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SPREADING OF CHARGED MICRO-DROPLETS

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Abstract. We consider the spreading of a charged microdroplet on a flat dielectric surface whose spreading is driven by surface tension and electrostatic repulsion. This leads to a third order nonlinear partial differential equation that gives the evolution of the height profile. Assuming the droplets are circular we are able to prove existence of solutions with infinite contact angle and in many cases we are able to prove nonexistence of solutions with finite contact angle.

1. Introduction

The interaction between a fluid and an electric field has received much attention recently due to its connection to potential technological applications in microfluidics, inkjet printing, electrospinning, and electrospray ionization. See for example [2]-[4]. The spreading of a charged droplet on an electrically insulating surface has received less attention. However, this is relevant to spray painting of insulating surfaces when the droplets are charged. A natural question is whether it is possible to accelerate the spreading by charging the drops and what the influence of the charge is on the shape of the drop [1].

We will study the spreading of a charged microdroplet using the *lubrication* approximation which assumes that the fluid spreads over a solid surface and that the droplet is thin so that the horizontal component of the velocity is much larger than the vertical component and that the stresses are mostly due to gradients of the velocity in the direction perpendicular to the surface. Using this approximation it is shown in [1] that the height profile $h(r,t)$ of a circular drop satisfies

$$h_t + \frac{1}{r}\frac{\partial}{\partial r}\Big[\frac{r}{3\mu}h^3\frac{\partial}{\partial r}\Big(\frac{Q^2}{2\epsilon_0(4\pi a(t))^2}\frac{1}{a^2(t)-r^2} + \gamma(h_{rr} + \frac{h_r}{r})\Big)\Big] = 0 \tag{1.1}$$

where $a(t)$ is the radius of the drop and the boundary conditions are:

$$h_r(0,t) = h_{rrr}(0,t) = 0 \quad \text{(due to the circular symmetry), and} \tag{1.2}$$

$$h(a(t),t) = 0. \tag{1.3}$$

Here γ is the free surface tension coefficient, ϵ_0 is the permittivity of the gas above the drop, μ is the viscosity, and Q is the total charge.

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We seek a self-similar solution such that the radius of the drop $a(t)$ satisfies a power law; i.e., $a(t) = At^{\beta}$. The height profile will then, by conservation of mass, be of the form

$$h(r,t) = \frac{1}{t^{2\beta}} H\big(\frac{r}{a(t)}\big) = \frac{1}{t^{2\beta}} H\big(\frac{r}{At^{\beta}}\big)$$

where $\rho = r/a(t)$ and $0 \le \rho \le 1$. This then gives for $\beta = 1/10$:

$$\Big[\rho H^3 \Big(H_{\rho\rho} + \frac{H_\rho}{\rho} + \frac{Y}{1-\rho^2}\Big)_\rho\Big]_\rho = Z(\rho^2 H_\rho + 2\rho H)$$

where:

$$Y = \frac{Q^2}{32\pi^2 \epsilon_0 \gamma A^2}, \quad Z = \frac{3\mu A^4}{10\gamma}.$$

Integrating once, using (1.2), and rewriting yields

$$\Big(H'' + \frac{H'}{\rho}\Big)' = \frac{Z\rho}{H^2} - \frac{2Y\rho}{(1-\rho^2)^2} \text{ for } 0 < \rho < 1, \tag{1.4}$$

$$H'(0) = 0, \tag{1.5}$$

$$H(1) = 0. \tag{1.6}$$

Note that Y and Z are positive constants. Also, note that

$$H(\rho) = \sqrt{Z/(2Y)}(1-\rho^2)$$

is one solution of (1.4)-(1.6). A natural question is whether there are other solutions of (1.4)-(1.6).

In attempting to solve (1.4)-(1.6), we first thought of using the *shooting* method. That is, we would solve (1.4) with:

$$H(0) = d > 0, \tag{1.7}$$

$$H'(0) = 0, \tag{1.8}$$

$$H''(0) = k \tag{1.9}$$

where k is arbitrary and then show that if k is sufficiently large then $H > 0$ on $[0,1)$ and if k is sufficiently small then H must have a zero on $[0,1)$. Then making an appropriate choice for k we could show that $H(1) = 0$. Therefore we conjectured that for each d there would be at least one value of k such that H was a solution. However, what we discovered is that the *shooting* method will not work for this problem. In fact, what turns out to be true is the following theorem.

Theorem 1.1. *Let $H \in C^3(\rho_0, 1)$ be a solution of (1.4) such that $0 \le \rho_0 < 1$ and $H(\rho_0) > 0$. Then $H > 0$ on $(\rho_0, 1)$.*

We were able to eventually show that if we look at a slightly different differential equation then it is possible to solve this new problem by the *shooting* method. The key turned out to be to look at the function

$$W = H - \sqrt{Z/(2Y)}(1-\rho^2). \tag{1.10}$$

Using (1.4) it is straightforward to see that

$$\begin{aligned}\Big(W'' + \frac{W'}{\rho}\Big)' &= \frac{-2Y\rho W\big(H + \sqrt{Z/(2Y)}(1-\rho^2)\big)}{H^2(1-\rho^2)^2} \\ &= \frac{-2Y\rho W\big(W + 2\sqrt{Z/(2Y)}(1-\rho^2)\big)}{\big(W + \sqrt{Z/(2Y)}(1-\rho^2)\big)^2(1-\rho^2)^2}\end{aligned} \tag{1.11}$$

for $0 < \rho < 1$. The initial conditions for W are related to (1.7)-(1.9) by (1.10),

$$W(0) = d - \sqrt{Z/(2Y)}, \tag{1.12}$$

$$W'(0) = 0, \tag{1.13}$$

$$W''(0) = k + \sqrt{2Z/Y}. \tag{1.14}$$

Theorem 1.2. *For each $d \geq \sqrt{Z/(2Y)}$ there is a $C^3[0,1)$ solution of (1.11) with $W'(0) = 0$ and $W(1) = 0$. In addition, if $d > \sqrt{Z/(2Y)}$ then $W > 0$ on $[0,1)$ and $W'(1) = -\infty$. (Thus W and hence H have infinite contact angle at $\rho = 1$). If $d = \sqrt{Z/(2Y)}$ then $W \equiv 0$ is a solution of (1.11). (Thus W and hence H have finite contact angle at $\rho = 1$ for this choice of d). So we see that there is a solution of (1.4)-(1.6) for these values of d.*

Note that if $0 < d < \sqrt{Z/(2Y)}$, then it is not clear that the argument we used in the proof of Theorem 1.2 can be extended to these values of d and it is not clear whether (1.4)-(1.6) can be solved for these values of d.

Next we attempted to determine if there are solutions of (1.4)-(1.6) other than $H = \sqrt{Z/(2Y)}(1-\rho^2)$ which have finite contact angle at $\rho = 1$.

Something which seemed feasible was to attempt to find a power series solution of (1.4)-(1.6) centered at $\rho = 1$ in the form

$$H(\rho) = \sum_{n=0}^{\infty} a_n(\rho - 1)^n \tag{1.15}$$

where of course

$$a_n = \frac{H^{(n)}(1)}{n!}.$$

We eventually discovered that requiring H to be smooth on $[0,1]$ and hence with finite contact angle at $\rho = 1$ allows there to be only one solution of (1.4)-(1.6). The following theorem will be restated and proved as Theorem 4.4 in section 4.

Theorem 1.3. *Let $H \in C^\infty[0,1]$ be a solution of (1.4)-(1.6) with $H(0) > 0$. Then*

$$H(\rho) \equiv \sqrt{Z/(2Y)}(1-\rho^2). \tag{1.16}$$

Despite the fact that there are no $C^\infty[0,1]$ solutions of (1.4)-(1.6) which are positive on all of $[0,1)$ other than (1.16), we still thought that there might be a power series solutions of (1.4) and (1.6) on $(1-\epsilon, 1]$ for some $\epsilon > 0$. Interestingly, there are some values of $\frac{Y^{3/2}}{(2Z)^{1/2}}$ which appear to allow power series solutions and others which do not. We will also prove the following result, which will be restated and proved as Theorem 4.5 in section 4.

