Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 206, pp. 1–29.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ASYMPTOTIC STABILITY OF FRACTIONAL IMPULSIVE NEUTRAL STOCHASTIC PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract. In this article, we study the asymptotical stability in p-th moment of mild solutions to a class of fractional impulsive partial neutral stochastic integro-differential equations with state-dependent delay in Hilbert spaces. We assume that the linear part of this equation generates an α-resolvent operator and transform it into an integral equation. Sufficient conditions for the existence and asymptotic stability of solutions are derived by means of the Krasnoselskii-Schaefer type fixed point theorem and properties of the α-resolvent operator. An illustrative example is also provided.

1. Introduction

Partial stochastic differential equations have attracted the considerable attention of researchers and many qualitative theories for the solutions of this kind have been derived; see [7, 8] and the references therein. In particular, the stability theory of stochastic differential equations has been popularly applied in variety fields of science and technology. Several authors have established the stability results of mild solutions for these equations by using various techniques; see, for example, Govindan [13] considered the existence and stability for mild solution of stochastic partial differential equations by applying the comparison theorem. Caraballo and Liu [5] proved the exponential stability for mild solution to stochastic partial differential equations with delays by utilizing the well-known Gronwall inequality. The exponential stability of the mild solutions to the semilinear stochastic delay evolution equations have been discussed by using Lyapunov functionals in Liu [16]. The author [17] considered the exponential stability for stochastic partial functional differential equations by means of the Razuminkhin-type theorem. Liu and Truman [18] investigated the almost sure exponential stability of mild solution for stochastic partial functional differential equation by using the analytic technique. Taniguchi [32] discussed the exponential stability for stochastic delay differential equations by the energy inequality. Using fixed point approach, Luo [20] studied the asymptotic stability of mild solutions of stochastic partial differential equations with finite delays. Further, Sakthivel et al. [22, 25, 26] established the asymptotic

2000 *Mathematics Subject Classification.* 34A37, 60H15, 35R60, 93E15, 26A33.
Key words and phrases. Asymptotic stability; impulsive neutral integro-differential equations; stochastic integro-differential equations; α-resolvent operator.

Submitted April 28, 2013. Published September 18, 2013.

stability and exponential stability of second-order stochastic evolution equations in Hilbert spaces.

Impulsive differential and integro-differential systems are occurring in the field of physics where it has been very intensive research topic since the theory provides a natural framework for mathematical modeling of many physical phenomena [3]. Moreover, various mathematical models in the study of population dynamics, biology, ecology and epidemic can be expressed as impulsive stochastic differential equations. In recent years, the qualitative dynamics such as the existence and uniqueness, stability for first-order impulsive partial stochastic differential equations have been extensively studied by many authors; for instance, Sakthivel and Luo [23, 24] studied the existence and asymptotic stability in p-th moment of mild solutions to impulsive stochastic partial differential equations through fixed point theory. Anguraj and Vinodkumar [1] investigated the existence, uniqueness and stability of mild solutions of impulsive stochastic semilinear neutral functional differential equations without a Lipschitz condition and with a Lipschitz condition. Chen [6], Long et al. [19] discussed the exponential p-stability of impulsive stochastic partial functional differential equations. He and Xu [14] studied the existence, uniqueness and exponential p-stability of a mild solution of the impulsive stochastic neutral partial functional differential equations by using Banach fixed point theorem.

On the other hand, fractional differential equations play an important role in describing some real world problems. This is caused both by the intensive development of the theory of fractional calculus itself and by applications of such constructions in various domains of science, such as physics, mechanics, chemistry, engineering, etc. For details, see [21] and references therein. The existence of solutions of fractional semilinear differential and integrodifferential equations are one of the theoretical fields that investigated by many authors [11, 30, 33, 34]. Recently, much attention has been paid to the differential systems involving the fractional derivative and impulses. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically fractional order differential equations rather than integer order differential equations. Consequently, there are many contributions relative to the solutions of various impulsive semilinear fractional differential and integrodifferential systems in Banach spaces; see [2, 9, 31]. The qualitative properties of fractional stochastic differential equations have been considered only in few publications [7, 12, 27, 28, 35]. More recently, Sakthivel et al. [29] studied the existence and asymptotic stability in p-th moment of a mild solution to a class of nonlinear fractional neutral stochastic differential equations with infinite delays in Hilbert spaces. However, up to now the existence and asymptotic stability of mild solutions for fractional impulsive neutral partial stochastic integro-differential equations with state-dependent delay have not been considered in the literature. In order to fill this gap, this paper studies the existence and asymptotic stability of the following nonlinear impulsive fractional stochastic integro-differential equation of the form

$$\begin{aligned} {}^cD^\alpha N(x_t) = AN(x_t) + \int_0^t R(t-s)N(x_s)ds + h(t, x(t-\rho_2(t)))dt \\ + f(t, x(t-\rho_3(t)))\frac{dw(t)}{dt}, \quad t \geq 0, t \neq t_k, \end{aligned} \tag{1.1}$$

$$x_0(\cdot) = \varphi \in \mathfrak{B}_{\mathcal{F}_0}([\tilde{m}(0), 0], H), \quad x'(0) = 0, \tag{1.2}$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \quad t = t_k,\ k = 1, \dots, m, \tag{1.3}$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1,2)$, A, $(R(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in $(H, \|\cdot\|_H)$, and $D_t^\alpha \sigma(t)$ represents the Caputo derivative of order $\alpha > 0$ defined by

$$D_t^\alpha \sigma(t) = \int_0^t \eta_{n-\alpha}(t-s) \frac{d^n}{ds^n}\sigma(s)ds,$$

where n is the smallest integer greater than or equal to α and $\eta_\beta(t) := t^{\beta-1}/\Gamma(\beta)$, $t > 0$, $\beta \geq 0$. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Wiener process with a covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w; and $N(x_t) = x(0) + g(t, x(t - \rho_1(t))), x \in H$, and $g, h : [0,\infty) \times H \to H, f : [0,\infty) \times H \to L(K,H)$, are all Borel measurable; $I_k : H \to H (k = 1, \dots, m)$, are given functions. Moreover, the fixed moments of time t_k satisfies $0 < t_1 < \dots < t_m < \lim_{k\to\infty} t_k = \infty$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively; $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump; let $\rho_i(t) \in C(\mathbb{R}^+, \mathbb{R}^+)(i = 1,2,3)$ satisfy $t - \rho_i(t) \to \infty$ as $t \to \infty$, and $\tilde{m}(0) = \max\{\inf_{s\geq 0}(s - \rho_i(s)), i = 1,2,3\}$. Here $\mathfrak{B}_{\mathcal{F}_0}([\tilde{m}(0), 0], H)$ denote the family of all almost surely bounded, $\mathcal{F}_0$- measurable, continuous random variables $\varphi(t) : [\tilde{m}(0), 0] \to H$ with norm $\|\varphi\|_{\mathfrak{B}} = \sup_{\tilde{m}(0)\leq t\leq 0} E\|\varphi(t)\|_H$.

To the best of our knowledge, most of the previous research on the existence and stability investigation for impulsive stochastic systems was based upon a Lipschitz condition. This condition turns out to be restrictive. In this paper, we establish sufficient conditions for the existence and asymptotic stability in p-th moment of mild for problem (1.1)–(1.3) by using Krasnoselskii-Schaefer type fixed point theorem [4] with the α-resolvent operator. The obtained result can be seen as a contribution to this emerging field.

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 aims to prove the main results. An example is presented in the last section.

2. Preliminaries

Let K and H be two real separable Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$, their inner products and by $\|\cdot\|_K, \|\cdot\|_H$ their vector norms, respectively.

Let $(\Omega, \mathcal{F}, P; \mathbb{F})(\mathbb{F} = \{\mathcal{F}\}_{t\geq 0})$ be a complete probability space satisfying that $\mathcal{F}_0$ contains all P-null sets. Let $\{e_i\}_{i=1}^\infty$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Brownian motion with a trace class operator Q, denote $\mathrm{Tr}(Q) = \sum_{i=1}^\infty \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^\infty \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^\infty$ are mutually independent one-dimensional standard Brownian motions. Then, the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. Let $L(K,H)$ denote the space of all bounded linear operators from K into H equipped with the usual operator norm $\|\cdot\|_{L(K,H)}$ and we abbreviate this notation to $L(H)$ when $H = K$. For $\varsigma \in L(K,H)$

we define

$$\|\varsigma\|_{L_2^0}^2 = \mathrm{Tr}(\varsigma Q \varsigma^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varsigma e_n\|^2.$$

If $\|\varsigma\|_{L_2^0}^2 < \infty$, then ς is called a Q-Hilbert-Schmidt operator, and let $L_2^0(K, H)$ denote the space of all Q-Hilbert-Schmidt operators $\varsigma : K \to H$. For a basic reference, the reader is referred to [8].

