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# EXISTENCE RESULTS FOR $n$-TH ORDER MULTIPOINT INTEGRAL BOUNDARY-VALUE PROBLEMS OF DIFFERENTIAL INCLUSIONS

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Abstract. In this article we study the existence of solutions for $n$-th order differential inclusions with nonlocal integral boundary conditions. Our results are based on some classical fixed point theorems for multivalued maps. Some illustrative examples are discussed.

## 1. Introduction

In this article, we discuss the existence of solutions for the boundary value problem of $n$-th order differential inclusions with multi-point integral boundary conditions

$$\begin{gathered} u^{(n)}(t) \in F(t, u(t)), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} u(s)ds, \quad 0 < \eta_i < 1, \end{gathered} \tag{1.1}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\alpha, \beta, \gamma_i, \eta_i$ $(i = 1, 2, \dots, m)$ are real constants to be chosen appropriately.

Boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary-value problems as special cases. Integral boundary-value problems occur in the mathematical modeling of a variety of physical and biological processes, and have recently received considerable attention. For some recent work on boundary-value problems with integral boundary conditions, we refer to [1]-[6], [8, 13, 14, 15, 17, 18, 21, 22], [25]-[30] and the references cited therein.

The present work is motivated by [6] which deals with a single-valued case of the problem (1.1). We aim to establish a variety of results for the inclusion problem (1.1) by considering the multivalued map involved to be convex as well as non-convex valued. The first result relies on Bohnenblust-Karlin fixed point theorem and the second one is based on the nonlinear alternative of Leray-Schauder type.

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In the third result, we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semi-continuous multivalued maps with nonempty closed and decomposable values, while the fourth result is obtained by using the fixed point theorem for contractive multivalued maps due to Covitz and Nadler.

The paper is organized as follows. In Section 2, we present an auxiliary lemma and recall some preliminary concepts of multivalued analysis that we need in the sequel. Section 3 contains the main existence results for the problem (1.1). In Section 4, some illustrative examples are discussed.

## 2. Preliminaries

2.1. **An auxiliary result.** In this subsection, we obtain an auxiliary result which is pivotal to define the solution of the problem (1.1).

**Lemma 2.1.** *Let $\alpha+(n-1)\beta \neq \frac{1}{n}\sum_{i=1}^{m}\gamma_i\eta_i^n$. For any $y \in C([0,1],\mathbb{R})$, the unique solution of the boundary-value problem*

$$\begin{gathered} u^{(n)}(t)=y(t), \quad t\in[0,1],\\ u(0)=0, \quad u'(0)=0, \quad u''(0)=0,\dots,u^{(n-2)}(0)=0,\\ \alpha u(1)+\beta u'(1)=\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}u(s)ds, \quad 0<\eta_i<1, \end{gathered} \tag{2.1}$$

*is given by*

$$\begin{aligned} u(t)=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)ds+\Lambda t^{n-1}\Big\{\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}y(s)ds\\ -\alpha\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)ds-\beta\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}y(s)ds\Big\}, \end{aligned} \tag{2.2}$$

*where*

$$\Lambda=\frac{1}{\alpha+(n-1)\beta-\frac{1}{n}\sum_{i=1}^{m}\gamma_i\eta_i^n}. \tag{2.3}$$

*Proof.* We know that the solution of the differential equation in (2.1) can be written as

$$u(t)=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)ds+c_0+c_1t+c_2t^2+\dots+c_{n-2}t^{n-2}+c_{n-1}t^{n-1}, \tag{2.4}$$

where $c_i, i=0,1,\dots n-1$ are arbitrary real constants. Using the given boundary conditions, we find that $c_0=c_1=c_2=\dots=c_{n-2}=0$, and

$$\begin{aligned} c_{n-1}=\Lambda\Big(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}y(s)ds-\alpha\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)ds\\ -\beta\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}y(s)ds\Big) \end{aligned}$$

where $\Lambda$ defined by (2.3). Substituting these values in (2.4), we get (2.2). This completes the proof. □

In view of Lemma 2.1, we define the solutions for (1.1) as follows.