Theorem 1.4. *Let $\epsilon > 0$. If $n(n-1)(n-2) \neq \frac{Y^{3/2}}{(2Z)^{1/2}}$ for every positive integer n and $H \in C^\infty(1-\epsilon, 1]$ is a solution of (1.4) and (1.6) with H positive on $(1-\epsilon, 1)$ then*

$$H(\rho) \equiv \sqrt{Z/(2Y)}(1-\rho^2).$$

Note: The

Conjecture: Let $\epsilon > 0$. If there is a positive integer $n_0 \geq 3$ such that $n_0(n_0 - 1)(n_0 - 2) = \frac{Y^{3/2}}{(2Z)^{1/2}}$, then there are power series solutions of (1.4) and (1.6) which are positive on $(1-\epsilon, 1)$ other than

$$H(\rho) = \sqrt{Z/(2Y)}(1-\rho^2).$$

What we show here is that a recurrence relation for the a_n in (1.15) can be solved but proving the convergence of the series is not at all clear or obvious.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We suppose by the way of contradiction that there exists $z_0 > 0$ with $\rho_0 < z_0 < 1$ such that $H(z_0) = 0$ and $H(\rho) > 0$ on $[\rho_0, z_0)$. Integrating (1.4) on (ρ_1, ρ) where $\rho_0 < \rho_1$ gives for some constant C_0,

$$H'' + \frac{H'}{\rho} + \frac{Y}{1-\rho^2} = C_0 + \int_{\rho_1}^{\rho} \frac{Zt}{H^2}\, dt. \tag{2.1}$$

Multiplying (2.1) by ρ and integrating on (ρ_1, ρ) gives for some constant C_1,

$$\rho H' = \frac{Y}{2}\ln(1-\rho^2) + C_1\rho^2 + \int_{\rho_1}^{\rho} t \int_{\rho_1}^{t} \frac{sZ}{H^2}\, ds. \tag{2.2}$$

The first two terms on the right-hand side of (2.2) have limits as $\rho \to z_0^-$ (since $z_0 < 1$) and the integral term on the right-hand side is an increasing function. Thus H' is bounded from below and in fact:

$$\lim_{\rho \to z_0^-} H'(\rho) \quad \text{exists (and is possibly } +\infty).$$

However, since $H(z_0) = 0$ and $H(\rho) > 0$ on $[\rho_0, z_0)$ we see that

$$\lim_{\rho \to z_0^-} H'(\rho) = -A \le 0 \quad \text{(and thus } A \text{ is finite).} \tag{2.3}$$

It then follows from L'Hopital's rule that

$$\lim_{\rho \to z_0^-} \frac{H(\rho)}{\rho - z_0} = -A$$

and thus

$$\lim_{\rho \to z_0^-} \frac{H^2(\rho)}{(\rho - z_0)^2} = A^2.$$

Suppose now that $A > 0$. Then there is a ρ_2 with $\rho_0 \le \rho_2 < z_0$ such that

$$H^2 \le 2A^2(\rho - z_0)^2 \text{ for } \rho_2 \le \rho < z_0.$$

Thus for $t \in (\rho_2, z_0)$ we have

$$\int_{\rho_2}^{t} \frac{Zs}{H^2}\, ds \ge \frac{Z\rho_2}{2A^2}\int_{\rho_2}^{t} \frac{1}{(s-z_0)^2}\, ds = \frac{Z\rho_2}{2A^2}\big[\frac{-1}{t-z_0} + \frac{1}{\rho_2 - z_0}\big].$$

Multiplying by t and integrating again gives

$$\begin{aligned} &\int_{\rho_2}^{\rho} t \int_{\rho_2}^{t} \frac{Zs}{H^2}\, ds \ge \frac{Z{\rho_2}^2}{2A^2}\int_{\rho_2}^{\rho} \big[\frac{-t}{t-z_0} + \frac{t}{\rho_2 - z_0}\big]\, dt \\ &= \frac{Z{\rho_2}^2}{2A^2}\big[-(\rho - \rho_2) - z_0\ln(\rho - z_0) + z_0\ln(\rho_2 - z_0) + \frac{\rho^2 - \rho_2^2}{2(\rho_2 - z_0)}\big]. \end{aligned} \tag{2.4}$$

We see that the expression $-z_0\ln(\rho - z_0)$ on the right-hand side of (2.4) goes to $+\infty$ as $\rho \to z_0^-$ which contradicts (2.2) and (2.3). Thus we see that it must be the case that $A = 0$. Thus

$$\lim_{\rho \to z_0^-} H'(\rho) = \lim_{\rho \to z_0^-} \frac{H(\rho)}{\rho - z_0} = 0. \tag{2.5}$$

Next, it is straightforward to show using (1.4) that

$$\big[H^2(H''+\frac{H'}{\rho})-HH'^2\big]'=\frac{2HH'^2}{\rho}+\rho\big[Z-\frac{2YH^2}{(1-\rho^2)^2}\big]-H'^3.$$

Integrating this on (ρ_2,ρ) gives

$$\begin{aligned}&H^2\big(H''+\frac{H'}{\rho}\big)-HH'^2\\&=H^2(\rho_2)\big(H''(\rho_2)+\frac{H'(\rho_2)}{\rho_2}\big)-H(\rho_2)H'^2(\rho_2)\\&\quad+\int_{\rho_2}^{\rho}\frac{2HH'^2}{t}\,dt+\int_{\rho_2}^{\rho}t[Z-\frac{2YH^2}{(1-t^2)^2}]\,dt-\int_{\rho_2}^{\rho}H'^3\,dt.\end{aligned}\tag{2.6}$$

It follows from (2.5) that $H^2H'\to 0$ and $HH'^2\to 0$ as $\rho\to z_0^-$. Also the integrals in (2.6) are finite because $z_0<1$. Thus it follows that

$$\lim_{\rho\to z_0^-}H^2H''=B.$$

We now want to show that $B=0$. Suppose then that $B\neq 0$. Then integrating on H^2H'' on (ρ,z_0) gives

$$H^2(\rho)H'(\rho)+\int_\rho^{z_0}2HH'^2\,dt=-\int_\rho^{z_0}H^2H''.$$

Dividing by $z_0-\rho$ and taking limits as $\rho\to z_0^-$ we see that the right-hand side limits to $-B\neq 0$ and the left-hand side limits to 0 by (2.5). This is a contradiction and therefore,

$$\lim_{\rho\to z_0^-}H^2H''=0.\tag{2.7}$$

Next, multiplying (1.4) by H^2, taking limits, and using (2.5) and (2.7) gives

$$\lim_{\rho\to z_0^-}H^2H'''=Zz_0>0.\tag{2.8}$$

Now integrating H^2H''' on (ρ,z_0) and using that $H(z_0)=0$, (2.5), and (2.7) gives

$$-H^2H''+HH'^2+\int_\rho^{z_0}H'^3\,dt=\int_\rho^{z_0}H^2H'''\,dt.$$

Dividing by $z_0-\rho$ and taking limits as $\rho\to z_0^-$ gives Zz_0 on the right (from (2.8)) while from (2.5) and the fact that $H(z_0)=0$, the second and third terms on the left have a limit of 0. Thus we see that

$$\lim_{\rho\to z_0^-}\frac{-H^2H''}{z_0-\rho}=Zz_0>0.\tag{2.9}$$

Therefore near z_0 we have

$$-H^2H''\geq\frac{Zz_0}{2}(z_0-\rho),$$

and after integrating on (ρ,z_0) and using (2.5) we see that

$$H^2H'+\int_\rho^{z_0}2HH'^2\,dt\geq\frac{Zz_0}{4}(z_0-\rho)^2.$$

Dividing by $(z_0-\rho)^2$ gives

$$\frac{H^2H' + \int_\rho^{z_0} 2HH'^2\,dt}{(z_0-\rho)^2} \ge \frac{Zz_0}{4}. \tag{2.10}$$

Finally, taking limits as $\rho \to z_0^-$ using (2.5) we see that the left-hand side of (2.10) limits to 0 and thus $Zz_0 = 0$ which contradicts that $Z > 0$ and $z_0 > 0$. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. We first prove existence of a solution of (1.11)-(1.14) on $(0,\rho_0)$ for some $\rho_0 > 0$. Assuming first that $W \in C^3[0,1]$ is a solution of (1.11)-(1.14) then by L'Hopital's rule $W''(0) = \lim_{\rho\to 0^+} \frac{W'(\rho)}{\rho}$. Using this, integrating (1.11) on $(0,\rho)$, and using (1.12)-(1.14) gives

$$W'' + \frac{W'}{\rho} = 2W''(0) - \int_0^\rho \frac{2YtW\big(H+\sqrt{Z/(2Y)}(1-t^2)\big)}{H^2(1-t^2)^2}\,dt. \tag{2.11}$$

Multiplying by ρ and integrating on $(0,\rho)$ gives

$$\rho W' = W''(0)\rho^2 - \int_0^\rho s\int_0^s \frac{2YtW\big(H+\sqrt{Z/(2Y)}(1-t^2)\big)}{H^2(1-t^2)^2}\,dt\,ds. \tag{2.12}$$

Dividing by ρ and integrating on $(0,\rho)$ gives

$$W = W(0) + W''(0)\frac{\rho^2}{2} - \int_0^\rho \frac{1}{x}\int_0^x s\int_0^s \frac{2YtW\big(H+\sqrt{Z/(2Y)}(1-t^2)\big)}{H^2(1-t^2)^2}\,dt\,ds\,dx. \tag{2.13}$$

Denoting the right-hand side of (2.13) as $T(W)$, it is straightforward to show that T is a contraction mapping on $C^3[0,\epsilon]$ for some $\epsilon > 0$ when $W(0) > 0$ and $W''(0)$ is arbitrary. Thus it follows from the contraction mapping principle [5] that there is a solution of (1.4)-(1.6) on $(0,\rho_0)$ for some $\rho_0 > 0$.