Let $\mathbb{Y}$ be the space of all $\mathcal{F}_0$-adapted process $\psi(t, \tilde{w}) : [\tilde{m}(0), \infty) \times \Omega \to \mathbb{R}$ which is almost certainly continuous in t for fixed $\tilde{w} \in \Omega$. Moreover $\psi(s, \tilde{w}) = \varphi(s)$ for $s \in [\tilde{m}(0), 0]$ and $E\|\psi(t, \tilde{w})\|_H^p \to 0$ as $t \to \infty$. Also $\mathbb{Y}$ is a Banach space when it is equipped with a norm defined by

$$\|\psi\|_{\mathbb{Y}} = \sup_{t \geq 0} E\|\psi(t)\|_H^p.$$

Now, we give knowledge on the α-resolvent operator which appeared in [30].

Definition 2.1. A one-parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t \geq 0}$ on H is called an α-resolvent operator for

$$^cD^\alpha x(t) = Ax(t) + \int_0^t R(t-s)x(s)ds, \tag{2.1}$$

$$x(0) = x_0 \in H, \quad x'(0) = 0, \tag{2.2}$$

if the following conditions are satisfied

(a) The function $\mathcal{R}_\alpha(\cdot) : [0, \infty) \to L(H)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in H$ and $\alpha \in (1, 2)$;

(b) For $x \in D(A)$, we have $\mathcal{R}_\alpha(\cdot)x \in C([0, \infty), [D(A)]) \cap C^1((0, \infty), H)$,

$$D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t R(t-s)\mathcal{R}_\alpha(s)x\,ds,$$

$$D_t^\alpha \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)R(s)x\,ds$$

for every $t \geq 0$.

In this work we use the following assumptions:

(P1) The operator $A : D(A) \subseteq H \to H$ is a closed linear operator with $[D(A)]$ dense in H. Let $\alpha \in (1, 2)$. For some $\phi_0 \in (0, \frac{\pi}{2}]$, for each $\phi < \phi_0$ there is a positive constant $C_0 = C_0(\phi)$ such that $\lambda \in \rho(A)$ for each

$$\lambda \in \Sigma_{0,\alpha\vartheta} = \{\lambda \in \mathbb{C}, \lambda \neq 0, |\arg(\lambda)| < \alpha\vartheta\},$$

where $\vartheta = \phi + \frac{\pi}{2}$ and $\|R(\lambda, A)\| \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,\alpha\vartheta}$.

(P2) For all $t \geq 0, R(t) : D(R(t)) \subseteq H \to H$ is a closed linear operator, $D(A) \subseteq D(R(t))$ and $R(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|R(t)x\|_H \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{R} : \Sigma_{0,\pi/2} \to L([D(A)], H)$ has an analytical extension (still denoted by $\widehat{R}$) to $\Sigma_{0,\vartheta}$ such that $\|\widehat{R}(\lambda)x\|_H \leq \|\widehat{R}(\lambda)\|_H\|x\|_1$ for all $x \in D(A)$, and $\|\widehat{R}(\lambda)\|_H = O(1/|\lambda|)$, as $|\lambda| \to \infty$.

(P1) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant $\widetilde{C}$ such that $A(D) \subseteq D(A), \widehat{R}(\lambda)(D) \subseteq D(A)$, and $\|A\widehat{R}(\lambda)x\|_H \leq \widetilde{C}\|x\|_H$ for every $x \in D$ and all $\lambda \in \Sigma_{0,\vartheta}$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \vartheta)$,

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C}, |\lambda| > r, |\arg(\lambda)| < \theta\},$$

for $\Gamma_{r,\theta}, \Gamma^i_{r,\theta}$, $i = 1, 2, 3$, are the paths

$$\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}, \quad \Gamma^2_{r,\theta} = \{te^{i\xi} : |\xi| \leq \theta\}, \quad \Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\},$$

and $\Gamma_{r,\theta} = \cup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise. In addition, $\rho_\alpha(G_\alpha)$ are the sets

$$\rho_\alpha(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{R}(\lambda))^{-1} \in L(X)\}.$$

We now define the operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{R}_\alpha(t) := \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda) d\lambda, & t > 0,\\ I, & t = 0. \end{cases}$$

Lemma 2.2 ([30])**.** *Assume that conditions* (P1)–(P3) *are fulfilled. Then there exists a unique α-resolvent operator for problem* (2.1)-(2.2).

Lemma 2.3 ([30])**.** *The function $\mathcal{R}_\alpha : [0, \infty) \to L(H)$ is strongly continuous and $\mathcal{R}_\alpha : (0, \infty) \to L(H)$ is uniformly continuous.*

Definition 2.4 ([30])**.** Let $\alpha \in (1, 2)$, we define the family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{S}_\alpha(t)x := \int_0^t g_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)ds$$

for each $t \geq 0$.

Lemma 2.5 ([30])**.** *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $L(H)$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $L(H)$.*

Lemma 2.6 ([30])**.** *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $L([D(A)])$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $L([D(A)])$.*

Lemma 2.7 ([30])**.** *If $R(\lambda_0^\alpha, A)$ is compact for some $\lambda_0^\alpha \in \rho(A)$, then $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$.*

Definition 2.8. A stochastic process $\{x(t), t \in [0, T]\}(0 \leq T < \infty)$ is called a mild solution of (1.1)-(1.3) if

(i) $x(t)$ is adapted to $\mathcal{F}_t, t \geq 0$.
(ii) $x(t) \in H$ has càdlàg paths on $t \in [0, T]$ a.s and for each $t \in [0, T]$, $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \mathcal{R}_\alpha(t)[\varphi(0) - g(0, \varphi(-\rho_1(0)))] + g(t, x(t - \rho_1(t))) \\ + \int_0^t \mathcal{S}_\alpha(t-s)h(s, x(s - \rho_2(s)))ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s, x(s - \rho_3(s)))dw(s), & t \in [0, t_1], \\ \mathcal{R}_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-)) - g(t_1, x(t_1^+ - \rho_1(t_1^+)))] \\ +g(t, x(t - \rho_1(t))) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)h(s, x(s - \rho_2(s)))ds \\ + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s, x(s - \rho_3(s)))dw(s), & t \in (t_1, t_2], \\ \dots \\ \mathcal{R}_\alpha(t - t_m)[x(t_m^-) + I_m(x(t_m^-)) - g(t_m, x(t_m^+ - \rho_1(t_m^+)))] \\ +g(t, x(t - \rho_1(t))) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)h(s, x(s - \rho_2(s)))ds \\ + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s, x(s - \rho_3(s)))dw(s), & t \in (t_m, T]. \end{cases} \tag{2.3}$$

Definition 2.9. Let $p \geq 2$ be an integer. Equation (2.3) is said to be stable in the p-th moment, if for any $\varepsilon > 0$, there exists a $\tilde{\delta} > 0$ such that $\|\varphi\|_{\mathfrak{B}} < \tilde{\delta}$ guarantees that

$$E\big[\sup_{t\geq 0}\|x(t)\|_H^p\big] < \varepsilon.$$

Definition 2.10. Let $p \geq 2$ be an integer. Equation (2.3) is said to be asymptotically stable in p-th moment if it stable in the p-th moment and for any $\varphi \in \mathfrak{B}_{\mathcal{F}_0}([\tilde{m}(0), 0], H)$,

$$\lim_{T\to+\infty} E\big[\sup_{t\geq T}\|x(t)\|_H^p\big] = 0.$$

Lemma 2.11 ([8])**.** *For any $p \geq 1$ and for arbitrary $L_2^0(K,H)$-valued predictable process $\phi(\cdot)$ such that*

$$\sup_{s\in[0,t]} E\Big\|\int_0^s \phi(v)dw(v)\Big\|_H^{2p} \leq C_p\Big(\int_0^t (E\|\phi(s)\|_{L_2^0}^{2p})^{1/p}ds\Big)^p, \quad t \in [0,\infty),$$

where $C_p = (p(2p-1))^p$. Next, we state a Krasnoselskii-Schaefer type fixed point theorem.

Lemma 2.12 ([4])**.** *Let Φ_1, Φ_2 be two operators such that:*

(a) Φ_1 *is a contraction, and*
(b) Φ_2 *is completely continuous.*

Then either

(i) *the operator equation $\Phi_1 x + \Phi_2 x$ has a solution, or*
(ii) *the set $\Upsilon = \{x \in H : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0,1)$.*

3. Main results

In this section we present our result on asymptotic stability in the p-th moment of mild solutions of system (1.1)-(1.3). for this, we state the following hypotheses:

(H1) The operator families $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$, and there exist constants $M > 0, \delta > 0$ such that $\|\mathcal{R}_\alpha(t)\|_{L(H)} \leq Me^{-\delta t}$ and $\|\mathcal{S}_\alpha(t)\|_{L(H)} \leq Me^{-\delta t}$ for every $t \geq 0$.