**Definition 2.2.** A function $x \in AC^{(n-1)}([0,1],\mathbb{R})$ is called a solution of problem (1.1) if there exists a function $g \in L^1([0,1],\mathbb{R})$ with $g(t) \in F(t,x(t))$, a.e. on $[0,1]$ such that $x^{(n)}(t) = g(t)$, a.e. on $[0,1]$ and $x(0) = 0$, $x'(0) = 0$, $x''(0) = 0, \dots, x^{(n-2)}(0) = 0$, $\alpha x(1) + \beta x'(1) = \sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} x(s)ds$, $0 < \eta_i < 1$.

2.2. **Basic concepts of multivalued analysis.** Here we outline some basic definitions and results for multivalued maps, [12, 16, 20].

Let $C([0,1],\mathbb{R})$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

For a normed space $(X, \|\cdot\|)$, let

$$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\},$$
$$\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\},$$
$$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\},$$
$$\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}.$$

A multivalued map $G : X \to \mathcal{P}(X)$ :

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;
(ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in $X$ for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);
(iii) is called *upper semi-continuous (upper semi-continuous)* on $X$ if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of $x_0$ such that $G(\mathcal{N}_0) \subseteq N$;
(iv) $G$ is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set $B$ in $E$;
(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;
(vi) is said to be *measurable* if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable;
(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator $G$ will be denoted by $FixG$.

For each $x \in C([0,1],\mathbb{R})$, define the set of selections of $F$ by

$$S_{F,x} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\}.$$

We define the graph of $G$ to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

**Lemma 2.3** ([12, Proposition 1.2])**.** *If $G : X \to \mathcal{P}_{cl}(Y)$ is upper semi-continuous then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if $G$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

**Lemma 2.4** ([24])**.** *Let $X$ be a separable Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be measurable with respect to $t$ for each $x \in X$ and upper semi-continuous with respect to $x$ for almost all $t \in [0,1]$ and $S_{F,x} \neq \emptyset$ for any $x \in C([0,1],X)$, and* *let $\Theta$ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F : C([0,1],X) \to \mathcal{P}_{cp,c}(C([0,1],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

*is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.*

Next, we state the well-known Bohnenblust-Karlin fixed point theorem and the nonlinear alternative of Leray-Schauder for multivalued maps.

**Lemma 2.5** (Bohnenblust-Karlin [7]). *Let $D$ be a nonempty subset of a Banach space $X$, which is bounded, closed, and convex. Suppose that $G : D \to 2^X \setminus \{0\}$ is upper semi-continuous with closed, convex values such that $G(D) \subset D$ and $\overline{G(D)}$ is compact. Then $G$ has a fixed point.*

**Lemma 2.6** (Nonlinear alternative for Kakutani maps [19]). . *Let $E$ be a Banach space, $C$ a closed convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semi-continuous compact map. Then either*

- (i) *$F$ has a fixed point in $\overline{U}$, or*
- (ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

**Definition 2.7.** Let $A$ be a subset of $I \times \mathbb{R}$. $A$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $A$ belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $I$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

**Definition 2.8.** A subset $\mathcal{A}$ of $L^1(I,\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset I$, the function $u\chi_{\mathcal{J}} + v\chi_{I-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

**Lemma 2.9** ([9]). *Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1(I,\mathbb{R}))$ be a lower semi-continuous (l.s.c.) multivalued operator with nonempty closed and decomposable values. Then $N$ has a continuous selection, that is, there exists a continuous function (single-valued) $h : Y \to L^1(I,\mathbb{R})$ such that $h(x) \in N(x)$ for every $x \in Y$.*

Let $(X,d)$ be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [23]).

**Definition 2.10.** A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

- (a) $\gamma$-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \quad \text{for each } x, y \in X;$$

- (b) a contraction if it is $\gamma$-Lipschitz with $\gamma < 1$.