Next, let us denote $(0,\rho_1)$ as the maximal open interval of existence for this solution. We claim now that $\rho_1 \ge 1$. So we suppose by the way of contradiction that $0 < \rho_1 < 1$. Integrating (1.4) on $(0,\rho)$ gives

$$H'' + \frac{H'}{\rho} + \frac{Y}{1-\rho^2} = 2k + Y + \int_0^\rho \frac{Zt}{H^2}\,dt.$$

Multiplying by ρ, integrating $(0,\rho)$, and simplifying gives

$$H' = \frac{Y}{2}\frac{\ln(1-\rho^2)}{\rho} + (k+\frac{Y}{2})\rho + \frac{1}{\rho}\int_0^\rho s\int_0^s \frac{Zt}{H^2}\,dt\,ds. \tag{2.14}$$

Integrating again on $(0,\rho)$ gives

$$H = H(0) + \frac{Y}{2}\int_0^\rho \frac{\ln(1-t^2)}{t}\,dt + (k+\frac{Y}{2})\frac{\rho^2}{2} + \int_0^\rho \frac{1}{x}\int_0^x t\int_0^t \frac{sZ}{H^2}\,ds\,dt\,dx. \tag{2.15}$$

From (2.14) we see that $\int_0^\rho s\int_0^s \frac{Zt}{H^2}\,dt\,ds$ is an increasing function and since $\rho_1 < 1$ we see that $\lim_{\rho\to\rho_1^-} H'(\rho)$ exists (and is possibly $+\infty$). Similarly from (2.15) we see that $\int_0^\rho \frac{1}{x}\int_0^x t\int_0^t \frac{sZ}{H^2}\,ds\,dt\,dx$ is increasing and thus $\lim_{\rho\to\rho_1^-} H(\rho)$ exists (and is possibly $+\infty$). We claim now that $\lim_{\rho\to\rho_1^-} H(\rho)$ exists and is finite.

First, if $\lim_{\rho\to\rho_1^-} H'(\rho)$ is finite then it follows that $\lim_{\rho\to\rho_1^-} H(\rho)$ is also finite. So suppose $\lim_{\rho\to\rho_1^-} H(\rho) = +\infty$. Then by contraposition it follows that $\lim_{\rho\to\rho_1^-} H'(\rho) = +\infty$. On the other hand, if $\lim_{\rho\to\rho_1^-} H(\rho) = +\infty$ then it follows that $\frac{1}{H}$ is bounded near $\rho = \rho_1 < 1$. Then from (2.14) it follows that H' is bounded contradicting that $\lim_{\rho\to\rho_1^-} H'(\rho) = +\infty$. So we see that $\lim_{\rho\to\rho_1^-} H(\rho)$ exists and is finite. As in the proof of Theorem 1.1 it is possible to show that $\lim_{\rho\to\rho_1^-} H(\rho) > 0$. Therefore $H(\rho) > 0$ on $[0, \rho_1]$ and since H is continuous then there exists a $c_0 > 0$ such that $H \geq c_0 > 0$ on $[0, \rho_1]$.

Using this estimate in (2.13) and using that $\rho_1 < 1$ we obtain the existence of a constant c_2 so that

$$|W| \leq |W(0)| + |W''(0)| + c_2 \int_0^\rho \frac{1}{x} \int_0^x s \int_0^s |W|\,dt\,ds\,dx.$$

Next, since $0 \leq \rho \leq 1$ we see that

$$\begin{aligned}\int_0^\rho \frac{1}{x} \int_0^x s \int_0^s |W|\,dt\,ds\,dx &\leq \int_0^\rho \frac{1}{x} \int_0^x s \int_0^\rho |W|\,dt\,ds\,dx \\ &= \frac{\rho^2}{4} \int_0^\rho |W|\,dt \\ &\leq \int_0^\rho |W|\,dt\end{aligned}$$

and therefore,

$$|W| \leq |W(0)| + |W''(0)| + c_2 \int_0^\rho |W|.$$

Then by the Gronwall inequality [5] it follows that W remains bounded on $[0, \rho_1]$. We can then apply the contraction mapping principle again and obtain existence on a slightly larger interval contradicting the maximality of ρ_1. Thus the assumption that $0 < \rho_1 < 1$ must be false and therefore we see that $\rho_1 \geq 1$. Hence we see W is a solution of (1.11) on the entire open interval $(0, 1)$.

Next, we observe from Theorem 1.1 that $H > 0$ on $[0, 1)$. Thus we see by (1.11) that when $W > 0$ then

$$\Big(W'' + \frac{W'}{\rho}\Big)' < 0.$$

Integrating on $(0, \rho)$ and using L'Hopital's rule again we see $W''(0) = \lim_{\rho\to 0^+} \frac{W'(\rho)}{\rho}$ and thus

$$W'' + \frac{W'}{\rho} < 2W''(0).$$

Multiplying by ρ and integrating on $(0, \rho)$ using (1.13) gives

$$\rho W' < W''(0)\rho^2 \tag{2.16}$$

and thus

$$W' < W''(0)\rho. \tag{2.17}$$

Integrating a final time on $(0, \rho)$ gives

$$W < W(0) + \frac{W''(0)}{2}\rho^2 = \Big(d - \sqrt{Z/(2Y)}\Big) + \Big(k + \sqrt{2Z/Y}\Big)\frac{\rho^2}{2}. \tag{2.18}$$

Now if $W > 0$ on $[0, 1]$ then the left-hand side of (2.18) is positive but we see that the right-hand side of (2.18) is negative if k is sufficiently negative. Thus we obtain a contradiction and so we see that if k is sufficiently negative then W has a zero on $[0, 1]$.

Next it follows from (2.15) that

$$H \geq d + \frac{Y}{2} \int_0^\rho \frac{\ln(1-t^2)}{t}\, dt + (k + \frac{Y}{2})\frac{\rho^2}{2}.$$

Thus by (1.10),

$$W \geq \Big(d - \sqrt{Z/(2Y)}\Big) + \frac{Y}{2} \int_0^\rho \frac{\ln(1-t^2)}{t}\, dt + \Big(k + \frac{Y}{2} + \sqrt{2Z/Y}\Big)\frac{\rho^2}{2}. \tag{2.19}$$

We see next by L'Hopital's rule that

$$\lim_{\rho\to 0^+} \frac{\frac{Y}{2}\int_0^\rho \frac{\ln(1-t^2)}{t}\, dt}{\rho^2} = -\frac{Y}{4}.$$

Therefore it follows that

$$\lim_{\rho\to 0^+} \frac{\frac{Y}{2}\int_0^\rho \frac{\ln(1-t^2)}{t}\, dt + (k + \frac{Y}{2} + \sqrt{2Z/Y}))\frac{\rho^2}{2}}{\rho^2} = \frac{k}{2} + \sqrt{Z/(2Y)}. \tag{2.20}$$

Also $\frac{\ln(1-t^2)}{t}$ is integrable at $t = 1$ so we see that if $(d - \sqrt{Z/(2Y)}) > 0$ and k is chosen sufficiently large then it follows from (2.19), (2.20), and the integrability of $\frac{\ln(1-t^2)}{t}$ that $W > 0$ on $[0, 1)$.

We now define W_k to be the solution of (1.11)-(1.14). We have shown that $W_k > 0$ on $[0, 1]$ if k is sufficiently large and that W_k has a zero on $[0, 1]$ if k is sufficiently negative.

Now we choose k_0 to be the infimum of all k such that $W_k > 0$ on $[0, 1]$. We claim that W_{k_0} is a positive solution of (1.11) with $W_{k_0}'(0) = 0$, $W_{k_0}(1) = 0$ and thus $H_{k_0} = W_{k_0} + \sqrt{Z/(2Y)}(1 - \rho^2)$ is a solution of (1.4)-(1.6).

First we observe from (1.10), (2.14), and (2.17) that

$$\frac{Y}{2}\frac{\ln(1-\rho^2)}{\rho} + \Big(k + \frac{Y}{2} + \sqrt{2Z/Y}\Big)\rho \leq W_k' \leq W_k''(0) = k + \sqrt{2Z/Y}.$$

Thus we see that there exists constants C_1 and C_2 (independent of k for k near k_0) such that $|W_k'| \leq C_1|\ln(1-\rho)| + C_2$ on $[0, 1]$ and therefore there is a C_3 (independent of k for k near k_0) such that

$$\int_0^1 W_k'^2(t)\, dt \leq C_3.$$

Then we see by the Holder inequality that

$$|W_k(x) - W_k(y)| = |\int_x^y W_k'(t)\, dt| \leq \sqrt{|x-y|}\sqrt{\int_0^1 W_k'^2(t)\, dt} \leq \sqrt{C_3}\sqrt{|x-y|}.$$

Thus the $\{W_k\}$ are equicontinuous on $[0, 1]$. Now if W_{k_0} is ever negative then W_k would have to be somewhere negative for some $k > k_0$, but by assumption if $k > k_0$ then $W_k > 0$. Thus we see that $W_{k_0} \geq 0$.