(H2) The function $g : [0,\infty) \times H \to H$ is continuous and there exists $L_g > 0$ such that

$$E\|g(t,\psi_1) - g(t,\omega_2)\|_H^p \leq L_g\|\psi_1 - \psi_2\|_H^p, \quad t \geq 0, \omega_1, \psi_2 \in H;$$
$$E\|g(t,\psi)\|_H^p \leq L_g\|\psi\|_H^p, \quad t \geq 0, \psi \in H.$$

(H3) The function $h : [0,\infty) \times H \to H$ satisfies the following conditions:
  (i) The function $h : [0,\infty) \times H \to H$ is continuous.
  (ii) There exist a continuous function $m_h : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $\Theta_h : [0,\infty) \to (0,\infty)$ such that

$$E\|h(t,\psi)\|_H^p \leq m_h(t)\Theta_h(E\|\psi\|_H^p), \quad t \geq 0, \psi \in H.$$

(H4) The function $f : [0,\infty) \times H \to L(K,H)$ satisfies the following conditions:
  (i) The function $f : [0,\infty) \times H \to L(K,H)$ is continuous.
  (ii) There exist a continuous function $m_f : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $\Theta_f : [0,\infty) \to (0,\infty)$ such that

$$E\|f(t,\psi)\|_H^p \leq m_f(t)\Theta_f(E\|\psi\|_H^p),\ t \geq 0, \psi \in H,$$

with
$$\int_1^\infty \frac{1}{\Theta_h(s)+\Theta_f(s)}ds = \infty. \tag{3.1}$$

(H5) The functions $I_k : H \to H$ are completely continuous and that there are constants d_k^j, $k = 1,2,\dots m$, $j = 1,2$, such that $E\|I_k(x)\|_H^p \le d_k^1 E\|x\|_H^p + d_k^2$, for every $x \in H$.

In the proof of the existence theorem, we need the following lemmas.

Lemma 3.1. *Assume that conditions* (H1), (H3) *hold. Let* Φ_1 *be the operator defined by: for each* $x \in \mathbb{Y}$,
$$(\Phi_1 x)(t) = \begin{cases} \int_0^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds, & t \in [0,t_1],\\ \int_{t_1}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds, & t \in (t_1,t_2],\\ \dots \\ \int_{t_m}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds, & t \in (t_m,t_{m+1}],\\ \dots \end{cases} \tag{3.2}$$
Then Φ_1 *is continuous and maps* $\mathbb{Y}$ *into* $\mathbb{Y}$.

Proof. We first prove that Φ_1 is continuous in the p-th moment on $[0,\infty)$. Let $x \in \mathbb{Y}$, $\tilde t \ge 0$ and $|\xi|$ be sufficiently small. Then for $\tilde t \in [0,t_1]$, by using Hölder's inequality, we have
$$\begin{aligned}
&E\|(\Phi_1 x)(\tilde t+\xi) - (\Phi_1 x)(\tilde t)\|_H^p\\
&\le 2^{p-1}E\Big\|\int_0^{\tilde t}[\mathcal{S}_\alpha(\tilde t+\xi-s)-\mathcal{S}_\alpha(\tilde t-s)]h(s,x(s-\rho_2(s)))ds\Big\|_H^p\\
&\quad + 2^{p-1}E\Big\|\int_{\tilde t}^{\tilde t+\xi}\mathcal{S}_\alpha(\tilde t+\xi-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p\\
&\le 2^{p-1}E\Big[\int_0^{\tilde t}\|[\mathcal{S}_\alpha(\tilde t+\xi-s)-\mathcal{S}_\alpha(\tilde t-s)]h(s,x(s-\rho_2(s)))\|_H ds\Big]^p\\
&\quad + 2^{p-1}M^pE\Big[\int_{\tilde t}^{\tilde t+\xi}e^{-\delta(\tilde t+\xi-s)}\|h(s,x(s-\rho_2(s)))\|_H ds\Big]^p\\
&\le 2^{p-1}\Big[\int_0^{\tilde t}\|\mathcal{S}_\alpha(\tilde t+\xi-s)-\mathcal{S}_\alpha(\tilde t-s)\|_{L(H)}^{(p/p-1)}ds\Big]^{p-1}\\
&\quad\times\int_0^{\tilde t}E\|h(s,x(s-\rho_2(s)))\|_H^p ds\\
&\quad + 2^{p-1}M^p\Big[\int_{\tilde t}^{\tilde t+\xi}e^{-(p\delta/p-1)(\tilde t+\xi-s)}ds\Big]^{p-1}\\
&\quad\times\int_{\tilde t}^{\tilde t+\xi}E\|h(s,x(s-\rho_2(s)))\|_H^p ds \to 0 \quad \text{as } \xi\to\infty.
\end{aligned}$$
Similarly, for any $\tilde t \in (t_k,t_{k+1}]$, $k = 1,2,\dots$, we have
$$\begin{aligned}
&E\|(\Phi_1 x)(\tilde t+\xi) - (\Phi_1 x)(\tilde t)\|_H^p\\
&\le 2^{p-1}\Big[\int_{t_k}^{\tilde t}\|\mathcal{S}_\alpha(\tilde t+\xi-s)-\mathcal{S}_\alpha(\tilde t-s)\|_{L(H)}^{-(p/p-1)}ds\Big]^{p-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{t_k}^{\tilde{t}} E\|h(s,x(s-\rho_2(s)))\|_H^p ds \\
&+ 2^{p-1}M^p\Big[\int_{\tilde{t}}^{\tilde{t}+\xi} e^{-(p\delta/p-1)(\tilde{t}+\xi-s)}ds\Big]^{p-1} \\
&\times \int_{\tilde{t}}^{\tilde{t}+\xi} E\|h(s,x(s-\rho_2(s)))\|_H^p ds \to 0 \quad \text{as} \quad \xi\to\infty.
\end{aligned}
$$

Then, for all $x(\tilde{t})\in\mathbb{Y},\tilde{t}\geq 0$, we have

$$
E\|(\Phi_1x)(\tilde{t}+\xi)-(\Phi_1x)(\tilde{t})\|_H^p \to 0 \quad \text{as } \xi\to\infty.
$$

Thus Φ_1 is continuous in the p-th moment on $[0,\infty)$.

Next we show that $\Phi_1(\mathbb{Y})\subset\mathbb{Y}$. By using (H1), (H3) and Hölder's inequality, we have for $t\in[0,t_1]$

$$
\begin{aligned}
E\|(\Phi_1x)(t)\|_H^p &\leq E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p \\
&\leq E\Big[\int_0^t \|\mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))\|_H ds\Big]^p \\
&\leq M^pE\Big[\int_0^t e^{-\delta(t-s)}\|h(s,x(s-\rho_2(s)))\|_H ds\Big]^p \\
&= M^pE\Big[\int_0^t e^{-(\delta(p-1)/p)(t-s)}e^{-(\delta/p)(t-s)}\|h(s,x(s-\rho_2(s)))\|_H ds\Big]^p \\
&\leq M^p\Big[\int_0^t e^{-\delta(t-s)}ds\Big]^{p-1}\int_0^t e^{-\delta(t-s)}E\|h(s,x(s-\rho_2(s)))\|_H^p ds \\
&\leq M^p\delta^{1-p}\int_0^t e^{-\delta(t-s)}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds.
\end{aligned}
$$

Similarly, for any $t\in(t_k,t_{k+1}]$, $k=1,2,\dots$, we have

$$
\begin{aligned}
E\|(\Phi_1x)(t)\|_H^p &\leq E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p \\
&\leq M^p\delta^{1-p}\int_{t_k}^t e^{-\delta(t-s)}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds.
\end{aligned}
$$

Then, for all $x(t)\in\mathbb{Y},t\in[\tilde{m}(0),\infty)$, we have

$$
E\|(\Phi_1x)(t)\|_H^p \leq M^p\delta^{p-1}\int_0^t e^{-\delta(t-s)}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds. \tag{3.3}
$$

However, for any any $\varepsilon>0$, there exists a $\tilde{\tau}_1>0$ such that $E\|x(s-\rho_2(s))\|_H^p<\varepsilon$ for $t\geq\tilde{\tau}_1$. Thus, we obtain

$$
\begin{aligned}
&E\|(\Phi_1x)(t)\|_H^p \\
&\leq M^p\delta^{1-p}e^{-\delta t}\int_0^t e^{\delta s}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds \\
&\leq M^p\delta^{1-p}e^{-\delta t}\int_0^{\tilde{\tau}_1} e^{\delta s}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds + M^p\delta^{1-p}L_h\Theta_h(\varepsilon),
\end{aligned}
$$

where $L_h = \sup_{t\geq 0}\int_{\tilde\tau_1}^t e^{-\delta(t-s)}m_h(s)ds$. As $e^{-\delta t}\to 0$ as $t\to\infty$ and, there exists $\tilde\tau_2\geq\tilde\tau_1$ such that for any $t\geq\tilde\tau_2$ we have

$$M^p\delta^{p-1}e^{-\delta t}\int_0^{\tilde\tau_1} e^{\delta s}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds<\varepsilon-M^p\delta^{p-1}L_h\Theta_h(\varepsilon).$$

From the above inequality, for any $t\geq\tilde\tau_2$, we obtain $E\|(\Phi_1x)(t)\|_H^p<\varepsilon$. That is to say $E\|(\Phi_1x)(t)\|_H^p\to 0$ as $t\to\infty$. So we conclude that $\Phi_1(\mathbb{Y})\subset\mathbb{Y}$. □