**Lemma 2.11** ([11]). *Let $(X,d)$ be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

## 3. Existence results

We will use the following assumptions:

(H1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is Carathéodory; i.e.,
  (i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
  (ii) $x \mapsto F(t,x)$ is upper semi-continuous for almost all $t \in [0,1]$;

(H2) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1],\mathbb{R}^+)$ such that
$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\rho(t)$$
for all $\|x\| \le \rho$ and for a.e. $t \in [0,1]$ and
$$\liminf_{\rho\to\infty} \frac{1}{\rho}\int_0^1 \varphi_\rho(t)dt = \mu.$$

(H3)
$$\mu\Big\{\frac{1}{n!} + |\Lambda|\Big(\frac{\sum_{i=1}^m |\gamma_i|\eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\Big)\Big\} < 1. \tag{3.1}$$

**Theorem 3.1** (Upper Semicontinuous case)**.** *Assume that* (H1)–(H3) *hold. Then the boundary value problem* (1.1) *has at least one solution on* $[0,1]$.

*Proof.* Define an operator $\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by
$$\begin{aligned} \mathcal{F}(x) = \Big\{h \in C([0,1],\mathbb{R}) : h(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)ds \\ &+ \Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!}g(s)ds \\ &- \alpha\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}g(s)ds - \beta\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}g(s)ds\Big\}, \end{aligned} \tag{3.2}$$
for $g \in S_{F,x}$. Observe that the fixed points of the operator $\mathcal{F}$ correspond to the solutions of the problem (1.1). We will show that $\mathcal{F}$ satisfies the assumptions of the Bohnenblust-Karlin fixed point theorem (Lemma 2.5). The proof consists of several steps.

**Step 1.** $\mathcal{F}(x)$ is convex for each $x \in C([0,1],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex ($F$ has convex values), and therefore we omit the proof.

**Step 2.** $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $C([0,1],\mathbb{R})$. For a positive number $\rho$, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C([0,1],\mathbb{R})$. We shall prove that there exists a positive number $\rho'$ such that $\mathcal{F}(B_{\rho'}) \subseteq B_{\rho'}$. If not, for each positive number $\rho$, there exists a function $x_\rho(\cdot) \in B_\rho$, however, $\|\mathcal{F}(x_\rho)\| > \rho$ for some $t \in [0,1]$ and
$$\begin{aligned} h_\rho(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g_\rho(s)ds + \Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!}g_\rho(s)ds \\ &- \alpha\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}g_\rho(s)ds - \beta\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}g_\rho(s)ds\Big\}, \end{aligned}$$
for some $g_\rho \in S_{F,x_\rho}$.

On the other hand,
$$\rho < \|\mathcal{F}(x_\rho)\|$$

$$\begin{aligned}
&\le \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\varphi_\rho(s)ds + |\Lambda t^{n-1}|\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^n}{n!}\varphi_\rho(s)ds \\
&\quad + |\alpha|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}\varphi_\rho(s)ds + |\beta|\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}\varphi_\rho(s)ds\Big\} \\
&\le \Big\{\frac{1}{n!} + |\Lambda|\Big(\frac{\sum_{i=1}^m |\gamma_i|\eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\Big)\Big\}\int_0^1 \varphi_\rho(s)ds.
\end{aligned}$$

Divide both sides of the above inequality by $\rho$, then taking the lower limit as $\rho \to \infty$, we obtain

$$\mu\Big\{\frac{1}{n!} + |\Lambda|\Big(\frac{\sum_{i=1}^m |\gamma_i|\eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\Big)\Big\} > 1,$$

which contradicts (3.1). Hence it follows that there exists a positive number $\rho'$ such that $\mathcal{F}(B_{\rho'}) \subseteq B_{\rho'}$.