If $W_{k_0} > 0$ on $[0, 1]$ then $W_k > 0$ on $[0, 1]$ for $k < k_0$ contradicting that W_k has a zero on $[0, 1]$ for $k < k_0$. Thus W_{k_0} must have a zero on $(0, 1]$. So suppose there exists z_0 with $0 < z_0 \leq 1$ such that $W_{k_0} > 0$ on $[0, z_0)$. If $z_0 = 1$ then we are done with this part of the proof so we suppose $z_0 < 1$.

Since we also know that $W_{k_0} \geq 0$ we see that if $z_0 < 1$ then it follows that W_{k_0} has a local minimum at z_0 and so $W_{k_0}'(z_0) = 0$.

Also, since $W_{k_0}(0) > 0$ and $W_{k_0}(z_0) = 0$ it follows that W_{k_0} must have a local maximum M with $0 \le M < z_0 < 1$. Integrating (1.11) on (M, ρ) gives

$$W''_{k_0} + \frac{W'_{k_0}}{\rho} = B - \int_M^\rho \frac{2YtW_{k_0}\Big(H_{k_0} + \sqrt{Z/(2Y)}(1-t^2)\Big)}{H_{k_0}^2(1-t^2)^2}\,dt \tag{2.21}$$

where $B = W''_{k_0}(M)$ if $M > 0$ or $B = 2W''_{k_0}(0)$ if $M = 0$. Whether the local max is at $M > 0$ or at 0 we see that in either case $B \le 0$.

Multiplying (2.21) by ρ and integrating on (M, ρ) gives

$$W'_{k_0} = \frac{B\rho^2}{2} - \frac{1}{\rho}\int_M^\rho x \int_M^x \frac{2YtW_{k_0}\Big(H_{k_0} + \sqrt{Z/(2Y)}(1-t^2)\Big)}{H_{k_0}^2(1-t^2)^2}\,dt\,dx. \tag{2.22}$$

Thus we see that

$$0 = W'_{k_0}(z_0) \le -\frac{1}{\rho}\int_M^{z_0} x \int_M^x \frac{2YtW_{k_0}\Big(H_{k_0} + \sqrt{Z/(2Y)}(1-t^2)\Big)}{H_{k_0}^2(1-t^2)^2}\,dt\,dx. \tag{2.23}$$

But $W_{k_0} > 0$ on (M, z_0) and also by Theorem 1.1 we know $H_{k_0} > 0$ on $[0, 1)$ and therefore the right-hand side of (2.23) must be negative. Thus we obtain a contradiction and we see that $z_0 = 1$.

We also see by (2.22) that $W'_{k_0} < 0$ on $(M, 1)$. It also follows from (2.22) and that $W_{k_0} > 0$ and $H_{k_0} > 0$ that $\lim_{\rho\to1^-} W'_{k_0}$ exists (and is possibly $-\infty$). This limit must be strictly negative because $B \le 0$ and the integrand in (2.22) is not identically zero on $(M, 1)$. If $W'_{k_0}(1) = -L > -\infty$ then $H'_{k_0}(1) = -L - \sqrt{2Z/Y}$ and since $W_{k_0}(1) = H_{k_0}(1) = 0$ then $\lim_{\rho\to1^-} \frac{W_{k_0}(\rho)}{1-\rho} = L$ and $\lim_{\rho\to1^-} \frac{H_{k_0}(\rho)}{1-\rho} = L + \sqrt{2Z/Y}$. Using (2.22) and that $W_{k_0} > 0$ and $H_{k_0} > 0$ we see that there is a $C_4 > 0$ and a $\delta > 0$ such that

$$W'_{k_0} \le -\frac{C_4}{\rho}\int_{1-\delta}^\rho x \int_{1-\delta}^x \frac{t}{(1-t)^2}\,dt\,dx.$$

This goes to $-\infty$ as $\rho \to 1^-$ contradicting that $W'_{k_0}(1) = -L > -\infty$. Thus it must be the case that $W'(1) = -\infty$. This completes the proof of Theorem 1.2. $\square$

3. Facts about the behavior of H near $\rho = 1$

We now begin to investigate the behavior of (1.4) and (1.6) in a neighborhood of $\rho = 1$ assuming H is a solution of (1.4) and (1.6) with finite contact angle at $\rho = 1$.

So let us assume that $H \in C^3(1-\epsilon, 1]$ for some $\epsilon > 0$ and that H is a positive solution of (1.4) and (1.6) on $(1-\epsilon, 1)$. Since $H(1) = 0$,

$$H'(1) = \lim_{\rho\to1^-} \frac{H(\rho)}{\rho - 1}. \tag{3.1}$$

Multiplying (1.4) by H^2 gives

$$H^2\big(H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2}\big) = Z\rho - \frac{2Y\rho H^2}{(1-\rho^2)^2}. \tag{3.2}$$

Taking limits as $\rho \to 1^-$, using that $H \in C^3(1-\epsilon, 1]$, $H(1) = 0$, and (3.1) gives

$$0 = Z - \frac{Y}{2}H'(1)^2.$$

Since $H > 0$ on $(1-\epsilon, 1)$ and $H(1) = 0$, it then follows that $H'(1) \le 0$ and thus

$$H'(1) = -\sqrt{2Z/Y} < 0. \tag{3.3}$$

Next by Taylor's theorem we have

$$H = H'(1)(\rho - 1) + \frac{H''(1)}{2}(\rho-1)^2 + o(1)(\rho-1)^2.$$

Using this on the right-hand side of (3.2) we obtain

$$Z\rho - \frac{2Y\rho H^2}{(1-\rho^2)^2} = \frac{[2H'(1)H''(1)Y - Z(3+\rho)]\rho(\rho-1)}{(1+\rho)^2} + o(1)(\rho-1). \tag{3.4}$$

Next dividing (3.2) and (3.4) by $(\rho - 1)$ we obtain

$$H\frac{H}{\rho-1}\big(H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2}\big) = \frac{[2H'(1)H''(1)Y - Z(3+\rho)]\rho}{(1+\rho)^2} + o(1).$$

Taking limits as $\rho \to 1^-$ using $H(1) = 0$ and (3.1) gives

$$0 = \frac{2H'(1)H''(1)Y - 4Z}{4}.$$

Thus, using (3.3) we obtain

$$H''(1) = H'(1) = -\sqrt{2Z/Y}. \tag{3.5}$$

Now integrating (1.4) on $(\rho, 1)$ and using (3.5) gives

$$2H'(1) - \big[H'' + \frac{H'}{\rho}\big] = \int_\rho^1 \big[\frac{Zx}{H^2} - \frac{2Yx}{(1-x^2)^2}\big]\,dx.$$

Multiplying by t and integrating on $(\rho, 1)$ gives

$$-H'(1)\rho^2 + \rho H' = \int_\rho^1 t \int_t^1 \big[\frac{Zx}{H^2} - \frac{2Yx}{(1-x^2)^2}\big]\,dx\,dt. \tag{3.6}$$

Dividing by ρ, integrating on $(\rho, 1)$, and using (3.5) gives

$$\sqrt{Z/(2Y)}(1-\rho^2) - H = \int_\rho^1 \frac{1}{s}\int_s^1 t \int_t^1 \big[\frac{Zx}{H^2} - \frac{2Yx}{(1-x^2)^2}\big]\,dx\,dt\,ds.$$

Rewriting we obtain

$$H - \sqrt{Z/(2Y)}(1-\rho^2) = \int_\rho^1 \frac{1}{s}\int_s^1 t\int_t^1 2Yx\big[\frac{H^2 - \frac{Z}{2Y}(1-x^2)^2}{H^2(1-x^2)^2}\big]\,dx\,dt\,ds. \tag{3.7}$$

Finally,

$$\begin{aligned} H - \sqrt{Z/(2Y)}(1-\rho^2) &= \int_\rho^1 \frac{1}{s}\int_s^1 t\int_t^1 2Yx[H - \sqrt{Z/(2Y)}(1-x^2)] \\ &\quad \times [H + \sqrt{Z/(2Y)}(1-x^2)]/\big(H^2(1-x^2)^2\big)\,dx\,dt\,ds. \end{aligned} \tag{3.8}$$

Often one can use an identity like (3.8) to prove local existence of a solution of (1.4) and (1.6) near $\rho = 1$. The usual procedure is to define a mapping, T, as

$$\begin{aligned} T(H) &= \sqrt{Z/(2Y)}(1-\rho^2) + \int_\rho^1 \frac{1}{s}\int_s^1 t \\ &\quad \times \int_t^1 \big[\frac{2Yx[H - \sqrt{Z/(2Y)}(1-x^2)][H + \sqrt{Z/(2Y)}(1-x^2)]}{H^2(1-x^2)^2}\big]\,dx\,dt\,ds. \end{aligned}$$

If we could show that T is a contraction mapping then by the contraction mapping principle T would have a unique fixed point which would be a solution of (1.4) and (1.6). However, due to the singular nature of (1.4) near $\rho = 1$, it turns out that T is *not* a contraction and so this method of proof of existence does not work. However, we are able to draw some conclusions from (3.8) about the behavior of solutions of (1.4).