Lemma 3.2. *Assume that conditions* (H1), (H4) *hold. Let* Φ_2 *be the operator defined by: for each* $x\in\mathbb{Y}$,

$$(\Phi_2x)(t)=\begin{cases}\int_0^t\mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), & t\in[0,t_1],\\ \int_{t_1}^t\mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), & t\in(t_1,t_2],\\ \dots\\ \int_{t_m}^t\mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), & t\in(t_m,t_{m+1}],\\ \dots\end{cases}$$

Then Φ_2 *is continuous on* $[0,\infty)$ *in the* p*-th mean and maps* $\mathbb{Y}$ *into itself.*

Proof. We first prove that Φ_2 is continuous in the p-th moment on $[0,\infty)$. Let $x\in\mathbb{Y}$, $\tilde\theta\geq 0$ and $|\xi|$ be sufficiently small. Then for $\tilde\theta\in[0,t_1]$, by using Hölder's inequality and Lemma 2.11, we have

$$\begin{aligned}
&E\|(\Phi_2x)(\tilde\theta+\xi)-(\Phi_2x)(\tilde\theta)\|_H^p\\
&\leq 2^{p-1}E\Big\|\int_0^{\tilde\theta}[\mathcal{S}_\alpha(\tilde\theta+\xi-s)-\mathcal{S}_\alpha(\tilde\theta-s)]f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p\\
&\quad+2^{p-1}E\Big\|\int_{\tilde\theta}^{\tilde\theta+\xi}\mathcal{S}_\alpha(\tilde\theta+\xi-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p\\
&\leq 2^{p-1}C_p\Big[\int_0^{\tilde\theta}(E\|[\mathcal{S}_\alpha(\tilde\theta+\xi-s)-\mathcal{S}_\alpha(\tilde\theta-s)]f(s,x(s-\rho_3(s)))\|_H^p)^{2/p}ds\Big]^{p/2}\\
&\quad+2^{p-1}C_p\Big[\int_{\tilde\theta}^{\tilde t+\xi}(E\|\mathcal{S}_\alpha(\tilde\theta+\xi-s)f(s,x(s-\rho_3(s)))\|_H^p)^{2/p}ds\Big]^{p/2}\to 0
\end{aligned}$$

as $\xi\to\infty$. Similarly, for any $\tilde\theta\in(t_k,t_{k+1}]$, $k=1,2,\dots$, we have

$$\begin{aligned}
&E\|(\Phi_2x)(\tilde\theta+\xi)-(\Phi_2x)(\tilde\theta)\|_H^p\\
&\leq 2^{p-1}C_p\Big[\int_{t_k}^{\tilde\theta}(E\|[\mathcal{S}_\alpha(\tilde\theta+\xi-s)-\mathcal{S}_\alpha(\tilde\theta-s)]f(s,x(s-\rho_3(s)))\|_H^p)^{2/p}ds\Big]^{p/2}\\
&\quad+2^{p-1}C_p\Big[\int_{\tilde\theta}^{\tilde t+\xi}(E\|\mathcal{S}_\alpha(\tilde\theta+\xi-s)f(s,x(s-\rho_3(s)))\|_H^p)^{2/p}ds\Big]^{p/2}\to 0
\end{aligned}$$

as $\xi\to\infty$. Then, for all $x(\tilde\theta)\in\mathbb{Y},\tilde\theta\geq 0$, we have

$$E\|(\Phi_2x)(\tilde\theta+\xi)-(\Phi_2x)(\tilde\theta)\|_H^p\to 0\quad\text{as }\xi\to\infty.$$

Thus Φ_2 is continuous in the p-th moment on $[0,\infty)$.

Next we show that $\Phi_2(\mathbb{Y})\subset\mathbb{Y}$. By using (H1), (H4) and Hölder's inequality, for $t\in[0,t_1]$, we have

$$E\|(\Phi_2x)(t)\|_H^p$$

$$
\begin{aligned}
&\le E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p\\
&\le C_p\Big[\int_0^t (E\|\mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))\|_H^p)^{(2/p)}ds\Big]^{p/2}\\
&\le C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}(E\|f(s,x(s-\rho_3(s)))\|_H^p)]^{2/p}ds\Big]^{p/2}\\
&\le C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p}ds\Big]^{p/2}\\
&= C_pM^p\Big[\int_0^t [e^{-(p-1)\delta(t-s)}e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p}ds\Big]^{p/2}\\
&\le C_pM^p\Big[\int_0^t e^{-[\frac{2(p-1)}{p-2}]\delta(t-s)}ds\Big]^{p/2-1}\\
&\quad\times\int_0^t e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds\\
&\le C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_0^t e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds.
\end{aligned}
$$

Similarly, for any $t\in(t_k,t_{k+1}]$, $k=1,2,\dots$, we have

$$
\begin{aligned}
&E\|(\Phi_2x)(t)\|_H^p\\
&\le E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s))dw(s)\Big\|_H^p\\
&\le C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{t_k}^t e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds.
\end{aligned}
$$

Then, for all $x(t)\in\mathbb{Y}, t\in[\tilde m(0),\infty)$, we have

$$
\begin{aligned}
&E\|(\Phi_2x)(t)\|_H^p\\
&\le C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_0^t e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds.
\end{aligned}\tag{3.4}
$$

However, for any any $\varepsilon>0$, there exists a $\tilde\theta_1>0$ such that $E\|x(s-\rho_3(s))\|_H^p<\varepsilon$ for $t\ge\tilde\theta_1$. Thus from (3.4) we obtain

$$
\begin{aligned}
&E\|(\Phi_2x)(t)\|_H^p\\
&\le C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}e^{-\delta t}\int_0^t e^{\delta s}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds\\
&\le C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}e^{-\delta t}\int_0^{\tilde t_1} e^{\delta s}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds\\
&\quad+C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}L_f\Theta_f(\varepsilon),
\end{aligned}
$$

where $L_f=\sup_{t\ge0}\int_{\tilde t_1}^t e^{-\delta(t-s)}m_f(s)ds$. As $e^{-\delta t}\to0$ as $t\to\infty$ and, there exists $\tilde\theta_2\ge\tilde\theta_1$ such that for any $t\ge\tilde\theta_2$ we have

$$
C_pM^p[\frac{2\delta(p-1)}{p-2}]^{1-p/2}\int_0^{\tilde t_1} e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)ds
$$

$$< \varepsilon - C_pM^p[\frac{2\delta(p-1)}{p-2}]^{1-p/2}L_f\Theta_f(\varepsilon).$$

From the above inequality, for any $t \geq \tilde{\theta}_2$, we obtain $E\|(\Phi_2 x)(t)\|_H^p < \varepsilon$. That is to say $E\|(\Phi_2 x)(t)\|_H^p \to 0$ as $t \to \infty$. So we conclude that $\Phi_2(\mathbb{Y}) \subset \mathbb{Y}$. $\square$

Now, we are ready to present our main result.

Theorem 3.3. *Assume the conditions* (H1)-(H5) *hold. Let $p \geq 2$ be an integer. Then the fractional impulsive stochastic differential equations* (1.1)–(1.3) *is asymptotically stable in the p-th moment, provided that*

$$\max_{1\leq k\leq m}\{12^{p-1}M^p(1+d_k^1+2^{p-1}L_g)+8^{p-1}L_g\} < 1. \tag{3.5}$$

Proof. We define the nonlinear operator $\Psi : \mathbb{Y} \to \mathbb{Y}$ as $(\Psi x)(t) = \varphi(t)$ for $t \in [\tilde{m}(0), 0]$ and for $t \geq 0$,

$$(\Psi x)(t) = \begin{cases} \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi(-\rho_1(0)))] + g(t,x(t-\rho_1(t))) \\ +\int_0^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds \\ +\int_0^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), \\ \quad t\in[0,t_1], \\ \mathcal{R}_\alpha(t-t_1)[x(t_1^-)+I_1(x(t_1^-)) - g(t_1,x(t_1^+-\rho_1(t_1^+)))] \\ +g(t,x(t-\rho_1(t))) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds \\ +\int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), \\ \quad t\in(t_1,t_2], \\ \dots \\ \mathcal{R}_\alpha(t-t_m)[x(t_m^-)+I_m(x(t_m^-)) - g(t_m,x(t_m^+-\rho_1(t_m^+)))] \\ +g(t,x(t-\rho_1(t))) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds \\ +\int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), \\ \quad t\in(t_m,t_{m+1}], \\ \dots \end{cases} \tag{3.6}$$

Using (H2)–(H5), and the proofs of Lemmas 3.1 and 3.2, it is clear that the nonlinear operator Ψ is well defined and continuous in p-th moment on $[0,\infty)$. Moreover, for all $t \in [0,t_1]$ we have

$$\begin{aligned} &E\|(\Psi x)(t)\|_H^p \\ &\leq 4^{p-1}E\|\mathcal{R}_\alpha(t)[\varphi(0)-g(0,\varphi(-\rho_1(0)))]\|_H^p + 4^{p-1}E\|g(t,x(t-\rho_1(t)))\|_H^p \\ &\quad + 4^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))dw(s)\Big\|_H^p \\ &\quad + 4^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p. \end{aligned}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1,2,\dots$, we have

$$\begin{aligned} E\|(\Psi x)(t)\|_H^p &\leq 4^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[x(t_k^-)+I_k(x(t_k^-)) - g(t_k,x(t_k^+-\rho_1(t_k^+)))]\|_H^p \\ &\quad + 4^{p-1}E\|g(t,x(t-\rho_1(t)))\|_H^p \\ &\quad + 4^{p-1}E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p \end{aligned}$$

$$+ 4^{p-1}E\Big\| \int_{t_k}^{t} \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p.$$