**Step 3.** $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in B_r$, where $B_r$ is a bounded set of $C([0,1],\mathbb{R})$. For each $h \in \mathcal{F}(u)$, we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| &\le \Big|\frac{1}{(n-1)!}\int_0^{t_1}[(t_2-s)^{n-1} - (t_1-s)^{n-1}]g(s)ds \\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{n-1}g(s)ds\Big| \\
&\quad + |\Lambda||t_2^{n-1} - t_1^{n-1}|\Big(\sum_{i=1}^m \gamma_i \int_0^{\eta_i}\frac{(\eta_i-s)^{n-1}}{n!}|g(s)|ds \\
&\quad + |\alpha|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}|g(s)|ds + |\beta|\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}|g(s)|ds\Big) \\
&\le \frac{1}{n!}|2(t_2-t_1)^n + t_1^n - t_2^n|\int_0^1 \varphi_\rho(s)ds \\
&\quad + |\Lambda||t_2^{n-1} - t_1^{n-1}|\Big(\frac{\sum_{i=1}^m |\gamma_i|\eta_i^n}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\Big)\int_0^1 \varphi_\rho(s)ds.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. In view of steps 1-3, the Arzelá-Ascoli theorem applies and hence $\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

**Step 4.** $\mathcal{F}(x)$ is closed for each $x \in C([0,1],\mathbb{R})$. Let $\{u_n\}_{n\ge 0} \in \mathcal{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $g_n \in S_{F,u_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
u_n(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g_n(s)ds + \Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}g_n(s)ds \\
&\quad - \alpha\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}g_n(s)ds - \beta\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}g_n(s)ds\Big\}.
\end{aligned}$$

As $F$ has compact values, we pass onto a subsequence (if necessary) to obtain that $g_n$ converges to $g$ in $L^1([0,1],\mathbb{R})$. Thus, $g \in S_{F,u}$ and for each $t \in [0,1]$, we have

$$u_n(t) \to u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)ds + \Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}g(s)ds$$

$$- \alpha \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g ds - \beta \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} g(s) ds \Big\}.$$

Hence, $u \in \mathcal{F}(x)$. By Lemma 2.3, $\mathcal{F}$ will be upper semi-continuous (upper semi-continuous) if we prove that it has a closed graph since $\mathcal{F}$ is already shown to be completely continuous.

**Step 5.**$\mathcal{F}$ has a closed graph. Let $x_n \to x_*, h_n \in \mathcal{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(x_*)$. Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} g \mapsto \Theta(g)(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s) ds + \Lambda t^{n-1} \Big\{ \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} g(s) ds \\ &\quad - \alpha \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s) ds - \beta \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} g(s) ds \Big\}. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \Big\| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} (g_n(s) - g_*(s)) ds \\ &\quad + \Lambda t^{n-1} \Big\{ \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} (g_n(s) - g_*(s)) ds \\ &\quad - \alpha \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} (g_n(s) - g_*(s)) ds \\ &\quad - \beta \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} (g_n(s) - g_*(s)) ds \Big\} \Big\| \to 0, \end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 2.4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g_*(s) ds + \Lambda t^{n-1} \Big\{ \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} g_*(s) ds \\ &\quad - \alpha \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g_*(s) ds - \beta \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} g_*(s) ds \Big\}, \end{aligned}$$

for some $g_* \in S_{F,x_*}$.

Hence, we conclude that $\mathcal{F}$ is a compact multivalued map, upper semi-continuous with convex closed values. In view of Lemma 2.5, we deduce that $\mathcal{F}$ has a fixed point which is a solution of the problem (1.1). This completes the proof. □

For the next theorem we use the assumptions:

(H4) there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \text{for each } (t,x) \in [0,1] \times \mathbb{R};$$

(H5) there exists a constant $M > 0$ such that

$$\frac{M}{\psi(M)\|p\|_{L^1} \Big\{ \frac{1}{n!} + |\Lambda| \Big( \frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \Big) \Big\}} > 1.$$

**Theorem 3.2.** *Assume that* (H1), (H4), (H5) *hold. Then* (1.1) *has at least one solution on* $[0,1]$.

*Proof.* Let $x \in \lambda\mathcal{F}(x)$ for some $\lambda \in (0,1)$, where $\mathcal{F}$ is defined by (3.2). Then we show there exists an open set $U \subseteq C(I,\mathbb{R})$ with $x \notin \mathcal{F}(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\mathcal{F}(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned} x(t) = \lambda \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s)ds + \lambda\Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} g(s)ds \\ - \alpha \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds - \beta \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} g(s)ds\Big\}. \end{aligned}$$

