\pi mtJ/T\big)\hat{x}_m$$

where $\hat{x}_m \in \mathbb{R}^{2N}$ and $\sum_{m\in\mathbb{Z}} |\hat{x}_m|^2 < \infty$. Consider the Sobolev space

$$E = H_T^{1/2} = \{x \in L^2(S^1, \mathbb{R}^{2N}) : \|x\|_{H_T^{1/2}} < \infty\}$$

where

$$\|x\|_{H_T^{1/2}} = [\sum_{m\in\mathbb{Z}} (1+|m|)|\hat{x}_m|^2]^{1/2}.$$

It is well known that the space $(E, \|\cdot\|_{H_T^{1/2}})$ is a Hilbert space. For $x \in E$, let

$$Q(x) = \frac{1}{2}\int_0^T J\dot{x}(t)\cdot x(t)dt.$$

Note that Q is a quadratic form and by an easy calculation we obtain

$$Q(x) = -\pi \sum_{m\in\mathbb{Z}} m|\hat{x}_m|^2, \quad \forall x \in E. \tag{2.1}$$

Therefore, Q is a continuous quadratic form on E.

Consider the subspaces of E:

$$\begin{gathered} E^0 = \mathbb{R}^{2N}, \\ E^- = \{x \in E : x(t) \simeq \sum_{m\geq 1} \exp\big(2\pi mtJ/T\big)\hat{x}_m, \text{ a.e. } t \in [0,T]\}, \\ E^+ = \{x \in E : x(t) \simeq \sum_{m\leq -1} \exp\big(2\pi mtJ/T\big)\hat{x}_m, \text{ a.e. } t \in [0,T]\}, \end{gathered}$$

here E^0 denotes the space of constant functions. Then $E = E^0 \oplus E^+ \oplus E^-$. In fact it is not difficult to verify that E^+, E^-, E^0 are respectively the subspaces of E on which Q is positive definite, negative definite and null; and these subspaces are orthogonal with respect to the bilinear form

$$B(x,y) = \frac{1}{2}\int_0^T J\dot{x}(t)\cdot y(t)dt, \quad x, y \in E,$$

associated with Q. If $x \in E^+$ and $y \in E^-$, then $B(x,y) = 0$ and $Q(x+y) = Q(x) + Q(y)$. It is also easy to check that E^+, E^- and E^0 are mutually orthogonal in $L^2(S^1, \mathbb{R}^{2N})$. It follows that if $x = x^+ + x^- + x^0 \in E$, then the expression

$$\|x\| = [Q(x^+) - Q(x^-) + |x^0|^2]^{1/2} \tag{2.2}$$

is an equivalent norm in E. So in the following, we will use the norm defined in (2.2) as the norm for E. The subspaces E^+, E^-, E^0 are mutually orthogonal with respect to the associated inner product.

To prove our main results, the following auxiliary result will be needed.

Proposition 2.1 ([8]). *For all $s \in [1,\infty[$, the space E is compactly embedded in $L^s(S^1, \mathbb{R}^{2N})$. In particular there is a constant $\alpha_s > 0$ such that for all $x \in E$,*

$$\|x\|_{L^s} \leq \alpha_s \|x\|.$$

The following result is a version of the saddle point theorem.

Theorem 2.2 ([10]). Let $E = E^1 \oplus E^2$ be a real Hilbert space with $E^2 = (E^1)^\perp$. Suppose that $f \in C^1(E, \mathbb{R})$ satisfies

- (a) $f(u) = \frac{1}{2}(Lu,u) + d(u)$ and $Lu = L_1P_1u + L_2P_2u$, with $L_i : E^i \to E^i$ bounded and self-adjoint, $i = 1, 2$;
- (b) $d'(u)$ is compact;
- (c) There exists $\beta \in \mathbb{R}$ such that $f(u) \leq \beta$, for all $u \in E^1$;
- (d) There exists $\gamma \in \mathbb{R}$ such that $f(u) \geq \gamma$ for all $u \in E^2$.

