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SOLVABILITY OF SECOND-ORDER BOUNDARY-VALUE PROBLEMS ON NON-SMOOTH CYLINDRICAL DOMAINS

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Abstract. In this note, we present an abstract approach for the study of a second-order boundary-value problem on cusp domain. This study is performed in the framework of anisotropic little Hölder spaces. Our strategy is to use of the commutative version of the well known sums of operators theory. This technique allows us to obtain the space regularity of the unique strict solution for our problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ a cusp domain defined by

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < d_0,\, \left(\frac{x_1}{(x_3)^\alpha}, \frac{x_2}{(x_3)^\alpha}\right) \in \Omega_0\right\},$$

where $\Omega_0 \subset \mathbb{R}^2$ is a smooth domain of class C^∞, $\alpha > 1$ and $d_0 > 0$. This article concerns the solvability of the boundary-value problem of the second-order differential equation

$$\partial_t^2 u(t,x) + \Delta u(t,x) - \lambda u(t,x) = h(t,x), \quad (t,x) \in \Pi =]0,1[\times\Omega, \tag{1.1}$$

subject to the following boundary value conditions

$$\begin{gathered} a(x)u(0,x) - b(x)\partial_t u(0,x) = 0 \quad x \in \Omega, \\ u(1,x) = 0 \quad x \in \Omega, \\ u(t,x) = 0 \quad (t,x) \in]0,1[\times\partial\Omega. \end{gathered} \tag{1.2}$$

Here, $x = (x_1, x_2, x_3)$ represents a generic point of $\mathbb{R}^3$ and λ is a fixed positive spectral parameter.

The main assumptions on the functions a and b are

$$a, b > 0, \quad a, b \in C^1(\overline{\Omega}).$$

We are especially interested with the case when the right hand side of (1.1) is taken in the anisotropic little Hölder space

$$h^{2\nu,2\sigma}(\overline{\Pi}) = h^{2\nu}([0,1]; h^{2\sigma}(\overline{\Omega})), \quad \nu, \sigma \in]0, 1/2[,$$

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more details about these spaces are given in Section 2. We assume also that the right hand side h satisfies the condition
$$h = 0 \quad \text{on } \partial\Pi. \tag{1.3}$$
Note that in our situation, the classical arguments such as the variational method does not apply. Consequently, we opt for the use of the technique of the sum of linear operators. Fore more details about this technique, we refer the reader to [6, 7, 10, 11, 12]. In the literature, we find several regularity results concerning elliptic and parabolic problems which have been obtained via this technique, see [1, 2, 8, 9]. In this paper, we will use the commutative version developed in [6]. Our main result on the existence, uniqueness and regularity of the strict solution of (1.1)-(1.2) is stated in the following theorem.

Theorem 1.1. *Let $h \in h^{2\nu,2\sigma}(\overline{\Pi})$ with $\nu, \sigma \in]0, 1/2[$, satisfying Assumption (1.3). Then, under conditions (1.2), Problem (1.1) has a unique strict solution u such that*
$$(x_3)^{4\sigma\alpha+2\alpha}\partial_t^2 u \text{ and } (x_3)^{4\sigma\alpha}(\Delta - \lambda)u \in h^{2\nu,2\sigma}(\overline{\Pi}).$$

This article is organized as follows: In section 2, we introduce the necessary notation and some definitions related to the functional framework of anisotropic little Hölder spaces. In section 3, we recall the main results of the sum's operators theory. In section 4, using a suitable change of variables our concrete problem is transformed into a new one posed in a cylindrical domain. Next, thanks to the sums technique, we will give a complete study of our transformed problem which allows us to justify our main result.