Note 1: From (3.8) it follows that if there is an $\epsilon > 0$ such that

$$H > \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } (1-\epsilon, 1)$$

then

$$H > \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } [0, 1).$$

The reason for this is that if there were a ρ_0 such that $H(\rho_0) - \sqrt{Z/(2Y)}(1-\rho_0^2) = 0$ and

$$H > \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } (\rho_0, 1)$$

then the left-hand side of (3.8) would be zero but the right-hand side of (3.8) would be positive, yielding a contradiction.

Note 2: Similarly, if there is an $\epsilon > 0$ such that $H(0) > 0$ and

$$H < \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } (1-\epsilon, 1)$$

then by Theorem 1.1, $H > 0$ on $[0, 1)$ and then by (3.8),

$$H < \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } [0, 1).$$

4. The case $n(n-1)(n-2) \neq \frac{Y^{3/2}}{(2Z)^{1/2}}$ for all positive integers n

Our next attempt at solving (1.4)-(1.6) was to look for solutions of (1.4) with $H(1) = 0$ and then trying to show $H'(0) = 0$. Consequently we attempted to find a power series solution of (1.4) and (1.6) centered at $\rho = 1$ in the form

$$H(\rho) = \sum_{n=0}^{\infty} a_n(\rho - 1)^n \tag{4.1}$$

where

$$a_n = \frac{H^{(n)}(1)}{n!}. \tag{4.2}$$

Lemma 4.1. *If $H \in C^{k_0}[0,1]$ is a solution of (1.4)-(1.6) with $H(0) > 0$ and $k_0 \geq 3$ then*

$$H^{(n)}(1) = 0 \quad \text{for all } 3 \leq n \leq k_0.$$

(In particular, if $H \in C^{\infty}[0,1]$ then $H^{(n)}(1) = 0$ for all $n \geq 3$).

Proof. Suppose $H \in C^{k_0}[0,1]$ with $H(0) > 0$, $k_0 \geq 3$, and $H'''(1) \neq 0$. Using Taylor's theorem and (3.5) we then have

$$H - \sqrt{Z/(2Y)}(1-\rho^2) = \frac{H'''(1)}{3!}(\rho-1)^3 + o(1)(\rho-1)^3 \tag{4.3}$$

and therefore if $H'''(1) > 0$ then we see from (4.3) that

$$H < \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } (1-\epsilon, 1) \text{ for some } \epsilon > 0$$

and thus (by Note 1 at the end of section 2),

$$H < \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } [0, 1). \tag{4.4}$$

A similar argument shows that if $H'''(1) < 0$ then (by Note 2 at the end of section 2) we obtain

$$H > \sqrt{Z/(2Y)}(1-\rho^2) \quad \text{on } [0,1). \tag{4.5}$$

In addition, by (3.5) and (3.6) we have

$$H' + \sqrt{2Z/Y}\,\rho = -\frac{1}{\rho}\int_\rho^1 t\int_t^1 2Yx\Big[\frac{H^2 - \frac{Z}{2Y}(1-x^2)^2}{H^2(1-x^2)^2}\Big]\,dx\,dt. \tag{4.6}$$

Thus

$$\rho H' + \sqrt{2Z/Y}\,\rho^2 = -\int_\rho^1 t\int_t^1 2Yx\Big[\frac{H^2 - \frac{Z}{2Y}(1-x^2)^2}{H^2(1-x^2)^2}\Big]\,dx\,dt.$$

Hence

$$\lim_{\rho\to 0^+}\rho H' = -\int_0^1 t\int_t^1 2Yx\Big[\frac{H^2 - \frac{Z}{2Y}(1-x^2)^2}{H^2(1-x^2)^2}\Big]\,dx\,dt. \tag{4.7}$$

However, since $H \in C^{k_0}[0,1]$ and $H'''(1) \neq 0$, then either (4.4) holds or (4.5) holds and therefore the right-hand side of (4.7) is nonzero. However, if $H \in C^{k_0}[0,1]$ then we see that the left-hand side of (4.7) is zero. Thus, we obtain a contradiction and so we cannot find a $C^{k_0}[0,1]$ solution of (1.4)-(1.6) unless $H'''(1) = 0$.

Assuming now that $H'''(1) = 0$ and $H \in C^{k_0}[0,1]$ with $k_0 \geq 4$ then again using Taylor's theorem and (3.5) we see that

$$H - \sqrt{Z/(2Y)}(1-\rho^2) = \frac{H''''(1)}{4!}(\rho-1)^4 + o(1)(\rho-1)^4$$

and arguing in a similar way as before we can show that $H''''(1) = 0$. Continuing in this way we see that all the higher derivatives of H up through order k_0 would have to be zero at $\rho = 1$. This completes the proof. □

Lemma 4.2. *Suppose H is a solution of (1.4) and (1.6) with $H \in C^{k_0}(1-\epsilon, 1]$ and $H > 0$ on $(1-\epsilon, 1)$ for some $\epsilon > 0$ and $k_0 \geq 3$. Also suppose that*

$$n(n-1)(n-2) \neq \frac{Y^{3/2}}{(2Z)^{1/2}} \quad \text{for all positive integers } n \text{ with } 3 \leq n \leq k_0. \tag{4.8}$$

Then

$$H^{(n)}(1) = 0 \text{ for all } 3 \leq n \leq k_0.$$

(In particular, if $H \in C^\infty(1-\epsilon, 1]$ and $H > 0$ on $(1-\epsilon, 1)$ for some $\epsilon > 0$ and H is a solution of (1.4) and (1.6) satisfying (4.8) then $H^{(n)}(1) = 0$ for all $n \geq 3$).

Proof. We will assume (4.8) and show that

$$H^{(n)}(1) = 0 \quad \text{for all } 3 \leq n \leq k_0.$$

So suppose $n \geq 3$, $H \in C^n(1-\epsilon, 1]$, and that H satisfies

$$H = a_1(\rho-1) + a_2(\rho-1)^2 + a_n(\rho-1)^n + o(1)(\rho-1)^n. \tag{4.9}$$

Then

$$H^2 = a_1^2(\rho-1)^2 + 2a_1a_2(\rho-1)^3 + a_2^2(\rho-1)^4 + 2a_1a_n(\rho-1)^{n+1} + o(1)(\rho-1)^{n+1}.$$

Therefore,

$$\frac{H^2}{(1-\rho)^2} = a_1^2 + 2a_1a_2(\rho-1) + a_2^2(\rho-1)^2 + 2a_1a_n(\rho-1)^{n-1} + o(1)(\rho-1)^{n-1}.$$

Then

$$\begin{aligned}\frac{2YH^2}{(1-\rho^2)^2} &= \frac{2Y}{(1+\rho)^2}[a_1^2 + 2a_1a_2(\rho-1) + a_2^2(\rho-1)^2] \\ &\quad + \frac{4Ya_1a_n(\rho-1)^{n-1}}{(1+\rho)^2} + o(1)(\rho-1)^{n-1}.\end{aligned}$$

Next,

$$\begin{aligned}\frac{2YH^2}{(1-\rho^2)^2} - Z &= \frac{2Y}{(1+\rho)^2}\big[a_1^2 + 2a_1a_2(\rho-1) + a_2^2(\rho-1)^2 - \frac{Z(1+\rho)^2}{2Y}\big] \\ &\quad + \frac{4Ya_1a_n(\rho-1)^{n-1}}{(1+\rho)^2} + o(1)(\rho-1)^{n-1}.\end{aligned} \tag{4.10}$$

Using (3.5) and (4.2) we see that the term in brackets on the right-hand side of (4.10) is identically zero and thus (4.10) reduces to

$$\frac{2YH^2}{(1-\rho^2)^2} - Z = \frac{4Ya_1a_n(\rho-1)^{n-1}}{(1+\rho)^2} + o(1)(\rho-1)^{n-1}. \tag{4.11}$$

Now multiplying by $\frac{\rho}{H^2}$ and using the fact that

$$\lim_{\rho\to 1^-}\frac{H(\rho)}{\rho-1} = H'(1) = a_1 = -\sqrt{2Z/Y} \tag{4.12}$$

as well as (1.4), (3.5), and (4.11) we obtain

$$\begin{aligned}-(H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2}) = \frac{2Y\rho}{(1-\rho^2)^2} - \frac{Z\rho}{H^2} \quad &= \frac{4Ya_1a_n\rho(\rho-1)^{n-3}}{(1+\rho)^2} \\ \frac{(\rho-1)^2}{H^2} + o(1)(\rho-1)^{n-3}.&\end{aligned} \tag{4.13}$$

First we consider the case $n=3$. Taking limits as $\rho\to 1^-$ of (4.13) using (4.12) we obtain

$$-3!a_3 = -H'''(1) = \frac{Ya_1a_3}{a_1^2} = \frac{Ya_3}{a_1} = -\frac{Y^{3/2}}{(2Z)^{1/2}}a_3. \tag{4.14}$$