Furthermore, if f satisfies the Palais-Smale condition $(PS)_c$ for all $c \geq \gamma$, then f possesses a critical value $c \in [\gamma, \beta]$.

Remark 2.3. In (c) we may replace E^2 by $\varphi_1 + E^2$, $\varphi_1 \in E^1$. Likewise in (b) we may have $\varphi_2 + E^1$, $\varphi_2 \in E^2$, in place of E^1.

3. Proofs of theorems

By making the change of variables $t \to t/k$, transforms system (1.1) into

$$J\dot{u}(t) + kH'(kt, u(t)) = 0. \tag{3.1}$$

Hence, to find kT-periodic solutions of (1.1), it suffices to find T-periodic solutions of $(\mathcal{H}_k)$. Consider the family of functionals $(\Phi_k)_{k\in\mathbb{B}}$ defined on the space E introduced above by

$$\Phi_k(u) = \int_0^T [\frac{1}{2}J\dot{u}(t)\cdot u(t) + kH(kt, u(t))]dt.$$

Since ω is bounded, we have by (H1)

$$|H'(t,x)| \le a \sup_{s\ge 0} \omega(s)|x| + g(t). \tag{3.2}$$

So, by [8, Proposition B.37], $\Phi_k \in C^1(E,\mathbb{R})$ and critical points of Φ_k on E correspond to the T-periodic solutions of $(\mathcal{H}_k)$, moreover one has

$$\Phi_k'(u)v = \int_0^T [J\dot{u}(t) + kH'(kt, u(t))]\cdot v(t)dt, \quad \forall u, v \in E. \tag{3.3}$$

The following lemma will be needed for the study of the geometry of the functionals Φ_k.

Lemma 3.1. *Assume that (H1), (H2) hold, then there exist a non-increasing function $\theta \in C(]0,\infty[,\mathbb{R}^+)$ and a positive constant c_0 satisfying the following conditions*

- (i) *$\theta(s) \to 0$, $\theta(s)s \to +\infty$ as $s \to \infty$,*
- (ii) *$\|H'(t,u)\|_{L^2} \le c_0[\theta(\|u\|)\|u\| + 1]$ for all $u \in E$,*
- (iii)

$$\frac{1}{[\theta(|u^0|)|u^0|]^2}\int_0^T H(t,u^0)dt \to +\infty \quad \text{as } |u^0| \to +\infty.$$

The proof of the above lemma is similar to that of [14, Lemma 2.1], and it is ommitted here. Now, we show that, for every positive integer k, one can find a critical point u_k of Φ_k. To this aim, we will apply the saddle point theorem to each of the Φ_k's. Let us fix k and consider the subspaces $E^1 = E^-$, $E^2 = E^0 \oplus E^+$ of E. First, we prove the Palais-Smale.

Lemma 3.2. *Assume that* (H1) *and* (H2) *hold. Then for every integer $k \ge 1$, the functional Φ_k satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a sequence of E such that $(\Phi_k(u_n))$ is bounded and $\Phi_k'(u_n) \to 0$ as $n \to \infty$. Let us denote $\tilde{u}_n = u_n^+ + u_n^-$. We have

$$\Phi_k'(u_n)(u_n^+ - u_n^-) = 2\|\tilde{u}_n\|^2 + k\int_0^T H'(kt, u_n)\cdot(u_n^+ - u_n^-)dt. \tag{3.4}$$

Since θ is non-increasing and $\|u\| \ge \max(|u^0|, \|\tilde{u}\|)$, we obtain

$$\theta(\|u\|) \le \min(\theta(|u^0|), \theta(\|\tilde{u}\|)). \tag{3.5}$$

So by Hölder's inequality, Proposition 2.1, Lemma 3.1 (ii) and property (3.5), we obtain a positive constant c_2 such that

$$\begin{aligned} \big|k\int_0^T H'(kt,u_n).(u_n^+-u_n^-)dt\big| &\leq k\|u_n^+-u_n^-\|_{L^2}[\int_0^T |H'(kt,u_n)|^2dt]^{1/2} \\ &\leq c_2\|\tilde{u}_n\|[\theta(\|u_n\|)\|u_n\|+1] \\ &\leq c_2\|\tilde{u}_n\|\big[\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\|+\theta(|u_n^0|)|u_n^0|+1\big]. \end{aligned} \tag{3.6}$$