In view of (H4), we have for each $t \in [0,1]$,

$$\begin{aligned} |x(t)| &\le \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |g(s)|ds + |\Lambda|\Big\{\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} |g(s)|ds \\ &\quad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |g(s)|ds + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} |g(s)|ds\Big\} \\ &\le \psi(\|x\|) \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} p(s)ds + |\Lambda|\psi(\|x\|)\Big\{\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} p(s)ds \\ &\quad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} p(s)ds + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} p(s)ds\Big\} \\ &\le \psi(\|u\|)\Big\{\frac{1}{n!} + |\Lambda|\Big(\frac{\sum_{i=1}^m |\gamma_i|\eta_i^n}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\Big)\Big\} \int_0^1 p(s)ds. \end{aligned}$$

Consequently,

$$\frac{\|x\|}{\psi(\|x\|)\|p\|_{L^1}\Big\{\frac{1}{n!} + |\Lambda|\Big(\frac{\sum_{i=1}^m |\gamma_i|\eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\Big)\Big\}} \le 1.$$

In view of (H5), there exists $M$ such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.$$

Proceeding as in the proof of Theorem 3.1, one can claim that the operator $\mathcal{F} : \overline{U} \to \mathcal{P}(C([0,1],\ \mathbb{R}))$ is a compact multivalued map, upper semi-continuous with convex closed values. From the choice of $U$, there is no $x \in \partial U$ such that $x \in \lambda\mathcal{F}(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.6), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

As a next result, we study the case when $F$ is not necessarily convex valued by combining the nonlinear alternative of Leray-Schauder type with the selection theorem due to Bressan and Colombo [9] for lower semi-continuous maps with decomposable values. We will use the following assumption

- (H6) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that
  - (a) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
  - (b) $x \mapsto F(t,x)$ is lower semi-continuous for each $t \in [0,1]$;

**Theorem 3.3** (The lower semi-continuous case)**.** *Assume that* (H4)–(H6) *hold. Then* (1.1) *has at least one solution on* $[0,1]$.

*Proof.* It follows from (H4) and (H6) that $F$ is of l.s.c. type [16]. Then, by Lemma 2.9, there exists a continuous function $f : AC^1([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$, where $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ is the Nemytskii operator associated with $F$, defined as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\}.$$

Consider the problem

$$\begin{gathered} x^{(n)}(t) = f(x(t)), \quad t \in [0,1], \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \dots, x^{(n-2)}(0) = 0, \\ \alpha x(1) + \beta x'(1) = \sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} x(s)ds, \quad 0 < \eta_i < 1. \end{gathered} \tag{3.3}$$

Observe that if $x \in AC^{(n-1)}([0,1],\mathbb{R})$ is a solution of (3.3), then $x$ is a solution to the problem (1.1). To transform problem (3.3) into a fixed point problem, we define an operator $\overline{\mathcal{F}}$ as

$$\begin{aligned} \overline{\mathcal{F}}x(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(x(s))ds + \Lambda t^{n-1} \Big\{ \sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} f(x(s))ds \\ - \alpha \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(x(s))ds - \beta \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(x(s))ds \Big\}. \end{aligned}$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.2. So we omit it. This completes the proof. □

Now we show the existence of solutions for (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [11]. We sue the assumptions:

- (H7) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.
- (H8) $H_d(F(t,x), F(t,\bar{x})) \le m(t)|x - \bar{x}|$ for almost all $t \in [0,1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0,1],\mathbb{R}^+)$ and $d(0, F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.