But from (4.8) we have that

$$\frac{Y^{3/2}}{(2Z)^{1/2}} \neq 6 = 3\cdot 2\cdot 1$$

and thus from (4.14) it follows that

$$a_3 = \frac{H'''(1)}{3!} = 0.$$

Now let us assume (4.8) with $3 < n \le k_0$ and suppose that

$$H'''(1) = H^{(4)}(1) = \dots = H^{(n-1)}(1) = 0. \tag{4.15}$$

Then (4.9) holds and therefore from (4.13) we see that

$$\frac{H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2}}{(\rho-1)^{n-3}} = \frac{-4Ya_1a_n\rho}{(1+\rho)^2}\frac{(\rho-1)^2}{H^2} + o(1). \tag{4.16}$$

Taking limits as $\rho\to 1^-$ of the right-hand side of (4.16) and using (4.12) we obtain

$$\lim_{\rho\to 1^-}\frac{H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2}}{(\rho-1)^{n-3}} = \frac{-Ya_1a_n}{a_1^2} = \frac{Y^{3/2}}{(2Z)^{1/2}}a_n. \tag{4.17}$$

Next we observe that

$$\lim_{\rho\to 1^-}\frac{H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2}}{(\rho-1)^{n-3}} = \lim_{\rho\to 1^-}\frac{\rho^2 H''' + \rho H'' - H'}{(\rho-1)^{n-3}}. \tag{4.18}$$

Using (4.15) and that $H \in C^n(1-\epsilon, 1]$, we may apply L'Hopital's rule $(n-3)$ times to the limit on the right-hand side of (4.18), and it is straightforward to show that

$$\begin{aligned} &\lim_{\rho\to 1^-}\frac{\rho^2 H''' + \rho H'' - H'}{(\rho-1)^{n-3}} \\ &= \lim_{\rho\to 1^-}\frac{\rho^2 H^{(n)} + (2n-5)\rho H^{(n-1)} + (n^2-6n+8)H^{(n-2)}}{(n-3)!} = \frac{H^{(n)}(1)}{(n-3)!}. \end{aligned} \tag{4.19}$$

Thus from (4.17)-(4.19) we get

$$\frac{Y^{3/2}}{(2Z)^{1/2}}a_n = \frac{H^{(n)}(1)}{(n-3)!} = \frac{n!a_n}{(n-3)!} = n(n-1)(n-2)a_n. \tag{4.20}$$

And again by (4.8) we see that since $n \ge 3$ then

$$n(n-1)(n-2) \neq \frac{Y^{3/2}}{(2Z)^{1/2}}$$

and therefore by (4.20),

$$a_n = \frac{H^{(n)}(1)}{n!} = 0.$$

This completes the proof. $\square$

Lemma 4.3. *If $H \in C^\infty(1-\epsilon, 1]$ with $H > 0$ on $(1-\epsilon, 1)$ for some $\epsilon > 0$ and H is a solution of* (1.4), (1.6), *with $H^{(n)}(1) = 0$ for every $n \ge 3$ then*

$$H \equiv \sqrt{Z/(2Y)}(1-\rho^2).$$

Proof. Let

$$W = H - \sqrt{Z/(2Y)}(1-\rho^2).$$

Since $H(1) = 0$, it follows from (3.5) and by assumption that

$$W^{(n)}(1) = 0 \text{ for all } n \ge 0.$$

Then by (3.8) we have

$$W = \int_\rho^1 \frac{1}{s}\int_s^1 t\int_t^1 \Big[\frac{2YxW[H + \sqrt{Z/(2Y)}(1-x^2)]}{H^2(1-x^2)^2}\Big]\,dx\,dt\,ds.$$

We rewrite this as follows:

$$W = \int_\rho^1 \frac{1}{s}\int_s^1 t\int_t^1 \frac{W}{(1-x)^3}\Big[\frac{2Yx}{(1+x)^2}\frac{[\frac{H}{1-x} + \sqrt{Z/(2Y)}(1+x)]}{(\frac{H}{1-x})^2}\Big]\,dx\,dt\,ds. \tag{4.21}$$

It follows then from (3.1) and (3.3) that the limit as $x \to 1^-$ of the term in brackets in (4.21) is $\frac{Y^{3/2}}{(2Z)^{1/2}}$ and so there is a ρ_0 with $0 < \rho_0 < 1$ such that the term in brackets in (4.21) is bounded by M where $M = \frac{2Y^{3/2}}{(2Z)^{1/2}}$. Then on $[\rho_0, 1]$ we have

$$\begin{aligned} |W| &\le \int_\rho^1 \frac{1}{s} \int_s^1 t \int_t^1 \frac{M|W|}{(1-x)^3}\,dx\,dt\,ds \\ &\le \int_\rho^1 \frac{1}{s} \int_\rho^1 t \int_\rho^1 \frac{M|W|}{(1-x)^3}\,dx\,dt\,ds \\ &\le \frac{M}{\rho_0}(1-\rho)^2 \int_\rho^1 \frac{|W|}{(1-x)^3}\,dx. \end{aligned} \tag{4.22}$$

Now we let

$$U = \int_\rho^1 \frac{|W|}{(1-x)^3}\,dx \quad \text{for } \rho_0 < \rho < 1, \tag{4.23}$$

and observe that

$$U' = -\frac{|W|}{(1-\rho)^3}.$$

It follows from (4.22) that

$$(1-\rho)U' + BU \ge 0 \quad \text{where } B = \frac{M}{\rho_0}.$$

This implies

$$\frac{U(\rho)}{(1-\rho)^B}$$

is increasing and thus if $0 < \rho_0 < \rho < \rho_2 < 1$ then

$$\frac{U(\rho)}{(1-\rho)^B} \le \frac{U(\rho_2)}{(1-\rho_2)^B}. \tag{4.24}$$

Now we know from the beginning of the proof that $W(1) = W'(1) = W''(1) = 0$ and by assumption we also know for $n \ge 3$ that $W^{(n)}(1) = H^{(n)}(1) = 0$. Thus W vanishes faster than any power of $(1-\rho)$ at $\rho = 1$. That is,

$$\lim_{\rho \to 1^-} \frac{W}{(1-\rho)^n} = 0 \quad \text{for every } n \ge 1.$$

It follows then from (4.23) that the same is true for U.

It follows then from (4.24) that

$$\frac{U(\rho)}{(1-\rho)^B} \le \lim_{\rho_2 \to 1^-} \frac{U(\rho_2)}{(1-\rho_2)^B} = 0.$$

Thus $U \le 0$ on $(\rho_1, 1)$ but clearly $U \ge 0$ (by (4.23)) and thus $U \equiv 0$ which implies $W \equiv 0$ and thus

$$H \equiv \sqrt{Z/(2Y)}(1-\rho^2).$$

This completes the proof. □

Theorem 4.4. *Let $H \in C^\infty[0,1]$ be a solution of* (1.4)-(1.6) *with $H(0) > 0$. Then*

$$H(\rho) \equiv \sqrt{Z/(2Y)}(1-\rho^2).$$

The above theorem follows from Lemmas 4.1 and 4.3.

Theorem 4.5. *Let $\epsilon > 0$. If $n(n-1)(n-2) \neq \frac{Y^{3/2}}{(2Z)^{1/2}}$ for every positive integer n and $H \in C^\infty(1-\epsilon, 1]$ is a solution of* (1.4) *and* (1.6) *with H positive on $(1-\epsilon, 1)$ then*

$$H(\rho) \equiv \sqrt{Z/(2Y)}(1-\rho^2).$$

The above theorem follows from Lemmas 4.2 and 4.3.

5. The case $n_0(n_0-1)(n_0-2) = \frac{Y^{3/2}}{(2Z)^{1/2}}$ for some positive integer $n_0 \geq 3$

So the question now is whether there are any power series solutions of (1.4) and (1.6) if

$$n_0(n_0-1)(n_0-2) = \frac{Y^{3/2}}{(2Z)^{1/2}} \quad \text{for some } n_0 \geq 3.$$