Thus, for n large enough

$$\begin{aligned} \|\tilde{u}_n\| &\geq \Phi_k'(u_n)(u_n^+-u_n^-) \\ &\geq 2\|\tilde{u}_n\|^2 - c_2\Big[\|\tilde{u}_n\|[\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\|+\theta(|u_n^0|)|u_n^0|+1\Big] \end{aligned}$$

and then

$$c_2\theta(|u_n^0|)|u_n^0| \geq \|\tilde{u}_n\|[2-c_2\theta(\|\tilde{u}_n\|)]-c_2-1. \tag{3.7}$$

Assume that $(\|\tilde{u}_n\|)$ is unbounded, then by going to a subsequence, if necessary, we can assume that $\|\tilde{u}_n\|\to\infty$ as $n\to\infty$. Since $\theta(s)\to 0$ as $s\to\infty$, we deduce from (3.7) that there exists a positive constant c_3 such that

$$\|\tilde{u}_n\| \leq c_3[\theta(|u_n^0|)|u_n^0|+1] \tag{3.8}$$

for n large enough. Since ω is bounded, then $|u_n^0|\to\infty$ as $n\to\infty$. Now, by the mean value theorem, Hölder's inequality and Lemma 3.1 (ii), we obtain

$$\begin{aligned} &|\int_0^T (H(kt,u_n)-H(kt,u_n^0))dt| \\ &= |\int_0^T\int_0^1 H'(kt,u_n^0+s\tilde{u}_n).\tilde{u}_n\,ds\,dt| \\ &\leq \|\tilde{u}_n\|_{L^2}\int_0^1\Big(\int_0^T |H'(kt,u_n^0+s\tilde{u}_n)|^2dt\Big)^{1/2}ds \\ &\leq c_0\|\tilde{u}_n\|_{L^2}\int_0^1[\theta(\|u_n^0+s\tilde{u}_n\|)\|u_n^0+s\tilde{u}_n\|+1]ds. \end{aligned} \tag{3.9}$$

Since θ is non-increasing and $\|u_n^0+s\tilde{u}_n\|\geq|u_n^0|$ for all $s\in[0,1]$, we deduce from Proposition 2.1, (3.8) and (3.9) that there exists a constant $c_4>0$ such that for n large enough

$$\begin{aligned} k|\int_0^T (H(kt,u_n)-H(kt,u_n^0))dt| &\leq c_0\|\tilde{u}_n\|_{L^2}\Big[\theta(|u_n^0|)|u_n^0|+\theta(|u_n^0|)\|\tilde{u}_n\|+1\Big] \\ &\leq c_4[\theta(|u_n^0|)|u_n^0|]^2, \end{aligned} \tag{3.10}$$

so there exists a positive constant c_5 such that for n large enough,

$$\begin{aligned} \Phi_k(u_n) &= \|u_n^+\|^2-\|u_n^-\|^2+k\int_0^T (H(kt,u_n)-H(kt,u_n^0))dt+k\int_0^T H(kt,u_n^0)dt \\ &\geq -\|\tilde{u}_n\|^2-c_4[\theta(|u_n^0|)|u_n^0|]^2+k\int_0^T H(kt,u_n^0)dt \\ &\geq [\theta(|u_n^0|)|u_n^0|]^2\Big[-c_5+\frac{k}{[\theta(|u_n^0|)|u_n^0|]^2}\int_0^T H(kt,u_n^0)dt\Big]. \end{aligned}$$

By Lemma 3.1, this implies $\Phi_k(u_n) \to +\infty$ as $n \to \infty$ and contradicts the boundedness of $(\Phi_k(u_n))$. So $(\|\tilde{u}_n\|)$ is bounded.