2. Little Hölder spaces

We briefly recall the definition of the anisotropic little Hölder spaces. We will denote by $C_b^{2\sigma}(\overline{\Omega})$ the space of the bounded and 2σ-Hölder continuous functions defined on $\overline{\Omega}$. The little Hölder space $h^{2\sigma}(\overline{\Omega})$ is defined by
$$h^{2\sigma}(\overline{\Omega}) = \Big\{ f \in C_b^{2\sigma}(\overline{\Omega}) : \lim_{\varepsilon \to 0^+} \sup_{x' \neq x} \frac{|f(x') - f(x)|}{\|x' - x\|^{2\sigma}} = 0 \Big\},$$
endowed with the norm
$$\|u\|_{h^{2\sigma}(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)| + \sup_{x' \neq x} \frac{|f(x') - f(x)|}{\|x' - x\|^{2\sigma}}.$$
The anisotropic little Hölder space $h^{2\nu,2\sigma}(\overline{\Pi})$ is defined by
$$h^{2\nu,2\sigma}(\overline{\Pi}) = \big\{ f \in C^{2\sigma}([0,1]; h^{2\sigma}(\overline{\Omega})) : \lim_{\varepsilon \to 0^+} \sup_{0 < |t - t'| \leq \varepsilon} \frac{\|f(t) - f(t')\|_{h^{2\sigma}(\overline{\Omega})}}{|t - t'|^{2\nu}} = 0 \big\}.$$
We endow $h^{2\nu,2\sigma}(\overline{\Pi})$ with the norm
$$\|u\|_{h^{2\nu,2\sigma}(\overline{\Pi})} = \max_{t \in [0,1]} \|u(t)\|_{h^{2\sigma}(\overline{\Omega})} + \sup_{t' \neq t} \frac{\|u(t') - u(t)\|_{h^{2\sigma}(\overline{\Omega})}}{|t' - t|^{2\nu}},$$
more details about little Hölder spaces are given in [13, 14].

Remark 2.1. It is necessary to note that any function of $h^{2\nu,2\sigma}(\Pi)$, can be extended to a function of $h^{2\nu,2\sigma}(\overline{\Pi})$. This is why we shall write in the sequel $h^{2\nu,2\sigma}(\Pi)$ or $h^{2\nu,2\sigma}(\overline{\Pi})$.

3. On the sum of linear operators

Let E a complex Banach space and A, B two closed linear operators with domains $D(A)$, $D(B)$. Let L be the operator defined by

$$\begin{gathered} Lu = Bu + Au, \\ u \in D(L) = D(A) \cap D(B). \end{gathered} \tag{3.1}$$

where A and B satisfy the assumptions

(H1) (i) $\rho(A) \supset \Sigma_A = \{\mu : |\mu| \geq r, |\operatorname{Arg}(\mu)| < \pi - \epsilon_A\}$,
$$\|(A - \mu I)^{-1}\|_{L(E)} \leq C_A/|\mu|, \quad \forall \mu \in \Sigma_A;$$
(ii) $\rho(B) \supset \Sigma_B = \{\mu : |\mu| \geq r, |\operatorname{Arg}(\mu)| < \pi - \epsilon_B\}$,
$$\|(B - \mu I)^{-1}\|_{L(E)} \leq C_B/|\mu|, \quad \forall \mu \in \Sigma_B;$$
(iii) $\epsilon_A + \epsilon_B < \pi$;
(iv) $\overline{D(A) + D(B)} = E$.

(H2) for all $\mu_1 \in \rho(A)$ and all $\mu_2 \in \rho(B)$,
$$(A-\mu_1 I)^{-1}(B-\mu_2 I)^{-1} - (B-\mu_2 I)^{-1}(A-\mu_1 I)^{-1} = [(A-\mu_1 I)^{-1}; (B-\mu_2 I)^{-1}] = 0,$$
where $\rho(A)$ and $\rho(B)$ are the resolvent sets of A and B.

The main result proved in [6] reads as follows:

Theorem 3.1. *Let $\varrho \in]0,1[$. Assume* (H1), (H2) *hold and $f \in D_A(\varrho)$. Then, the problem*
$$Au + Bu = f,$$
has a unique strict solution $u \in D(A) \cap D(B)$, given by
$$u = -\frac{1}{2i\pi} \int_\Gamma (B+\mu)^{-1}(A-\mu)^{-1} f \, d\mu,$$
where Γ is a sectorial curve lying in $(\Sigma_A) \cap (\Sigma_{-B})$ oriented from $\infty e^{+i\theta_0}$ to $\infty e^{-i\theta_0}$ with $\epsilon_B < \theta_0 < \pi - \epsilon_A$. Moreover, $Au, Bu \in D_A(\varrho)$.

Remark 3.2. The interpolation spaces $D_A(\rho)$, with $\varrho \in]0,1[$, are defined as follows
$$D_A(\rho) = \{\xi \in E : \lim_{r \to 0^+} \|r^\rho A(A - rI)^{-1}\xi\|_E = 0\};$$
for more details, see [13, 14].