Then, for all $t \geq 0$, we have

$$\begin{aligned}
E\|(\Psi x)(t)\|_H^p &\leq 4^{p-1}E\|\mathcal{R}_\alpha(t)[\varphi(0)-g(0,\varphi(-\rho_1(0)))]\|_H^p \\
&\quad + 4^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[x(t_k^-)+I_k(x(t_k^-))-g(t_k,x(t_k^+-\rho_1(t_k^+)))]\|_H^p \\
&\quad + 4^{p-1}E\|g(t,x(t-\rho_1(t)))\|_H^p \\
&\quad + 4^{p-1}E\Big\| \int_0^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p \\
&\quad + 4^{p-1}E\Big\| \int_0^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p.
\end{aligned}$$

By (H1)–(H5), Lemmas 3.1 and 3.2 again, we obtain

$$\begin{gathered}
4^{p-1}E\|\mathcal{R}_\alpha(t)[\varphi(0)-g(0,\varphi(-\rho_1(0)))]\|_H^p \\
\leq 8^{p-1}M^pe^{-p\delta t}[E\|\varphi(0)\|_H^p + L_gE\|\varphi(-\rho_1(0))\|_H^p] \to 0 \quad \text{as} \quad t\to\infty, \\
4^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[x(t_k^-)+I_k(x(t_k^-))-g(t_k,x(t_k^+-\rho_1(t_k^+)))]\|_H^p \\
\leq 12^{p-1}M^pe^{-p\delta t}[E\|x(t_k^-)\|_H^p + E\|I_k(x(t_k^-))\|_H^p \\
+ L_gE\|x(t_k^+-\rho_1(t_k^+)))\|_H^p] \to 0 \quad \text{as } t\to\infty, \\
4^{p-1}E\|g(t,x(t-\rho_1(t)))\|_H^p \leq 4^{p-1}L_gE\|x(t-\rho_1(t)))\|_H^p \to 0 \quad \text{as } t\to\infty, \\
4^{p-1}E\Big\| \int_0^t \mathcal{S}_\alpha(t-s)h(s,x(t-\rho_2(t)))ds\Big\|_H^p \to 0 \quad \text{as } t\to\infty, \\
4^{p-1}E\Big\| \int_0^t \mathcal{S}_\alpha(t-s)f(s,x(t-\rho_3(t)))dw(s)\Big\|_H^p \to 0 \quad \text{as} \quad t\to\infty.
\end{gathered}$$

That is to say $E\|(\Psi x)(t)\|_H^p \to 0$ as $t\to\infty$. So Ψ maps $\mathbb{Y}$ into itself.

Next we prove that the operator Ψ has a fixed point, which is a mild solution of the problem (1.1)-(1.3). To see this, we decompose Ψ as $\Psi_1+\Psi_2$ for $t\in[0,T]$, where

$$(\Psi_1 x)(t) = \begin{cases}
-\mathcal{R}_\alpha(t)g(0,\varphi(-\rho_1(0))) + g(t,x(t-\rho_1(t))), & t\in[0,t_1], \\
-\mathcal{R}_\alpha(t-t_1)g(t_1,x(t_1^+-\rho_1(t_1^+))) + g(t,x(t-\rho_1(t))), & t\in(t_1,t_2], \\
\dots \\
-\mathcal{R}_\alpha(t-t_m)g(t_m,x(t_m^+-\rho_1(t_m^+))) + g(t,x(t-\rho_1(t))), & t\in(t_m,T],
\end{cases}$$

and

$$(\Psi_2 x)(t) = \begin{cases} \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), & t\in[0,t_1], \\ \mathcal{R}_\alpha(t-t_1)[x(t_1^-)+I_1(x(t_1^-))] \\ + \int_{t_1}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds \\ + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s,x(t-\rho_3(t)))dw(s), & t\in(t_1,t_2], \\ \dots \\ \mathcal{R}_\alpha(t-t_m)[x(t_m^-)+I_m(x(t_m^-))] \\ + \int_{t_m}^t \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds \\ + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s), & t\in(t_m,T]. \end{cases}$$

To use Lemma 2.12, we will verify that Ψ_1 is a contraction while Ψ_2 is a completely continuous operator. For better readability, we break the proof into a sequence of steps.

Step1. Ψ_1 is a contraction on $\mathbb{Y}$. Let $t\in[0,t_1]$ and $x,y\in\mathbb{Y}$. From (H2), we have

$$\begin{aligned} E\|(\Psi_1 x)(t)-(\Psi_1 y)(t)\|_H^p &\le E\|g(t,x(t-\rho_1(t)))-g(t,y(t-\rho_1(t)))\|_H^p \\ &\le L_g E\|x(t-\rho_1(t))-y(t-\rho_1(t))\|_H^p \\ &\le L_g\|x-y\|_{\mathbb{Y}}. \end{aligned}$$

Similarly, for any $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, we have

$$\begin{aligned} &E\|(\Psi_1 x)(t)-(\Psi_1 y)(t)\|_H^p \\ &\le 2^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[-g(t_k,x(t_k^+-\rho_1(t_k^+)))+g(t_k,y(t_k^+-\rho_1(t_k^+)))]\|_H^p \\ &\quad + 2^{p-1}E\|g(t,x(t-\rho_1(t)))-g(t,y(t-\rho_1(t)))\|_H^p \\ &\le 2^{p-1}M^pL_gE\|x(t_k^+-\rho_1(t_k^+))-y(t_k^+-\rho_1(t_k^+))\|_H^p \\ &\quad + 2^{p-1}L_gE\|x(t-\rho_1(t))-y(t-\rho_1(t))\|_H^p \\ &\le 2^{p-1}L_g(M^p+1)\|x-y\|_{\mathbb{Y}}. \end{aligned}$$

Thus, for all $t\in[0,T]$,

$$E\|(\Psi_1 x)(t)-(\Psi_1 y)(t)\|_H^p \le 2^{p-1}L_g(M^p+1)\|x-y\|_{\mathbb{Y}}.$$

Taking supremum over t,

$$\|\Psi_1 x-\Psi_1 y\|_{\mathbb{Y}} \le L_0\|x-y\|_{\mathbb{Y}},$$

where $L_0 = 2^{p-1}L_g(M^p+1) < 1$. By (3.5), we see that $L_0<1$. Hence, Ψ_1 is a contraction on $\mathbb{Y}$.

Step 2. Ψ_2 maps bounded sets into bounded sets in $\mathbb{Y}$. Indeed, it is sufficient to show that there exists a positive constant $\mathcal{L}$ such that for each $x\in B_r=\{x: \|x\|_{\mathbb{Y}}\le r\}$ one has $\|\Psi_2 x\|_{\mathbb{Y}}\le\mathcal{L}$. Now, for $t\in[0,t_1]$ we have

$$\begin{aligned} (\Psi_2 x)(t) &= \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)h(s,x(t-\rho_2(t)))ds \\ &\quad + \int_0^t \mathcal{S}_\alpha(t-s)f(s,x(t-\rho_3(t)))dw(s). \end{aligned} \tag{3.7}$$

If $x \in B_r$, from the definition of $\mathbb{Y}$, it follows that

$$\begin{aligned} E\|x(s-\rho_i(s))\|_H^p &\leq 2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\sup_{s\in[0,T]} E\|x(s)\|_H^p \\ &\leq 2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}r := r^*, \quad i=1,2,3. \end{aligned}$$

By (H1)-(H4), from (3.7) and Hölder's inequality, for $t\in[0,t_1]$, we have

$$\begin{aligned} &E\|(\Psi_2 x)(t)\|_H^p \\ &\leq 3^{p-1}E\|\mathcal{R}_\alpha(t)\varphi(0)\|_H^p + 3^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)h(s,x(t-\rho_2(t)))ds\Big\|_H^p \\ &\quad + 3^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)f(s,x(t-\rho_3(t)))dw(s)\Big\|_H^p \\ &\leq 3^{p-1}M^pE\|\varphi(0)\|_H^p + 3^{p-1}M^pE\Big[\int_0^t e^{-\delta(t-s)}\|h(s,x(t-\rho_2(t)))\|_H ds\Big]^p \\ &\quad + 3^{p-1}C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}(E\|f(s,x(t-\rho_3(t)))\|_H^p)]^{2/p}ds\Big]^{p/2} \\ &\leq 3^{p-1}M^pE\|\varphi(0)\|_H^p + 3^{p-1}M^p\Big[\int_0^t e^{-\delta(t-s)}ds\Big]^{p-1} \\ &\quad \times \int_0^t e^{-\delta(t-s)}E\|h(s,x(t-\rho_2(t)))\|_H^p ds \\ &\quad + 3^{p-1}C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(s))\|_H^p)]^{2/p}ds\Big]^{p/2} \\ &\leq 3^{p-1}M^pE\|\varphi(0)\|_H^p + 3^{p-1}M^p\delta^{1-p} \\ &\quad \times \int_0^t e^{-\delta(t-s)}m_h(s)\Theta_h(E\|x(t-\rho_2(t))\|_H^p)ds + 3^{p-1}C_pM^p \\ &\quad \times \Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_0^t e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(t-\rho_3(t))\|_H^p)ds \\ &\leq 3^{p-1}M^pE\|\varphi(0)\|_H^p + 3^{p-1}M^p\delta^{1-p}\Theta_h(r^*)\int_0^{t_1} e^{-\delta(t-s)}m_h(s)ds \\ &\quad + 3^{p-1}C_pM^p\Theta_f(r^*)\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_0^{t_1} e^{-\delta(t-s)}m_f(s)ds := \mathcal{L}_0. \end{aligned}$$