**Theorem 3.4** (The Lipschitz case)**.** *Assume* (H7), (H8) *hold. Then* (1.1) *has at least one solution on* $[0,1]$ *if*

$$\|m\|_{L^1} \Big\{ \frac{1}{n!} + |\Lambda| \Big( \frac{\sum_{i=1}^{m} |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \Big) \Big\} < 1.$$

*Proof.* We transform the problem (1.1) into a fixed point problem by means of the operator $\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined by (3.2) and show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 2.11. The proof will be given in two steps.

**Step 1.** $\mathcal{F}(x)$ is nonempty and closed for every $v \in S_{F,x}$. Since the set-valued map $F(\cdot, x(\cdot))$ is measurable with the measurable selection theorem (e.g., [10, Theorem III.6]), it admits a measurable selection $v : [0,1] \to \mathbb{R}$. Moreover, by the assumption (H8), we have

$$|v(t)| \le m(t) + m(t)|x(t)|,$$

that is, $v \in L^1([0,1],\mathbb{R})$ and hence $F$ is integrably bounded. Therefore, $S_{F,y} \neq \emptyset$. Moreover $\mathcal{F}(x) \in \mathcal{P}_{cl}(C([0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, as proved in Step 4 of Theorem 3.1.

**Step 2.** Next we show that there exists $\delta < 1$ such that

$$H_d(\mathcal{F}(x),\mathcal{F}(\bar{x})) \le \delta\|x-\bar{x}\| \quad \text{for each } x,\bar{x} \in AC^{(n-1)}([0,1],\mathbb{R}).$$

Let $x,\bar{x} \in AC^{(n-1)}([0,1],\mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}v_1(s)ds + \Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^n}{n!}v_1(s)ds \\ - \alpha\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}v_1(s)ds - \beta\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}v_1(s)ds\Big\}. \end{aligned}$$

By (H8), we have

$$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w(t) \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w(t)| \le m(t)|x(t)-\bar{x}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w(t)| \le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t,\bar{x}(t))$ is measurable [10, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for $U$. So $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t)-v_2(t)| \le m(t)|x(t)-\bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}v_2(s)ds + \Lambda t^{n-1}\Big\{\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^n}{n!}v_2(s)ds \\ - \alpha\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}v_2(s)ds - \beta\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}v_2(s)ds\Big\}. \end{aligned}$$

Thus,

$$\begin{aligned} &|h_1(t)-h_2(t)| \\ &\le \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|v_1(s)-v_2(s)|(s)ds \\ &\quad + |\Lambda|t^{n-1}\Big\{\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i-s)^n}{n!}|v_1(s)-v_2(s)|(s)ds \\ &\quad + |\alpha|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)\|x-\overline{x}\|ds + |\beta|\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}m(s)\|x-\overline{x}\|ds\Big\} \\ &\le \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}m(s)\|x-\overline{x}\|ds \\ &\quad + |\Lambda|t^{n-1}\Big\{\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i-s)^n}{n!}m(s)\|x-\overline{x}\|ds \\ &\quad + |\alpha|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)\|x-\overline{x}\|ds + |\beta|\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}m(s)\|x-\overline{x}\|ds\Big\} \end{aligned}$$

$$\leq \|m\|_{L^1}\Big\{\frac{1}{n!}+|\Lambda|\Big(\frac{\sum_{i=1}^m|\gamma_i|\eta_i^n}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\Big)\Big\}\|x-\overline{x}\|.$$

Hence,

$$\|h_1-h_2\|\leq \|m\|_{L^1}\Big\{\frac{1}{n!}+|\Lambda|\Big(\frac{\sum_{i=1}^m|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\Big)\Big\}\|x_1-x_2\|.$$

Analogously, interchanging the roles of $x$ and $\overline{x}$, we obtain

$$\begin{aligned} &H_d(\mathcal{F}(x),\mathcal{F}(\bar{x}))\leq \delta\|x-\bar{x}\|\\ &\leq \|m\|_{L^1}\Big\{\frac{1}{n!}+|\Lambda|\Big(\frac{\sum_{i=1}^m|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\Big)\Big\}\|x-\overline{x}\|. \end{aligned}$$