4. Applications of the sums theory

4.1. **Change of variables.** As in [1], consider the change of variables $T : \Pi \to Q$,
$$(t, x_1, x_2, x_3) \mapsto (t, \xi_1, \xi_2, \xi_3) = (t, \frac{x_1}{(x_3)^\alpha}, \frac{x_2}{(x_3)^\alpha}, \frac{1}{\alpha - 1}(x_3)^{1-\alpha}),$$
where
$$Q =]0,1[\times D, \quad D = \Omega_0 \times]d_1, +\infty[, \quad d_1 = \frac{1}{\alpha - 1}(d_0)^{1-\alpha} > 0,$$
Let us introduce the following change of functions
$$\begin{gathered} v(t,\xi) = u(t,x), \quad g(t,\xi) = h(t,x), \\ \widetilde{a}(\xi) = a(x), \quad \widetilde{b}(\xi) = b(x). \end{gathered}$$

Consequently, our problem (1.1) becomes

$$\begin{gathered}
\phi(\xi_3)\partial_t^2 v(t,\xi) + [P - \lambda\phi(\xi_3)]v(t,\xi) = f(t,\xi), \quad (t,\xi) \in Q, \\
\widetilde{a}(\xi)v(0,\xi) - \widetilde{b}(\xi)\partial_t v(0,\xi) = 0, \quad \xi \in D, \\
v(1,\xi) = 0, \quad \xi \in D, \\
v(.,\xi) = 0, \quad \xi \in \partial D,
\end{gathered} \tag{4.1}$$

with

$$\xi = (\xi_1, \xi_2, \xi_3), \quad f(t,\xi) = \phi(\xi_3)g(t,\xi), \quad \phi(\xi_3) = (\xi_3)^{\frac{2\alpha}{1-\alpha}}.$$

Here P is the second order differential operator with C^∞-bounded coefficients on $\overline{Q}$ given by

$$\begin{aligned}
Pv(t,\xi) &= (\alpha-1)^{\frac{2\alpha}{\alpha-1}}(\partial_{\xi_1}^2 v + \partial_{\xi_2}^2 v + \partial_{\xi_3}^2 v) \\
&\quad + (\alpha-1)^{\frac{2\alpha}{\alpha-1}}\big\{(\frac{\alpha}{\alpha-1})^2\big\{(\frac{\xi_1}{\xi_3})^2\partial_{\xi_1}^2 v + (\frac{\xi_2}{\xi_3})^2\partial_{\xi_2}^2 v\big\}\big\} \\
&\quad + (\alpha-1)^{\frac{2\alpha}{\alpha-1}}\big\{2(\frac{\alpha}{\alpha-1})^2\frac{\xi_1\xi_2}{(\xi_3)^2}\partial_{\xi_1\xi_2}^2 v\big\} \\
&\quad + (\alpha-1)^{\frac{2\alpha}{\alpha-1}}\big\{\frac{2\alpha}{\alpha-1}\big\{\frac{\xi_1}{\xi_3}\partial_{\xi_1\xi_3}^2 v + \frac{\xi_1}{\xi_3}\partial_{\xi_1\xi_3}^2 v + \frac{\xi_2}{\xi_3}\partial_{\xi_2\xi_3}^2 v\big\}\big\} \\
&\quad + (\alpha-1)^{\frac{2\alpha}{\alpha-1}}\big\{\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\big\{\frac{\xi_1}{(\xi_3)^2}\partial_{\xi_1} v + \frac{\xi_2}{(\xi_3)^2}\partial_{\xi_2} v\big\}\big\} \\
&\quad + (\alpha-1)^{\frac{2\alpha}{\alpha-1}}\big\{\frac{\alpha}{\alpha-1}\frac{1}{\xi_3}\partial_{\xi_3} v\big\}.
\end{aligned} \tag{4.2}$$

Remark 4.1. Observe that the functions $\widetilde{a}$ and $\widetilde{b}$ are necessarily bounded on $\overline{Q}$. In fact, one has

$$\begin{aligned}
|\widetilde{a}(\xi_1,\xi_2,\xi_3)| &= |a(((\alpha-1)\xi_3)^{\frac{\alpha}{1-\alpha}}\xi_1, ((\alpha-1)\xi_3)^{\frac{\alpha}{1-\alpha}}\xi_2, ((\alpha-1)\xi_3)^{\frac{1}{1-\alpha}})| \\
&\le C \max_{(x_1,x_2,x_3)\in\overline{\Omega}} |a(x_1,x_2,x_3)|.
\end{aligned}$$

The following lemma specifies the impact of the change of variables on the functional framework

Lemma 4.2. *Let $\nu, \sigma \in]0, 1/2[$. Then*

1. $h \in h^{2\nu,2\sigma}(\overline{\Pi}) \Rightarrow g \in h^{2\nu,2\sigma}(\overline{Q})$;
2. $h \in h^{2\nu,2\sigma}(\overline{\Pi}) \Rightarrow f \in h^{2\nu,2\sigma}(\overline{Q})$;
3. $f \in h^{2\nu,2\sigma}(\overline{Q}) \Rightarrow (x_3)^{4\sigma\alpha} h \in h^{2\nu,2\sigma}(\overline{\Pi})$;
4. $a \in C^1(\overline{\Omega}) \Rightarrow \widetilde{a} \in C^1(\overline{D})$;
5. $b \in C^1(\overline{\Omega}) \Rightarrow \widetilde{b} \in C^1(\overline{D})$;

For the prof of the above lemma, it suffices to use the same arguments as in [5, Proposition 3.1].