Similarly, for any $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, we have

$$\begin{aligned} &(\Psi_2 x)(t) \\ &= \mathcal{R}_\alpha(t-t_k)[x(t_k^-)+I_k(x(t_k))] + \int_{t_k}^t \mathcal{S}_\alpha(t-s)h(s,x(t-\rho_2(t)))ds \\ &\quad + \int_{t_k}^t \mathcal{S}_\alpha(t-s)f(s,x(t-\rho_3(t)))dw(s). \end{aligned} \tag{3.8}$$

By (H1)–(H5), from (3.8) and Hölder's inequality, we have for $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$,

$$\begin{aligned} &E\|(\Psi_2 x)(t)\|_H^p \\ &\leq 3^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[x(t_k^-)+I_k(x(t_k^-))]\|_H^p \end{aligned}$$

$$
\begin{aligned}
&+ 3^{p-1}E\Big\| \int_{t_k}^{t} \mathcal{S}_\alpha(t-s)h(s,x(t-\rho_2(t)))ds\Big\|_H^p \\
&+ 3^{p-1}E\Big\| \int_{t_k}^{t} \mathcal{S}_\alpha(t-s)f(s,x(t-\rho_3(t)))dw(s)\Big\|_H^p \\
&\le 6^{p-1}M^p[E\|x(t_k^-)\|_H^p + E\|I_k(x(t_k^-))\|_H^p] \\
&+ 3^{p-1}M^pE\Big[\int_{t_k}^{t} e^{-\delta(t-s)}\|h(s,x(t-\rho_2(t)))\|_H ds\Big]^p \\
&+ 3^{p-1}C_pM^p\Big[\int_{t_k}^{t}[e^{-p\delta(t-s)}(E\|f(s,x(t-\rho_3(t)))\|_H^p)]^{2/p}ds\Big]^{p/2} \\
&\le 6^{p-1}M^p(r + d_k^1E\|x(t_k^-)\|_H^p + d_k^2) + 3^{p-1}M^p\Big[\int_{t_k}^{t} e^{-\delta(t-s)}ds\Big]^{p-1} \\
&\times \int_{t_k}^{t} e^{-\delta(t-s)}E\|h(s,x(t-\rho_2(t)))\|_H^p ds \\
&+ 3^{p-1}C_pM^p\Big[\int_{t_k}^{t}[e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(t-\rho_3(t))\|_H^p)]^{2/p}ds\Big]^{p/2} \\
&\le 6^{p-1}M^p(r + d_k^1r + d_k^2) + 3^{p-1}M^p\delta^{p-1} \\
&\times \int_{t_k}^{t} e^{-\delta(t-s)}m_h(s)\Theta_h(E\|x(t-\rho_2(t))\|_H^p)ds \\
&+ 3^{p-1}C_pM^p\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{t_k}^{t} e^{-\delta(t-s)}m_f(s)\Theta_f(E\|x(t-\rho_3(t))\|_H^p)ds \\
&\le 6^{p-1}M^p(r + d_k) + 3^{p-1}M^p\delta^{1-p}\Theta_h(r^*)\int_{t_k}^{t_{k+1}} e^{-\delta(t-s)}m_h(s)ds \\
&+ 3^{p-1}C_pM^p\Theta_f(r^*)\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{t_k}^{t_{k+1}} e^{-\delta(t-s)}m_f(s)ds := \mathcal{L}_k.
\end{aligned}
$$

Take $\mathcal{L} = \max_{0\le k\le m}\mathcal{L}_k$, for all $t \in [0,T]$, we have $E\|(\Psi_2x)(t)\|_H^p \le \mathcal{L}$. Then for each $x \in B_r$, we have $\|\Psi_2x\|_{\mathbb{Y}} \le \mathcal{L}$.

Step 3. $\Psi_2 : \mathbb{Y} \to \mathbb{Y}$ is continuous. Let $\{x_n(t)\}_{n=0}^\infty \subseteq \mathbb{Y}$ with $x_n \to x(n \to \infty)$ in $\mathbb{Y}$. Then there is a number $r > 0$ such that $E\|x_n(t)\|_H^p \le r$ for all n and a.e. $t \in [0,T]$, so $x_n \in B_r$ and $x \in B_r$. By the assumption (H3) and $I_k, k = 1,2,\dots,m$, are completely continuous, we have

$$
\begin{gathered}
E\|h(s,x_n(s-\rho_2(s))) - h(s,x(s-\rho_2(s)))\|_H^p \to 0 \quad \text{as } n \to \infty, \\
E\|f(s,x_n(s-\rho_3(s))) - f(s,x(s-\rho_3(s)))\|_H^p \to 0 \quad \text{as } n \to \infty
\end{gathered}
$$

for each $s \in [0,t]$, and since

$$
\begin{gathered}
E\|h(s,x_n(s-\rho_2(s))) - h(s,x(s-\rho_2(s)))\|_H^p \le 2m_h(t)\Theta_h(r^*), \\
E\|f(s,x_n(s-\rho_3(s))) - f(s,x(s-\rho_3(s)))\|_H^p \le 2m_f(t)\Theta_f(r^*).
\end{gathered}
$$

Then by the dominated convergence theorem, for $t \in [0,t_1]$, we have

$$
\begin{aligned}
&E\|(\Psi_2x_n)(t) - (\Psi_2x)(t)\|_H^p \\
&\le 2^{p-1}E\Big\| \int_0^t \mathcal{S}_\alpha(t-s)[h(s,x_n(s-\rho_2(s))) - h(s,x(s-\rho_2(s)))]ds\Big\|_H^p
\end{aligned}
$$

$$
\begin{aligned}
&+ 2^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)[f(s,x_n(s-\rho_3(s)))-f(s,x(s-\rho_3(s)))]dw(s)\Big\|_H^p\\
&\le 2^{p-1}M^pE\Big[\int_0^t e^{-\delta(t-s)}\|h(s,x_n(s-\rho_2(s)))-h(s,x(s-\rho_2(s)))\|_H ds\Big]^p\\
&+ 2^{p-1}C_pM^p\Big[\int_0^t (E\|\mathcal{S}_\alpha(t-s)[f(s,x_n(s-\rho_3(s)))\\
&- f(s,x(s-\rho_3(s)))]\|_H^p)^{2/p}ds\Big]^{p/2}\\
&\le 2^{p-1}M^p\delta^{1-p}\int_0^t e^{-\delta(t-s)}E\|h(s,x_n(s-\rho_2(s)))-h(s,x(s-\rho_2(s)))\|_H^p ds\\
&+ 2^{p-1}C_pM^p\Big[\int_0^t e^{-2\delta(t-s)}(E\|f(s,x_n(s-\rho_3(s)))\\
&- f(s,x(s-\rho_3(s)))\|_H^p)^{2/p}ds\Big]^{p/2} \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Similarly, for any $t\in(t_k,t_{k+1}], k=1,2,\dots,m$, we have

$$
\begin{aligned}
&E\|(\Psi_2x_n)(t)-(\Psi_2x)(t)\|_H^p\\
&\le 3^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[x_n(t_k^-)-x(t_k^-)+I_k(x_n(t_k^-))-I_k(x(t_k^-))]\|_H^p\\
&+ 3^{p-1}E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)[h(s,x_n(s-\rho_2(s)))-h(s,x(s-\rho_2(s)))]ds\Big\|_H^p\\
&+ 3^{p-1}E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)[f(s,x_n(s-\rho_3(s)))-f(s,x(s-\rho_3(s)))]dw(s)\Big\|_H^p\\
&\le 6^{p-1}M^p[E\|x_n(t_k^-)-x(t_k^-)\|_H^p+E\|I_k(x_n(t_k^-))-I_k(x(t_k^-))\|_H^p]\\
&+ 3^{p-1}M^p\delta^{1-p}\int_0^t e^{-\delta(t-s)}E\|h(s,x_n(s-\rho_2(s)))-h(s,x(s-\rho_2(s)))\|_H^p ds\\
&+ 3^{p-1}C_pM^p\Big[\int_{t_k}^t e^{-2\delta(t-s)}(E\|f(s,x_n(s-\rho_3(s)))\\
&- f(s,x(s-\rho_3(s)))\|_H^p)^{2/p}ds\Big]^{p/2} \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Then, for all $t\in[0,T]$ we have

$$\|\Psi_2x_n-\Psi_2x\|_{\mathbb{Y}}\to 0 \quad \text{as } n\to\infty.$$

Therefore, Ψ_2 is continuous on B_r.