Lemma 5.1. *Let n_0 be an integer with $n_0 \geq 3$ such that*

$$n_0(n_0-1)(n_0-2) = \frac{Y^{3/2}}{(2Z)^{1/2}}.$$

Suppose that

$$H = \sum_{n=1}^{\infty} a_n(\rho-1)^n$$

is a power series solution of (1.4) *and* (1.6). *Let*

$$b_n = n(n-1)(n-2)a_n + (n-1)(n-2)(2n-5)a_{n-1} + (n-2)^2(n-4)a_{n-2} \tag{5.1}$$

for $n \geq 3$, and

$$c_n = \sum_{k=1}^{n} a_k a_{n+1-k} \quad \text{for } n \geq 1. \tag{5.2}$$

Then

$$a_1 = -\sqrt{2Z/Y} \text{ and } a_2 = -\frac{1}{2}\sqrt{2Z/Y}. \tag{5.3}$$

In addition

$$\big[6 - \frac{Y^{3/2}}{(2Z)^{1/2}}\big]a_3 = 0, \tag{5.4}$$

$$\big[24 - \frac{Y^{3/2}}{(2Z)^{1/2}}\big]a_4 = -18a_3 - \frac{5Y^2}{2Z}a_2a_3 - \frac{Y}{2Z}b_3c_2, \tag{5.5}$$

$$\big[60 - \frac{Y^{3/2}}{(2Z)^{1/2}}\big]a_5 = -60a_4 - 9a_3 - \frac{Y^2}{4Z}\sum_{k=2}^{4} a_k a_{6-k} - \frac{Y}{2Z}\sum_{k=3}^{4} b_k c_{6-k} \tag{5.6}$$

$$+ \frac{Y^2}{4Z}\big[-2c_4 - \frac{3}{4}c_3 + \sum_{k=0}^{1}\frac{(-1)^{1-k}(17-7k)c_{k+1}}{2^{4-k}}\big], \tag{5.7}$$

and

$$\big[n(n-1)(n-2) - \frac{Y^{3/2}}{(2Z)^{1/2}}\big]a_n$$

$$= -(n-1)(n-2)(2n-5)a_{n-1} - (n-2)^2(n-4)a_{n-2} - \frac{Y^2}{4Z}\sum_{k=2}^{n-1} a_k a_{n+1-k} \tag{5.8}$$

$$
- \frac{Y}{2Z} \sum_{k=3}^{n-1} b_k c_{n+1-k} + \frac{Y^2}{4Z} \Big[-2c_{n-1} - \frac{3}{4} c_{n-2} + \sum_{k=1}^{n-3} \frac{(-1)^{n-k}(n-k-5)c_k}{2^{n-k}} \Big] \tag{5.9}
$$

for $n \geq 6$.

Further, if $n_0 > 3$ then $H^{(n)}(1) = 0$ for $3 \leq n \leq n_0 - 1$. Also, if $n_0 = 4$ then the right-hand side of (5.5) is zero. If $n_0 = 5$ then the right-hand side of (5.6)-(5.7) is zero. Finally, if $n = n_0 \geq 6$, then the right-hand side of (5.8)-(5.9) is 0.

Proof. We suppose

$$
H = \sum_{n=1}^{\infty} a_n (\rho - 1)^n
$$

where

$$
a_n = \frac{H^{(n)}(1)}{n!}. \tag{5.10}
$$

It follows from (3.5) that

$$
a_1 = -\sqrt{2Z/Y}, \quad a_2 = -\frac{1}{2}\sqrt{2Z/Y}. \tag{5.11}
$$

Then

$$
\begin{gathered}
H' = \sum_{n=1}^{\infty} n a_n (\rho - 1)^{n-1}, \\
H'' = \sum_{n=1}^{\infty} n(n-1) a_n (\rho - 1)^{n-2} = \sum_{n=1}^{\infty} (n+1) n a_{n+1} (\rho - 1)^{n-1}, \\
H''' = \sum_{n=1}^{\infty} n(n-1)(n-2) a_n (\rho - 1)^{n-3} = \sum_{n=1}^{\infty} (n+2)(n+1) n a_{n+2} (\rho - 1)^{n-1}.
\end{gathered} \tag{5.12}
$$

Also

$$
(\rho - 1) H'' = \sum_{n=1}^{\infty} n(n-1) a_n (\rho - 1)^{n-1}.
$$

Therefore,

$$
\rho H'' = H'' + (\rho - 1) H'' = \sum_{n=1}^{\infty} [(n+1) n a_{n+1} + n(n-1) a_n] (\rho - 1)^{n-1}. \tag{5.13}
$$

Also

$$
2(\rho-1) H''' = \sum_{n=1}^{\infty} 2n(n-1)(n-2) a_n (\rho-1)^{n-2} = \sum_{n=1}^{\infty} 2(n+1)n(n-1) a_{n+1} (\rho-1)^{n-1}
$$

and

$$
(\rho - 1)^2 H''' = \sum_{n=1}^{\infty} n(n-1)(n-2) a_n (\rho - 1)^{n-1}.
$$

Therefore,

$$\begin{aligned}\rho^2 H''' &= (\rho-1)^2 H''' + 2(\rho-1)H''' + H''' \\ &= \sum_{n=1}^{\infty}\Big[(n+2)(n+1)na_{n+2} + 2(n+1)n(n-1)a_{n+1} \\ &\quad + n(n-1)(n-2)a_n\Big](\rho-1)^{n-1}.\end{aligned} \tag{5.14}$$

Finally, combining (5.12)-(5.14), we obtain

$$\rho^2 H''' + \rho H'' - H' = \sum_{n=1}^{\infty} b_{n+2}(\rho-1)^{n-1} \tag{5.15}$$

where

$$b_{n+2} = (n+2)(n+1)na_{n+2} + (n+1)n(2n-1)a_{n+1} + n^2(n-2)a_n \tag{5.16}$$

for $n \geq 1$. After reindexing this is (5.1). Also, we have

$$H^2 = \sum_{n=2}^{\infty} c_{n-1}(\rho-1)^n \tag{5.17}$$

where

$$c_n = \sum_{k=1}^{n} a_k a_{n+1-k} \quad \text{for } n \geq 1. \tag{5.18}$$

This is (5.2). Multiplying (5.15) and (5.17) gives

$$H^2(\rho^2 H''' + \rho H'' - H') = \sum_{n=3}^{\infty}\Big(\sum_{k=1}^{n-2} b_{k+2}c_{n-k-1}\Big)(\rho-1)^{n-1}. \tag{5.19}$$

Next

$$\frac{H^2}{(\rho-1)^2} = \sum_{n=2}^{\infty} c_{n-1}(\rho-1)^{n-2} = \sum_{n=0}^{\infty} c_{n+1}(\rho-1)^n. \tag{5.20}$$

Also

$$\frac{1}{1+\rho} = \frac{1}{2+(\rho-1)} = \frac{1}{2(1+\frac{\rho-1}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{\rho-1}{2}\Big)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(\rho-1)^n$$

for $|\rho-1| < 2$. Differentiating we obtain

$$-\frac{1}{(1+\rho)^2} = \sum_{n=1}^{\infty}\frac{(-1)^n n}{2^{n+1}}(\rho-1)^{n-1} \quad \text{for } |\rho-1| < 2. \tag{5.21}$$

Multiplying (5.20) and (5.21) we see that

$$-\frac{H^2}{(\rho^2-1)^2} = \sum_{n=1}^{\infty}\Big(\sum_{k=0}^{n-1}\frac{(-1)^{n-k}(n-k)c_{k+1}}{2^{n-k+1}}\Big)(\rho-1)^{n-1}.$$

Thus

$$-\frac{2YH^2}{(\rho^2-1)^2} = \sum_{n=1}^{\infty}\Big(\sum_{k=0}^{n-1}\frac{(-1)^{n-k}(n-k)c_{k+1}Y}{2^{n-k}}\Big)(\rho-1)^{n-1}.$$

Next, using (3.5) we see that

$$Z-\frac{2YH^2}{(\rho^2-1)^2}=\sum_{n=3}^{\infty}\Big(\sum_{k=0}^{n-1}\frac{(-1)^{n-k}(n-k)c_{k+1}Y}{2^{n-k}}\Big)(\rho-1)^{n-1}. \tag{5.22}$$

Thus after reindexing,

$$3(\rho-1)[Z-\frac{2YH^2}{(\rho^2-1)^2}]=\sum_{n=4}^{\infty}\Big(\sum_{k=0}^{n-2}\frac{3(-1)^{n-1-k}(n-1-k)c_{k+1}Y}{2^{n-1-k}}\Big)(\rho-1)^{n-1}$$

and

$$3(\rho-1)^2[Z-\frac{2YH^2}{(\rho^2-1)^2}]=\sum_{n=5}^{\infty}\Big(\sum_{k=0}^{n-3}\frac{3(-1)^{n-2-k}(n-2-k)c_{k+1}Y}{2^{n-2-k}}\Big)(\rho-1)^{n-1}$$

and

$$(\rho-1)^3[Z-\frac{2YH^2}{(\rho^2-1)^2}]=\sum_{n=6}^{\infty}\Big(\sum_{k=0}^{n-4}\frac{(-1)^{n-3-k}(n-3-k)c_{k+1}Y}{2^{n-3-k}}\Big)(\rho-1)^{n-1}. \tag{5.23}$$

Now using that $\rho^3=1+3(\rho-1)+3(\rho-1)^2+(\rho-1)^3$, combining (5.22)-(5.23), using (3.5), and (5.1) as well as some tedious algebra gives

$$\begin{aligned}
&\rho^3[Z-\frac{2YH^2}{(\rho^2-1)^2}]\\
&=-a_1a_3Y(\rho-1)^2-[\frac{1}{2}c_4+2a_1a_3]Y(\rho-1)^3\\
&\quad+\frac{Y}{2}\Big[-c_5-2c_4-\frac{3}{4}c_3+\sum_{k=0}^{1}\frac{(-1)^{1-k}(17-7k)c_{k+1}}{2^{4-k}}\Big](\rho-1)^4\\
&\quad+\sum_{n=6}^{\infty}\frac{Y}{2}\Big[-c_n-2c_{n-1}-\frac{3}{4}c_{n-2}\\
&\quad+\sum_{k=0}^{n-4}\frac{(-1)^{n-1-k}(n-k-6)c_{k+1}}{2^{n-1-k}}\Big](\rho-1)^{n-1}.
\end{aligned} \tag{5.24}$$