4.2. **Abstract formulation of the transformed problem.** Let $E = h^{2\sigma}(\overline{\Omega})$, we choose $X = C([0,1]; E)$ equipped with its natural norm

$$\|f\|_X = \sup_{0\le t\le 1} \|f\|_E.$$

Let us define the vector-valued functions

$$v : [0,1] \to E;\ t \to v(t); \quad v(t)(\xi) = v(t,\xi),$$

$$f : [0,1] \to E;\ t \to f(t); \quad f(t)(\xi) = f(t,\xi).$$

Let $(P, D(P))$ be the linear operator given by (4.2) where

$$D(P) = \{\psi \in C(\overline{D}) \cap W^{2,q}(D), q \geq 3 : P\psi \in C(\overline{D}),\ \psi = 0 \text{ on } \partial D\}.$$

Now, for $\xi \in D$ and $0 \leq t \leq 1$, define the two operators A and B by

$$\begin{gathered} D(A) = \{v \in X : \phi(\xi_3)\partial_t^2 v \in X,\ \widetilde{a}(\xi)v(0) - \widetilde{b}(\xi)\partial_t v(0) = 0, v(1) = 0\}, \\ (Av)(t) = \phi(\xi_3)\partial_t^2 v(t), \quad \xi_3 \geq d_1, \end{gathered}$$

and

$$\begin{gathered} D(B) = \{v \in X : v(t) \in D(P)\} \\ (Bv)(t) = [P - \lambda\phi(\xi_3)](v(t)). \end{gathered} \tag{4.3}$$

Consequently, the abstract version of Problem (1.1) is

$$Av + Bv = f. \tag{4.4}$$

Proposition 4.3. *The operator B satisfy Assumption* (H1).

Proof. The operator B has the same properties as the operator $P - \lambda\phi(\xi_3)I$. We are then concerned with the study of the spectral problem

$$\begin{gathered} (P - \lambda\phi(\xi_3))v - \mu v = \varphi \in h^\sigma(\overline{D}) \\ v = 0 \quad \text{on } \partial D \end{gathered}$$

with the necessary condition

$$\varphi = 0 \quad \text{on } \partial D. \tag{4.5}$$

Due to [4], there exist $K > 0$ and $C > 0$ such that for $\operatorname{Re}\mu > 0$ one has

$$\|v\|_{h^\sigma(D)} \leq \frac{K}{|C\lambda + \mu| + 1}\|\varphi\|_{h^\sigma(D)} \leq \frac{K}{|\mu| + 1}\|\varphi\|_{h^\sigma(D)}$$

which implies that the operator (4.3) is the generator of an analytic semigroup $(T(s))_{s\geq 0}$ strongly continuous, therefore there exists $\epsilon_B \in]0, \frac{\pi}{2}[$ such that B satisfies (H1). $\square$

Proposition 4.4. *The operator A satisfies Assumption* (H1).

Proof. For simplicity, we use the same argument as in [3]. The study of operator A given by (4.3) is based essentially on the study of the spectral problem

$$\begin{gathered} v''(t) - zv(t) = \phi(t) \\ \widetilde{a}(\xi)v(0) - \widetilde{b}(\xi)\partial_t v(0) = 0, \\ v(1) = 0 \end{gathered} \tag{4.6}$$

For $z \in \mathbb{C} \setminus \mathbb{R}+-$ the unique solution v is

$$v(t) = (A - z)^{-1}\phi = \int_0^1 K_{\sqrt{z}}(t,\xi,s)\varphi(s)ds, \tag{4.7}$$