Step 4. Ψ_2 maps bounded sets into equicontinuous sets of $\mathbb{Y}$.

Let $0<\tau_1<\tau_2\le t_1$. Then, by using Hölder's inequality and Lemma 2.11, for each $x\in B_r$, we have

$$
\begin{aligned}
&E\|(\Psi_2x)(\tau_2)-(\Psi_2x)(\tau_1)\|_H^p\\
&\le 7^{p-1}E\|[\mathcal{R}_\alpha(\tau_2)-\mathcal{R}_\alpha(\tau_1)]\varphi(0)\|_H^p\\
&+ 7^{p-1}E\Big\|\int_0^{\tau_1-\varepsilon}[\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]h(s,x(s-\rho_2(s)))ds\Big\|_H^p\\
&+ 7^{p-1}E\Big\|\int_{\tau_1-\varepsilon}^{\tau_1}[\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]h(s,x(s-\rho_2(s)))ds\Big\|_H^p
\end{aligned}
$$

$$
\begin{aligned}
&+ 7^{p-1}E\Big\| \int_{\tau_1}^{\tau_2} \mathcal{S}_\alpha(\tau_2-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p \\
&+ 7^{p-1}E\Big\| \int_{0}^{\tau_1-\varepsilon} [\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p \\
&+ 7^{p-1}E\Big\| \int_{\tau_1-\varepsilon}^{\tau_1} [\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p \\
&+ 7^{p-1}E\Big\| \int_{\tau_1}^{\tau_2} \mathcal{S}_\alpha(\tau_2-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p \\
&\le 7^{p-1}E\|[\mathcal{R}_\alpha(\tau_2)-\mathcal{R}_\alpha(\tau_1)]\varphi(0)\|_H^p \\
&+ 7^{p-1}E\Big[\int_0^{\tau_1-\varepsilon} \|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}\|h(s,x(s-\rho_1(s)))\|_H ds\Big]^p \\
&+ 7^{p-1}E\Big[\int_{\tau_1-\varepsilon}^{\tau_1} \|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}\|h(s,x(s-\rho_1(s)))\|_H ds\Big]^p \\
&+ 7^{p-1}E\Big[\int_{\tau_1}^{\tau_2} \|\mathcal{S}_\alpha(\tau_2-s)\|_{L(H)}\|h(s,x(s-\rho_1(s)))\|_H ds\Big]^p \\
&+ 7^{p-1}C_p\Big[\int_0^{\tau_1-\varepsilon} [\|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p \\
&\times \big(E\|f(s,x(s-\rho_3(s)))\|_H^p\big)]^{2/p} ds\Big]^{p/2} \\
&+ 7^{p-1}C_p\Big[\int_{\tau_1-\varepsilon}^{\tau_1} [\|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p \\
&\times \big(E\|f(s,x(s-\rho_3(s)))\|_H^p\big)]^{2/p} ds\Big]^{p/2} \\
&+ 7^{p-1}C_p\Big[\int_{\tau_1}^{\tau_2} [\|\mathcal{S}_\alpha(\tau_2-s)\|_{L(H)}^p (E\|f(s,x(s-\rho_3(s)))\|_H^p)]^{2/p} ds\Big]^{p/2} \\
\le\ & 7^{p-1}E\|[\mathcal{R}_\alpha(\tau_2)-\mathcal{R}_\alpha(\tau_1)]\varphi(0)\|_H^p \\
&+ 7^{p-1}T^p\int_0^{\tau_1-\varepsilon} \|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p m_h(s)\Theta_h(E\|x(s-\rho_1(s))\|_H^p)ds \\
&+ 14^{p-1}M^p\Big[\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta(\tau_1-s)}ds\Big]^{p-1}\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta(\tau_1-s)}m_h(s) \\
&\times \Theta_h(E\|x(s-\rho_1(s))\|_H^p)ds \\
&+ 7^{p-1}M^p\Big[\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\Big]^{p-1}\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds \\
&+ 4^{p-1}C_p\Big[\int_0^{\tau_1-\varepsilon} [\|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p \\
&\times m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p} ds\Big]^{p/2} \\
&+ 14^{p-1}C_pM^p\Big[\int_{\tau_1-\varepsilon}^{\tau_1} [e^{-p\delta(\tau_1-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p} ds\Big]^{p/2} \\
&+ 7^{p-1}C_pM^p\Big[\int_{\tau_1}^{\tau_2} [e^{-p\delta(\tau_2-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p} ds\Big]^{p/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le 7^{p-1}E\|[\mathcal{R}_\alpha(\tau_2)-\mathcal{R}_\alpha(\tau_1)]\varphi(0)\|_H^p\\
&\quad + 7^{p-1}T^p\Theta_h(r^*)\int_0^{\tau_1-\varepsilon}\|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p m_h(s)ds\\
&\quad + 14^{p-1}M^p\Theta_h(r^*)\delta^{1-p}\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta(\tau_1-s)}m_h(s)ds\\
&\quad + 7^{p-1}M^p\Theta_h(r^*)\delta^{1-p}\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}m_h(s)ds\\
&\quad + 7^{p-1}C_p\Theta_f(r^*)\Big[\int_0^{\tau_1-\varepsilon}[\|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p m_f(s)]^{2/p}ds\Big]^{p/2}\\
&\quad + 14^{p-1}C_pM^p\Theta_f(r^*)\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta(\tau_1-s)}m_f(s)ds\\
&\quad + 4^{p-1}C_pM^p\Theta_f(r^*)\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}m_f(s)ds.
\end{aligned}
$$

Similarly, for any $\tau_1,\tau_2\in(t_k,t_{k+1}]$, $\tau_1<\tau_2$, $k=1,\dots,m$, we have

$$
\begin{aligned}
&(\Psi_2x)(t)\\
&= \mathcal{R}_\alpha(t-t_k)[\bar{x}(t_k^-)+I_k(x(t_k))]+\int_{t_k}^t\mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds\\
&\quad + \int_{t_k}^t\mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s).
\end{aligned}
\tag{3.9}
$$

Then

$$
\begin{aligned}
&E\|(\Psi_2x)(\tau_2)-(\Psi_2x)(\tau_1)\|_H^p\\
&\le 7^{p-1}E\|[\mathcal{R}_\alpha(\tau_2)-\mathcal{R}_\alpha(\tau_1)][x(t_k^-)+I_k(x(t_k^-))]\|_H^p\\
&\quad + 7^{p-1}E\Big\|\int_{t_k}^{\tau_1-\varepsilon}[\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]h(s,x(s-\rho_2(s)))ds\Big\|_H^p\\
&\quad + 7^{p-1}E\Big\|\int_{\tau_1-\varepsilon}^{\tau_1}[\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]h(s,x(s-\rho_2(s)))ds\Big\|_H^p\\
&\quad + 7^{p-1}E\Big\|\int_{\tau_1}^{\tau_2}\mathcal{S}_\alpha(\tau_2-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p\\
&\quad + 7^{p-1}E\Big\|\int_{t_k}^{\tau_1-\varepsilon}[\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p\\
&\quad + 7^{p-1}E\Big\|\int_{\tau_1-\varepsilon}^{\tau_1}[\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)]f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p\\
&\quad + 7^{p-1}E\Big\|\int_{\tau_1}^{\tau_2}\mathcal{S}_\alpha(\tau_2-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p\\
&\le 4^{p-1}E\|[\mathcal{R}_\alpha(\tau_2)-\mathcal{R}_\alpha(\tau_1)][x(t_k^-)+I_k(x(t_k^-))]\|_H^p\\
&\quad + 7^{p-1}T^p\Theta_h(r^*)\int_{t_k}^{\tau_1-\varepsilon}\|\mathcal{S}_\alpha(\tau_2-s)-\mathcal{S}_\alpha(\tau_1-s)\|_{L(H)}^p m_h(s)ds\\
&\quad + 14^{p-1}M^p\Theta_h(r^*)\delta^{1-p}\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta(\tau_1-s)}m_h(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ 7^{p-1} M^p \Theta_h(r^*) \delta^{1-p} \int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} m_h(s) ds \\
&+ 4^{p-1} C_p \Theta_f(r^*) \Big[\int_{t_k}^{\tau_1 - \varepsilon} [\|\mathcal{S}_\alpha(\tau_2 - s) - \mathcal{S}_\alpha(\tau_1 - s)\|_{L(H)}^p m_f(s)]^{2/p} ds \Big]^{p/2} \\
&+ 8^{p-1} C_p M^p \Theta_f(r^*) \Big[\frac{2\delta(p-1)}{p-2} \Big]^{1-p/2} \int_{\tau_1 - \varepsilon}^{\tau_1} e^{-\delta(t-s)} m_f(s) ds \\
&+ 4^{p-1} C_p M^p \Theta_f(r^*) \Big[\frac{2\delta(p-1)}{p-2} \Big]^{1-p/2} \int_{\tau_1}^{\tau_2} e^{-\delta s} m_f(s) ds.
\end{aligned}
$$

The fact of I_k, $k = 1, 2, \dots, m$, are completely continuous in H and the compactness of $\mathcal{R}_\alpha(t)$, $\mathcal{S}_\alpha(t)$ for $t > 0$ imply the continuity in the uniform operator topology. So, as $\tau_2 - \tau_1 \to 0$, with ε is sufficiently small, the right-hand side of the above inequality is independent of $x \in B_r$ and tends to zero. The equicontinuities for the cases $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2 \le T$ are very simple. Thus the set $\{\Psi_2 x : x \in B_r\}$ is equicontinuous.