So we see that a power series solution of (1.4) is equivalent to equating the coefficients in (5.19) and (5.24) and so we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-2}b_{k+2}c_{n-1-k}\\
&=\frac{Y}{2}\Big[-c_n-2c_{n-1}-\frac{3}{4}c_{n-2}+\sum_{k=0}^{n-4}\frac{(-1)^{n-1-k}(n-k-6)c_{k+1}}{2^{n-1-k}}\Big]
\end{aligned} \tag{5.25}$$

for $n\geq 6$. In addition, when $n=3$,

$$b_3c_1=-a_1a_3Y.$$

(Using (5.1), (5.2), and (5.11), this is (5.4)). When $n=4$,

$$b_3c_2+b_4c_1=-\frac{1}{2}c_4Y-2a_1a_3Y.$$

(Using (5.1), (5.2), and (5.11), this is (5.5)). And when $n = 5$,

$$b_3c_3 + b_4c_2 + b_5c_1 = \frac{Y}{2}\Big[-c_5 - 2c_4 - \frac{3}{4}c_3 + \sum_{k=0}^{1} \frac{(-1)^{1-k}(17-7k)c_{k+1}}{2^{4-k}}\Big].$$

(Using (5.1), (5.2), and (5.11), this is (5.6) and (5.7)).

Next, rewriting and reindexing (5.25) we see that

$$\begin{aligned} &b_nc_1 + \frac{Y}{2}c_n \\ &= -\sum_{k=3}^{n-1} b_kc_{n+1-k} + \frac{Y}{2}\Big[-2c_{n-1} - \frac{3}{4}c_{n-2} + \sum_{k=1}^{n-3} \frac{(-1)^{n-k}(n-k-5)c_k}{2^{n-k}}\Big] \end{aligned} \tag{5.26}$$

for $n \geq 6$.

From (5.2) and (5.3) we recall that $a_1 = -\sqrt{2Z/Y}$ and $c_1 = a_1^2$. Therefore, multiplying both sides of (5.26) by $\frac{1}{c_1} = \frac{Y}{2Z}$ we see that

$$\begin{aligned} b_n + \frac{Y^2}{4Z}c_n = \Big(&-\frac{Y}{2Z}\sum_{k=3}^{n-1} b_kc_{n+1-k} + \frac{Y^2}{4Z}\Big[-2c_{n-1} - \frac{3}{4}c_{n-2} \\ &+ \sum_{k=1}^{n-3} \frac{(-1)^{n-k}(n-k-5)c_k}{2^{n-k}}\Big]\Big) \quad \text{for } n \geq 6. \end{aligned} \tag{5.27}$$

Rewriting (5.2) gives

$$c_n = 2a_1a_n + \sum_{k=2}^{n-1} a_ka_{n+1-k}. \tag{5.28}$$

Combining (5.1) and (5.28) we see that the left-hand side of (5.27) can be written as

$$\begin{aligned} &b_n + \frac{Y^2}{4Z}c_n \\ &= \big[n(n-1)(n-2) - \frac{Y^{3/2}}{(2Z)^{1/2}}\big]a_n + (n-1)(n-2)(2n-5)a_{n-1} \\ &\quad + (n-2)^2(n-4)a_{n-2} + \frac{Y^2}{4Z}\sum_{k=2}^{n-1} a_ka_{n+1-k}. \end{aligned} \tag{5.29}$$

Finally combining (5.27) and (5.29) we obtain

$$\begin{aligned} &\big[n(n-1)(n-2) - \frac{Y^{3/2}}{(2Z)^{1/2}}\big]a_n \\ &= -(n-1)(n-2)(2n-5)a_{n-1} - (n-2)^2(n-4)a_{n-2} - \frac{Y^2}{4Z}\sum_{k=2}^{n-1} a_ka_{n+1-k} \\ &\quad - \frac{Y}{2Z}\sum_{k=3}^{n-1} b_kc_{n+1-k} + \frac{Y^2}{4Z}\Big[-2c_{n-1} - \frac{3}{4}c_{n-2} + \sum_{k=1}^{n-3} \frac{(-1)^{n-k}(n-k-5)c_k}{2^{n-k}}\Big] \end{aligned} \tag{5.30}$$

for $n \geq 6$, which is (5.9).

We note that by (5.1)-(5.2) all the terms on the right-hand side of (5.5)-(5.9) can be expressed in terms of a_k where $1 \leq k \leq n-1$.

Note that if n is an integer and $3 \le n < n_0$ then $n(n-1)(n-2) \neq \frac{Y^{3/2}}{(2Z)}$ and so in a completely analogous way as in the proof of Lemma 4.2 in section 4 we can show that

$$a_3 = a_4 = \dots = a_{n_0-1} = 0. \tag{5.31}$$

Thus if $n_0 > 3$ then $H^{(n)}(1) = 0$ for $3 \le n \le n_0 - 1$.

Next, we need to show that if $n_0 = 4$ then the right-hand side of (5.5) is zero, if $n_0 = 5$ then the right-hand side of (5.7) is zero, and if $n = n_0 \ge 6$, then the right-hand side of (5.9) is 0.

Suppose first that $n_0 = 4$. Then $\frac{Y^{3/2}}{(2Z)^{1/2}} = 4 \cdot 3 \cdot 2 = 24$ and so we see from (5.5) that $a_3 = 0$. We also see from (5.1) and (5.3) that $b_3 = 6a_3 + 2a_2 - a_1 = 0$. Hence the right-hand side of (5.5) is 0.

Next suppose $n_0 = 5$. Then $\frac{Y^{3/2}}{(2Z)^{1/2}} = 5 \cdot 4 \cdot 3 = 60$ and so from (5.5) we see that $a_3 = 0$. We also see from (5.1) and (5.3) that $b_3 = 0$. Then from (5.5) we see that $a_4 = 0$. Then (5.1) and (5.2) imply that $b_4 = 0$ and $c_4 = 0$.

Since $a_3 = 0$ it follows then from (5.2) and (5.3) that

$$c_1 = a_1^2, \quad c_2 = 2a_1a_2 = a_1^2, \quad c_3 = 2a_1a_3 + a_2^2 = \frac{1}{4}a_1^2. \tag{5.32}$$

Substituting these values into the right-hand side of (5.6)-(5.7) gives

$$\frac{Y^2}{4Z}\big[-\frac{3}{4}c_3 - \frac{17}{16}c_1 + \frac{5}{4}c_2\big] = \frac{Y^2}{4Z}\big[-\frac{3}{16}a_1^2 - \frac{17}{16}a_1^2 + \frac{5}{4}a_1^2\big] = 0. \tag{5.33}$$

Thus, the right-hand side of (5.6)-(5.7) is zero.

Now suppose that $n_0 \ge 6$. It then follows from (5.31) that

$$\sum_{k=2}^{n_0-1} a_k a_{n_0+1-k} = 0.$$

It addition, it also follows from (5.1) and (5.31) that

$$b_k = 0 \quad \text{for } 3 \le k \le n_0 - 1$$

and therefore,

$$\sum_{k=3}^{n_0-1} b_k c_{n+1-k} = 0.$$

It also follows from (5.2) and (5.3) that (5.32) holds. In addition, by (5.31),

$$c_k = \sum_{l=1}^{k} a_l a_{k+1-l} = \sum_{l=1}^{2} a_l a_{k+1-l} = a_1a_k + a_2a_{k-1} = 0 \tag{5.34}$$

for $4 \le k \le n_0 - 1$.

Then using (5.31), (5.32), and (5.34) we see that if $n_0 \ge 6$, then the right-hand side of (5.8)-(5.9) reduces to

$$\begin{aligned}
&\sum_{k=1}^{3} \frac{(-1)^{n_0-k}(n_0-k-5)c_k}{2^{n_0-k}} \\
&= \frac{(-1)^{n_0}}{2^{n_0}}[-2(n_0-6)c_1 + 4(n_0-7)c_2 - 8(n_0-8)c_3] \\
&= \frac{(-1)^{n_0}}{2^{n_0}}[-2(n_0-6)a_1^2 + 4(n_0-7)a_1^2 - 2(n_0-8)a_1^2] = 0.
\end{aligned}$$

This completes the proof of the lemma. □

It follows then that we are free to choose a_{n_0} to be any nonzero value and this appears to indicate that there might be power series solutions of (1.4)-(1.6) in the case when $n_0(n_0-1)(n_0-2) = \frac{Y^{3/2}}{(2Z)^{1/2}}$ but we still need to show that the series

$$\sum_{n=1}^{\infty} a_n(\rho-1)^n$$

with the a_n chosen as in Lemma 4.1 of section 4 converges in some neighborhood of $\rho = 1$.

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