Since $\mathcal{F}$ is a contraction, it follows by Lemma 2.11 that $\mathcal{F}$ has a fixed point $x$ which is a solution of (1.1). This completes the proof. □

## 4. Examples

Consider the boundary-value problem

$$\begin{gathered} x'''(t)\in F(t,x(t)),\quad \text{a.e. } t\in[0,1],\\ x(0)=0,\quad x'(0)=0,\quad x(1)+x'(1)=\sum_{i=1}^{3}\gamma_i\int_0^{\eta_i}x(s)ds,\quad 0<\eta_i<1, \end{gathered}\tag{4.1}$$

where $n=3$, $\alpha=1$, $\beta=1$, $\eta_1=1/4$, $\eta_2=1/2$, $\eta_3=3/4$, $\gamma_1=1$, $\gamma_2=1/3$, $\gamma_3=2/3$. In (4.1), $F(t,x(t))$ will be chosen according to the requirement at hand. With the given data, it is found that

$$\begin{gathered} \Lambda=\frac{1}{\alpha+(n-1)\beta-\frac{1}{n}\sum_{i=1}^m\gamma_i\eta_i^n}\approx 0.346362,\\ Q=\frac{1}{n!}+|\Lambda|\Big(\frac{\sum_{i=1}^m|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\Big)\approx 0.400976. \end{gathered}$$

**Example 4.1.** Let $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x\to F(t,x)=\Big[\frac{x^2e^{-x^2}}{x^2+3},\ \frac{t|x|\sin|x|}{|x|+1}\Big].\tag{4.2}$$

For $f\in F$, we have

$$|f|\leq \max\Big(\frac{x^2e^{-x^2}}{x^2+3},\frac{t|x|\sin|x|}{|x|+1}\Big)\leq t|x|+1,\quad x\in\mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}}:=\sup\{|y|\,:\,y\in F(t,x)\}\leq \rho t+1=\varphi_\rho(t),\quad \|x\|\leq\rho.$$

We can find that $\liminf_{\rho\to\infty}\frac{1}{\rho}\int_0^1\varphi_\rho(s)ds=\mu=1/2$ and $\Lambda\mu\approx 0.173181<1$. Therefore, all the conditions of Theorem 3.1 are satisfied. So, the problem (4.1) with $F(t,x)$ given by (4.2) has at least one solution on $[0,1]$.

**Example 4.2.** If $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x\to F(t,x)=\Big[\frac{x^4}{x^4+2}+e^{-x^2}+t+2,\ \frac{|x|}{|x|+1}+t^2+\frac{1}{2}\Big].\tag{4.3}$$

For $f \in F$, we have

$$|f| \leq \max\Big(\frac{x^4}{x^4+2} + e^{-x^2} + t + 2,\ \frac{|x|}{|x|+1} + t^2 + \frac{1}{2}\Big) \leq 5, \quad x \in \mathbb{R}.$$

Here $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 5 = p(t)\psi(\|x\|)$, $x \in \mathbb{R}$, with $p(t) = 1$, $\psi(\|x\|) = 5$. It is easy to verify that $M > 2.00488$. Then, by Theorem 3.2, the problem (4.1) with $F(t,x)$ given by (4.3) has at least one solution on $[0,1]$.

**Example 4.3.** Consider the multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$x \to F(t,x) = \big[0,\ (t+1)\sin x + \frac{2}{3}\big]. \tag{4.4}$$

Then we have

$$\sup\{|u| : u \in F(t,x)\} \leq (t+1) + \frac{2}{3},$$
$$H_d(F(t,x),\ F(t,\overline{x})) \leq (t+1)|x - \overline{x}|.$$

Let $m(t) = t + 1$. Then $H_d(F(t,x), F(t,\overline{x})) \leq m(t)|x - \overline{x}|$, and $\|m\|_{L^1}\Lambda \approx 0.519543 < 1$. By Theorem 3.4, problem (4.1) with $F(t,x)$ given by (4.4) has at least one solution on $[0,1]$.

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