Assume that $(|u_n^0|)$ is unbounded. Up to a subsequence, if necessary, we can assume that $|u_n^0| \to \infty$, as $n \to \infty$. As above, we obtain for n large enough

$$\begin{aligned}|k \int_0^T (H(kt, u_n) - H(kt, u_n^0))dt| &\le c_0 \|\tilde{u}_n\|_{L^2} \big[\theta(|u_n^0|)|u_n^0| + \theta(|u_n^0|)\|\tilde{u}_n\| + 1\big] \\ &\le c_6 \theta(|u_n^0|)|u_n^0|\end{aligned}$$

where c_6 is a positive constant. So there exists a constant $c_7 > 0$ such that for n large enough

$$\begin{aligned}\Phi_k(u_n) &\ge -c_7 \theta(|u_n^0|)|u_n^0| + k \int_0^T H(kt, u_n^0)dt \\ &\ge [\theta(|u_n^0|)|u_n^0|]^2 \Big(- \frac{c_7}{\theta(|u_n^0|)|u_n^0|} + \frac{k}{[\theta(|u_n^0|)|u_n^0|]^2} \int_0^T H(kt, u_n^0)dt \Big)\end{aligned} \tag{3.11}$$

so $\Phi_k(u_n) \to +\infty$ as $n \to \infty$, which also contradicts the boundedness of $(\Phi_k(u_n))$. Then $(|u_n^0|)$ is also bounded and therefore $(\|u_n\|)$ is bounded. By a standard argument, we conclude that (u_n) possesses a convergent subsequence. The proof is complete. □

Now, let $u = u^0 + u^+ \in E^2 = E^0 \oplus E^+$ with $u^0 \ne 0$, then as in (3.15) there exists a positive constant c_8 such that

$$k| \int_0^T (H(kt, u) - H(kt, u^0))dt| \le c_8 \|u^+\| \big[\theta(|u^0|)|u^0| + \theta(|u^0|)\|u^+\| + 1\big]. \tag{3.12}$$

So

$$\Phi_k(u) \ge \|u^+\|^2 - c_8 \|u^+\| \Big[\theta(|u^0|)|u^0| + \theta(|u^0|)\|u^+\| + 1\Big] + k \int_0^T H(kt, u^0)dt. \tag{3.13}$$

Let $0 < \epsilon < 1$, Then we have

$$c_8 \theta(|u^0|)|u^0| \|u^+\| \le \frac{1}{\epsilon^2} c_8^2 [\theta(|u^0|)|u^0|]^2 + \epsilon^2 \|u^+\|^2. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned}\Phi_k(u) \ge \big[1 - \epsilon^2 - c_8 \theta(|u^0|)\big] \|u^+\|^2 - c_8 \|u^+\| + [\theta(|u^0|)|u^0|]^2 \\ \times \Big[- \frac{c_8^2}{\epsilon^2} + \frac{k}{[\theta(|u^0|)|u^0|]^2} \int_0^t H(kt, u^0)dt \Big]\end{aligned}$$

so

$$\Phi_k(u) \to +\infty \quad \text{as } \|u\| \to \infty, \ u \in E^2. \tag{3.15}$$

On the other hand, let $\xi \in \mathbb{R}^{2N}$, $|\xi| > 0$. By the mean value theorem, for $u \in E^1 = E^-$, we have

$$\begin{aligned}
&|\int_0^T (H(kt,u) - H(kt,\xi))dt| \\
&= |\int_0^T \int_0^1 H'(kt, \xi + s(u-\xi)).(u-\xi)\,ds\,dt| \\
&\le \|u-\xi\|_{L^2} \int_0^1 \Big[\int_0^T |H'(kt,\xi+s(u-\xi))|^2 dt\Big]^{1/2} ds \\
&\le \|u-\xi\|_{L^2} \int_0^1 \Big[\int_0^T \Big(a\omega(|\xi+s(u-\xi)|)|\xi+s(u-\xi)| + g(kt)\Big)^2 dt\Big]^{1/2} ds \\
&\le \|u-\xi\|_{L^2} \Big[a\int_0^1 \Big(\int_0^T [\omega(|\xi+s(u-\xi)|)|\xi+s(u-\xi)|]^2 dt\Big)^{1/2} ds + \|g\|_{L^2}\Big].
\end{aligned} \tag{3.16}$$