where

$$K_{\sqrt{z}}(t,\xi,s) = \begin{cases} \frac{\sinh\sqrt{z}(1-t)\ [\widetilde{a}(\xi)\sinh\sqrt{z}s + \widetilde{b}(\xi)\sqrt{z}\cosh\sqrt{z}s]}{\sqrt{z}[\widetilde{a}(\xi)\sinh\sqrt{z} + \widetilde{b}(\xi)\sqrt{z}\cosh\sqrt{z}]} & \text{if } 0 \leq s \leq t \\ \frac{\sinh\sqrt{z}(1-s)\ [\widetilde{a}(\xi)\sinh\sqrt{z}t + \widetilde{b}(\xi)\sqrt{z}\cosh\sqrt{z}t]}{\sqrt{z}[\widetilde{a}(\xi)\sinh\sqrt{z} + \widetilde{b}(\xi)\sqrt{z}\cosh\sqrt{z}]} & \text{if } t \leq s \leq 1, \end{cases}$$

with $\Re\sqrt{z} > 0$. One has

$$
\begin{aligned}
&\Big|\frac{1}{2}\frac{\widetilde{a}(\xi)}{\widetilde{b}(\xi)}\sinh\sqrt{z}(\exp(\sqrt{z})-\exp(-\sqrt{z}))+\frac{\sqrt{z}}{2}(\exp(\sqrt{z})+\exp(-\sqrt{z}))\Big| \\
&\geq \Big|\frac{\widetilde{a}(\xi)}{\widetilde{b}(\xi)}+\operatorname{Re}\sqrt{z}\Big|\sinh\operatorname{Re}\sqrt{z}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Big|\int_0^1 K_{\sqrt{z}}(t,\xi,s)\varphi(s)ds\Big| \\
&\leq \frac{\cosh\operatorname{Re}\sqrt{z}(1-t)\int_0^t[\widetilde{a}(\xi)\cosh\operatorname{Re}\sqrt{z}s+\widetilde{b}(\xi)|\sqrt{z}|\cosh\operatorname{Re}\sqrt{z}s]ds}{\widetilde{b}(\xi)|\sqrt{z}||(\frac{\widetilde{a}(\xi)}{\widetilde{b}(\xi)}+\operatorname{Re}\sqrt{z})|\sinh\operatorname{Re}\sqrt{z}} \\
&\quad+\frac{[\widetilde{a}(\xi)\cosh\operatorname{Re}\sqrt{z}t+\widetilde{b}(\xi)|\sqrt{z}|\cosh\operatorname{Re}\sqrt{z}t]\int_t^1\cosh\operatorname{Re}\sqrt{z}(1-s)ds}{\widetilde{b}(\xi)|\sqrt{z}||(\frac{\widetilde{a}(\xi)}{\widetilde{b}(\xi)}+\operatorname{Re}\sqrt{z})|\sinh\operatorname{Re}\sqrt{z}}
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int_0^1 K_{\sqrt{z}}(t,\xi,s)\varphi(s)ds\Big| &\leq \frac{\widetilde{b}(\xi)(\frac{\widetilde{a}(\xi)}{\widetilde{b}(\xi)}+\operatorname{Re}\sqrt{z})}{\widetilde{b}(\xi)|\sqrt{z}||(\frac{\widetilde{a}(\xi)}{\widetilde{b}(\xi)}+\operatorname{Re}\sqrt{z})|\operatorname{Re}\sqrt{z}|\sqrt{z}|} \\
&\leq \frac{1}{\cos(\theta/2)|z|},
\end{aligned}
$$

which means that Hypothesis (H1) is satisfied with $\epsilon_A \in]0,\pi/2[$. □

Remark 4.5. It is important to note that: 1. Thanks to [13], we have

$$D_A(\nu)=\left\{\varphi\in h^{2\nu}([0,1];E):\varphi(0)=\varphi(1)=0\right\}.$$

2. Hypothesis (H2) is checked in a similar way as in [6] and [12].

Applying the sums technique, we obtain the following maximal regularity results

Proposition 4.6. *Let $f \in h^{2\nu}([0,1];h^{2\sigma}(\overline{D}))$, $\nu,\sigma \in]0;1/2[$. Then, for $\lambda > 0$ Problem* (4.4) *has a unique strict solution v satisfying*

$$
\begin{gathered}
Av \in D_A(\nu) \\
Bv \in D_A(\nu)).
\end{gathered}
$$

As in [8], to prove our main result, that is theorem 1.1, it suffices to use the inverse change of variables $T^{-1}: Q \to \Pi$,

$$(t,\xi_1,\xi_2,\xi_3)\mapsto(t,x_1,x_2,x_3)=(t,((\alpha-1)\xi_3)^{\frac{\alpha}{1-\alpha}}\xi_1,((\alpha-1)\xi_3)^{\frac{\alpha}{1-\alpha}}\xi_2,((\alpha-1)\xi_3)^{\frac{1}{1-\alpha}}).$$

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