For $s \in [0,1]$,

$$A(s) = \{t \in [0,1] : |\xi + s(u(t) - \xi)| \ge |\xi|\}.$$

By a classical calculation as in the proof of Lemma 3.1, we obtain some positive constants c_9 and $c(\xi)$ such that

$$k|\int_0^T (H(kt,u) - H(kt,\xi))dt| \le c_9\omega(|\xi|)\|u\|^2 + c(\xi)(\|u\|+1). \tag{3.17}$$

This implies

$$\Phi_k(u) \le -\|u\|^2 + c_9\omega(|\xi|)\|u\|^2 + c(\xi)(\|u\|+1) + k\int_0^T H(t,\xi)dt. \tag{3.18}$$

Since $\omega(s) \to 0$ as $s \to \infty$, there exists $|\xi| > 0$ such that $c_9\omega(|\xi|) \le \frac{1}{2}$ and then we have for a constant $c_{10} > 0$,

$$\Phi_k(u) \le -\frac{1}{2}\|u\|^2 + c_{10}(\|u\|+1) + k\int_0^T H(t,\xi)dt;$$

therefore

$$\Phi_k(u) \to -\infty \quad \text{as } \|u\| \to \infty,\ u \in E^1. \tag{3.19}$$

Property (3.2) and [7, Proposition B.37] imply that the derivative of the functional $g_k(u) = k\int_0^T H(kt,u)dt$ is compact. Thus Lemma 3.2 and properties (3.15), (3.19) imply that for all integer $k \ge 1$, Φ_k satisfies all the assumptions of the saddle point theorem and then by Remark 2.3, Φ_k possesses a critical point $u_k \in E$ satisfying

$$C_k = \Phi_k(u_k) \ge \inf_{u\in E^1} \Phi_k(\sqrt{k}\varphi + u) \tag{3.20}$$

where $\varphi(t) = \frac{1}{\sqrt{\pi}}\exp(\frac{2\pi}{T}tJ)e_1$ and e_1 design the first element in the standard basis of $\mathbb{R}^{2N}$. Therefore, for all integer $k \ge 1$, $x_k(t) = u_k(\frac{t}{k})$ is a kT-periodic solution for (1.1). Theorem 1.1 and the first part of Theorem 1.3 are proved.

Proof of Theorem 1.3. It remains to prove (1.8). For this, we will prove that the sequence (u_k) obtained above satisfies

$$\lim_{k\to\infty} \frac{1}{k}\Phi_k(u_k) = +\infty. \tag{3.21}$$

This will be done by using estimates on the levels C_k of Φ_k. For this aim the following two lemmas will be needed.

Lemma 3.3 ([11])**.** *If* (H3) *holds, then for every* $\delta > 0$*, there exists a measurable subset* C_δ *of* C *with* $\operatorname{meas}(C - C_\delta) < \delta$ *such that*

$$H(t,x) \to +\infty \quad \text{as } |x| \to \infty, \text{ uniformly in } t \in C_\delta.$$

Lemma 3.4. *Assume that* H *satisfies* (H3)*, then*

$$\lim_{k\to\infty} \inf_{u\in E^2} \frac{\Phi_k(\sqrt{k}\varphi + u)}{k} = +\infty. \tag{3.22}$$

Proof. Arguing by contradiction, assume that there exist sequences $k_n \to \infty$, $(u_n) \subset E^2$ and a constant c_{11} such that

$$\Phi_{k_n}(\sqrt{k_n}\varphi + u_n) \le k_n c_{11}, \quad \forall n \in \mathbb{N}. \tag{3.23}$$

Taking $u_n = \sqrt{k_n}(u_n^+ + u_n^0)$, we obtain by an easy calculation

$$\Phi_{k_n}(\sqrt{k_n}\varphi + u_n) = k_n[\|u_n^+\|^2 - 1 + \int_0^T H(k_n t, \sqrt{k_n}(\varphi + u_n^+ + u_n^0))dt]. \tag{3.24}$$

On the other hand, by (H3) (ii), we have

$$\int_0^T H(k_n t, \sqrt{k_n}(\varphi + u_n^+ + u_n^0))dt \ge \int_0^T h(k_n t)dt = \int_0^T h(t)dt \tag{3.25}$$

so there exists a positive constant c_{12} such that

$$\Phi_{k_n}(\sqrt{k_n}\varphi + u_n) \ge k_n(\|u_n^+\|^2 - c_{12}). \tag{3.26}$$

Inequalities (3.23) and (3.26) imply that (u_n^+) is a bounded sequence in E. Up to a subsequence, if necessary, we can find $u^+ \in E^+$ such that

$$u_n^+(t) \to u^+(t) \quad \text{as } n \to \infty, \text{ a.e. } t \in [0,T]. \tag{3.27}$$

We claim that (u_n^0) is also bounded in E. Indeed, if we assume otherwise, then by using a subsequence, if necessary, (3.27) implies that

$$|\sqrt{k_n}(\varphi(t) + u_n^+(t) + u_n^0)| \to \infty \quad \text{as } n \to \infty, \text{ a.e. } t \in [0,T]. \tag{3.28}$$

Let $\delta = \frac{1}{2}\operatorname{meas}(C)$ and C_δ be as defined in Lemma 3.3. For all positive integer n, let us define the subset C_δ^n of $[0,T]$ by

$$C_\delta^n = \frac{1}{k_n} \cup_{p=0}^{k_n - 1} (C_\delta + pT).$$

It is easy to verify that $\operatorname{meas}(C_\delta^n) = \operatorname{meas}(C_\delta)$ and

$$H(k_n t, x) \to +\infty \quad \text{as } |x| \to \infty, \text{ uniformly in } t \in [0,T].] \tag{3.29}$$

By (H3) (ii), we have

$$\begin{aligned}
&\int_0^T H(k_n t, \sqrt{k_n}(\varphi + u_n^+ + u_n^0))dt \\
&\ge \int_{C_\delta^n} H(k_n t, \sqrt{k_n}(\varphi + u_n^+ + u_n^0))dt + \int_{[0,T]-C_\delta^n} h(k_n t)dt \\
&\ge \int_{C_\delta^n} H(k_n t, \sqrt{k_n}(\varphi + u_n^+ + u_n^0))dt - \int_o^T |h(t)|dt.
\end{aligned}$$

Therefore, by (3.28) and (3.29), we obtain

$$\int_0^T H(k_n t, \sqrt{k_n}(\varphi + u_n^+ + u_n^0))dt \to +\infty \quad \text{as } n \to \infty. \tag{3.30}$$

We deduce from (3.24) and (3.30) that

$$\frac{\Phi_{k_n}(\sqrt{k_n}\varphi + u_n)}{k_n} \to +\infty \quad \text{as } n \to \infty \tag{3.31}$$

which contradicts (3.23) and proves our claim. Hence, by taking a subsequence, if necessary, we can assume that there exists $u^0 \in E^0$ such that

$$\varphi(t) + u_n^+(t) + u_n^0 \to u(t) = \varphi(t) + u^+(t) + u^0 \quad \text{as } n \to \infty, \text{ a.e. } t \in [0,T]. \tag{3.32}$$

By Fourier analysis, we have $u(t) \neq 0$ for almost every $t \in [0,T]$. Therefore,

$$|\sqrt{k_n}(\varphi(t) + u_n^+(t) + u_n^0)| \to \infty \quad \text{as } n \to \infty, \text{ a.e. } t \in [0,T] \tag{3.33}$$

and by using (3.24), (3.29) and (3.33), we obtain (3.31) as above, which contradicts (3.23). This concludes the proof of Lemma 3.4. □

By (3.20) and Lemma 3.4, we have

$$\frac{C_k}{k} \to \infty \ as\ k \to \infty. \tag{3.34}$$

We claim that $\|u_k\|_\infty \to \infty$ as $k \to \infty$. Indeed, if we suppose otherwise, (u_k) possesses a bounded subsequence (u_{k_p}). Since

$$\frac{\Phi_{k_p}(u_{k_p})}{k_p} = -\frac{1}{2}\int_0^T H'(k_p t, u_{k_p}) \cdot u_{k_p} dt + \int_0^T H(k_p t, u_{k_p})dt \tag{3.35}$$

the sequence (C_{k_p}/k_p) is bounded, which contradicts (3.34). Consequently, we have $\|u_k\|_\infty \to \infty$ *as* $k \to \infty$. For $k \geq 1$, define $x_k(t) = u_k(t/k)$. Then x_k is a kT-periodic solution of the system (1.1) satisfying:

$$\|x_k\|_\infty = \|u_k\|_\infty \to \infty \quad \text{as } k \to \infty \tag{3.36}$$

which completes the proof of Theorem 1.3. □

For the proof of Theorem 1.6, we need the following lemma.

Lemma 3.5. *Let* (H1) *and* (H5) *hold. Then for all* $\rho > 0$*, there exists a constant* $c_\rho \geq 0$ *such that for all* $x \in \mathbb{R}^{2N}$, $|x| > 1$ *and for a.e.* $t \in [0,T]$,

$$H(t,x) \geq H(t,0) + \frac{\rho}{2}[\omega(|x|)|x|]^2(1 - \frac{1}{|x|^2}) - c_\rho \ln(|x|) - \frac{1}{2}a \sup_{r\geq 0} \omega(r) - g(t). \tag{3.37}$$

The proof of the above lemma is similar to that of [14, Lemma 2.5], it is omitted here.

Proof of Theorem 1.3. Since $a_0 = \liminf_{s\to\infty} \frac{\omega(s)}{\omega(s^{1/2})} > 0$, then for s large enough, we have

$$\frac{1}{\omega(s)} \leq \frac{1}{a_0 \omega(\sqrt{s})}. \tag{3.38}$$

this implies that for $|x|$ large enough

$$\frac{\ln(|x|)}{[\omega(|x|)|x|]^2} \leq \frac{\ln(|x|)}{|x|}\frac{1}{a_0^2[\omega(|x|^{1/2})|x|^{1/2}]^2} \to 0 \ as\ |x| \to \infty. \tag{3.39}$$

Combining (3.36), (3.39) we obtain (H4). By applying Corollary 1.4, we obtain a sequence (x_k) of kT-periodic solutions of (1.1) such that $\lim_{k\to\infty}\|x_k\|_\infty=+\infty$.

It remains to prove that, for every sufficiently large prime integer k, x_k has kT as minimal period. Let Ψ_k be the functional defined on the space $H^{1/2}_{kT}$ by

$$\Psi_k(x)=\frac{1}{2}\int_0^{kT}J\dot{x}(t)\cdot x(t)dt+\int_o^{kT}H(t,x(t))dt.$$

It is easy to see that $u\in E=H^{1/2}_T$ is a critical point of Φ_k if and only if $x(t)=u(\frac{t}{k})$ belongs to $H^{1/2}_{kT}$ and is a critical point of Ψ_k. Let $x_k\in H^{1/2}_{kT}$ be the critical point of Ψ_k associated to the critical point u_k of Φ_k obtained above, the property (3.34) is written as

$$\lim_{k\to\infty}\frac{1}{k}\Psi_k(x_k)=+\infty. \tag{3.40}$$

On the other hand, let us denote by S_T the set of T-periodic solutions of (1.1). We claim that S_T is bounded in $H^{1/2}_T$. Indeed, assume by contradiction that there exists a sequence $(x_n)\subset S_T$ such that $\|x_n\|\to\infty$ as $n\to\infty$. Let $x_n=x_n^++x_n^-+x_n^0$. Multiplying both sides of the identity

$$J\dot{x}_n+H'(t,x_n(t))=0 \tag{3.41}$$

by x_n^+ and integrating, we obtain

$$2\|x_n^+\|^2+\int_0^T H'(t,x_n(t))\cdot x_n^+dt=0. \tag{3.42}$$

Using Lemma 3.1 (ii), we can find a positive constant c_{13} such that

$$\|x_n^+\|\le c_{13}(\theta(\|x_n\|)\|x_n\|+1). \tag{3.43}$$

By Lemma 3.1 (i) and (3.43), we obtain

$$\frac{\|x_n^+\|}{\|x_n\|}\to 0\ as\ n\to\infty. \tag{3.44}$$

Similarly, we have

$$\frac{\|x_n^-\|}{\|x_n\|}\to 0\ as\ n\to\infty. \tag{3.45}$$

Taking $y_n=\frac{x_n}{\|x_n\|}$ and using (3.44) and (3.45), we may assume without loss of generality that $y_n\to y_0\in E^0$, with $|y_0|=1$. Since the embedding $E\to L^2$, $u\longmapsto u$ is compact, we can assume, by taking a subsequence if necessary that

$$y_n(t)\to y_0\quad\text{as } n\to\infty,\text{ a.e. } t\in[0,T], \tag{3.46}$$

and consequently

$$|x_n(t)|\to+\infty\quad\text{as } n\to\infty,\text{ a.e. } t\in[0,T]. \tag{3.47}$$

Fatou's lemma, property (a) of ω and (3.47) imply

$$\int_0^T[\omega(|x_n(t)|)|x_n(t)|]^2dt\to\infty\quad\text{as } n\to\infty. \tag{3.48}$$

On the other hand, by (H5), for all $\rho>0$, there exists a constant $c_\rho\ge 0$ such that

$$H'(t,x).x\ge\rho[\omega(|x|)|x|]^2-c_\rho \tag{3.49}$$

so we obtain

$$\rho\int_0^T[\omega(|x_n(t)|)|x_n(t)|]^2dt \le \int_0^T H'(t,x_n(t))\cdot x_n(t)dt + c_\rho T. \tag{3.50}$$

Furthermore, by [4, Proposition 3.2], we have

$$\int_0^T H'(t,x_n(t))\cdot x_n(t)dt \le \frac{T}{2\pi}\int_0^T |H'(t,x_n(t))|^2dt \tag{3.51}$$

Combining (3.50) with (3.51), and using assumption (H1), we obtain

$$\begin{aligned}\rho\int_0^T[\omega(|x_n(t)|)|x_n(t)|]^2dt &\le \frac{T}{2\pi}\int_0^T[a\omega(|x_n(t)|)|x_n(t)|+g(t)]^2dt + c_\rho T\\ &\le \frac{T}{\pi}(a^2\int_0^T[\omega(|x_n(t)|)|x_n(t)|]^2dt + \|g\|_{L^2}^2) + c_\rho T.\end{aligned} \tag{3.52}$$

Since $\rho > 0$ is arbitrary chosen, (3.52) implies that $(\int_0^T[\omega(|x_n(t)|)|x_n(t)|]^2dt)$ is bounded, which contradicts (3.48). Hence S_T is bounded and as a consequence $\Psi_1(S_T)$ is bounded. Since for any $x \in S_T$ one has $\Psi_k(x) = k\Psi_1(x)$, then there exists a positive constant c_{14} such that

$$\frac{|\Psi_k(x)|}{k} \le c_{14} \quad \forall x \in S_T,\ \forall k \ge 1\,. \tag{3.53}$$

Consequently, by (3.40) and (3.53), we deduce that, for k sufficiently large, $x_k \notin S_T$. By assumption (H1'), if k is chosen to be prime, the minimal period of x_k has to be kT. This completes the proof of Theorem 1.6. □

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