Step 5. The set $W(t) = \{(\Psi_2 x)(t) : x \in B_r\}$ is relatively compact in H. To this end, we decompose Ψ_2 by $\Psi_2 = \Gamma_1 + \Gamma_2$, where

$$
(\Gamma_1 x)(t) = \begin{cases}
\int_0^t S_\alpha(t-s) h(s, x(s - \rho_2(s))) ds & \\
+ \int_0^t S_\alpha(t-s) f(s, x(s - \rho_3(s))) dw(s), & t \in [0, t_1], \\
\int_{t_1}^t S_\alpha(t-s) h(s, x(s - \rho_2(s))) ds & \\
+ \int_{t_1}^t S_\alpha(t-s) f(s, x(s - \rho_3(s))) dw(s), & t \in (t_1, t_2], \\
\dots & \\
\int_{t_m}^t S_\alpha(t-s) h(s, x(s - \rho_2(s))) ds & \\
+ \int_{t_m}^t S_\alpha(t-s) f(s, x(s - \rho_3(s))) dw(s), & t \in (t_m, T],
\end{cases}
$$

and

$$
(\Gamma_2 x)(t) = \begin{cases}
\mathcal{R}_\alpha(t) \varphi(0), & t \in [0, t_1], \\
\mathcal{R}_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-))], & t \in (t_1, t_2], \\
\dots & \\
\mathcal{R}_\alpha(t - t_m)[x(t_m^-) + I_m(x(t_m^-))], & t \in (t_m, T].
\end{cases}
$$

We now prove that $\Gamma_1(B_r)(t) = \{(\Gamma_1 x)(t) : x \in B_r\}$ is relatively compact for every $t \in [0, T]$. Let $0 < t \le s \le t_1$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_r$, we define

$$
\begin{aligned}
(\Gamma_1^\varepsilon x)(t)(t) &= \int_0^{t-\varepsilon} \mathcal{S}_\alpha(t-s) h(s, x(s - \rho_2(s))) ds \\
&\quad + \int_0^{t-\varepsilon} \mathcal{S}_\alpha(t-s) f(s, x(s - \rho_3(s))) dw(s).
\end{aligned}
$$

Using the compactness of $\mathcal{S}_\alpha(t)$ for $t > 0$, we deduce that the set $U_\varepsilon(t) = \{(\Gamma_1^\varepsilon x)(t) : x \in B_r\}$ is relatively compact in H for every $\varepsilon, 0 < \varepsilon < t$. Moreover, by using Hölder's inequality, we have for every $x \in B_r$

$$
\begin{aligned}
&E\|(\Gamma_1 x)(t)(t) - (\Gamma_1^\varepsilon x)(t)(t)\|_H^p \\
&\le 2^{p-1} E\Big\| \int_{t-\varepsilon}^t \mathcal{S}_\alpha(t-s) h(s, x(s - \rho_2(s))) ds \Big\|_H^p
\end{aligned}
$$

$$
\begin{aligned}
&+ 2^{p-1}E\Big\| \int_{t-\varepsilon}^{t} \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p \\
&\le 2^{p-1}M^pE\Big[\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}\|h(s,x(s-\rho_2(s)))\|_H ds\Big]^p \\
&\quad + 2^{p-1}C_pM^p\Big[\int_{t-\varepsilon}^{t}[e^{-p\delta(t-s)}E\|f(s,x(s-\rho_3(s)))\|_H^p]^{2/p}ds\Big]^{p/2} \\
&\le 2^{p-1}M^p\delta^{1-p}\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds \\
&\quad + 2^{p-1}C_pM^p\Big[\int_{t-\varepsilon}^{t}[e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p}ds\Big]^{p/2} \\
&\le 2^{p-1}M^p\delta^{1-p}\Theta_h(r^*)\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}m_h(s)ds \\
&\quad + 2^{p-1}C_pM^p\Theta_f(r^*)\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}m_f(s)ds.
\end{aligned}
$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$. Let $t_k < t \le s \le t_{k+1}$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_r$, we define

$$
(\Gamma_1^\varepsilon x)(t) = \int_{t_k}^{t-\varepsilon} \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds + \int_{t_k}^{t-\varepsilon} \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s).
$$

Using the compactness of $\mathcal{S}_\alpha(t)$ for $t > 0$, we deduce that the set $U_\varepsilon(t) = \{(\Gamma_1^\varepsilon x)(t) : x \in B_r\}$ is relatively compact in H for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $x \in B_r$ we have

$$
\begin{aligned}
&E\|(\Gamma_1 x)(t)(t) - (\Gamma_1^\varepsilon x)(t)(t)\|_H^p \\
&\le 2^{p-1}E\Big\| \int_{t-\varepsilon}^{t} \mathcal{S}_\alpha(t-s)h(s,x(s-\rho_2(s)))ds\Big\|_H^p \\
&\quad + 2^{p-1}E\Big\| \int_{t-\varepsilon}^{t} \mathcal{S}_\alpha(t-s)f(s,x(s-\rho_3(s)))dw(s)\Big\|_H^p \\
&\le 2^{p-1}M^p\delta^{1-p}\Theta_h(r^*)\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}m_h(s)ds \\
&\quad + 2^{p-1}C_pM^p\Theta_f(r^*)\Big[\frac{2\delta(p-1)}{p-2}\Big]^{1-p/2}\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}m_f(s)ds.
\end{aligned}
$$

There are relatively compact sets arbitrarily close to the set $W(t) = \{(\Gamma_1 x)(t) : x \in B_r\}$, and $W(t)$ is a relatively compact in H. It is easy to see that $\Gamma_1(B_r)$ is uniformly bounded. Since we have shown $\Phi_1(B_r)$ is equicontinuous collection, by the Arzelá-Ascoli theorem it suffices to show that Γ_1 maps B_r into a relatively compact set in H.

Next, we show that $\Gamma_2(B_r)(t) = \{(\Gamma_2 x)(t) : x \in B_r\}$ is relatively compact for every $t \in [0, T]$. For all $t \in [0, t_1]$, since $(\Gamma_2 x)(t) = \mathcal{R}_\alpha(t)\varphi(0)$, by (H1), it follows that $\{(\Gamma_2 x)(t) : t \in [0, t_1], x \in B_r\}$ is a compact subset of H. On the other hand, for $t \in (t_k, t_{k+1}], k = 1, \dots, m$, and $x \in B_r$, and that the interval $[0, T]$ is divided into finite subintervals by t_k, $k = 1, 2, \dots, m$, so that we need to prove that

$$
W = \{\mathcal{R}_\alpha(t-t_k)[x(t_k^-) + I_k(x(t_k^-))], \quad t \in [t_k, t_{k+1}],\ x \in B_r\}
$$

is relatively compact in $C([t_k, t_{k+1}], H)$. In fact, from (H1) and (H5), it follows that the set $\{\mathcal{R}_\alpha(t-t_k)[x(t_k^-)+I_k(x(t_k^-))],\ x \in B_r\}$ is relatively compact in H, for all $t \in [t_k, t_{k+1}]$, $k = 1, \dots, m$. Also, we see that the functions in W are equicontinuous due to the compactness of I_k and the strong continuity of the operator $\mathcal{R}_\alpha(t)$, for all $t \in [0, T]$. Now an application of the Arzelá-Ascoli theorem justifies the relatively compactness of W. Therefore, we conclude that operator Γ_2 is also a compact map.

Step 6. We shall show the set $\Upsilon = \{x \in \mathbb{Y} : \lambda\Psi_1(\frac{x}{\lambda}) + \lambda\Psi_2(x) = x$ for some $\lambda \in (0,1)\}$ is bounded on $[0, T]$. To do this, we consider the nonlinear operator equation

$$x(t) = \lambda\Psi x(t),\ 0 < \lambda < 1, \tag{3.10}$$

where Ψ is already defined. Next we gives a priori estimate for the solution of the above equation. Indeed, let $x \in \mathbb{Y}$ be a possible solution of $x = \lambda\Psi(x)$ for some $0 < \lambda < 1$. This implies by (3.10) that for each $t \in [0, T]$ we have

$$x(t) = \begin{cases} \lambda\mathcal{R}_\alpha(t)[\varphi(0) - g(0, \varphi(-\rho_1(0)))] + \lambda g(t, x(t-\rho_1(t))) \\ +\lambda\int_0^t \mathcal{S}_\alpha(t-s)h(s, x(s-\rho_2(s)))ds \\ +\lambda\int_0^t \mathcal{S}_\alpha(t-s)f(s, x(s-\rho_3(s)))dw(s), & t \in [0, t_1], \\ \lambda\mathcal{R}_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-)) - g(t_1, x(t_1^+ - \rho_1(t_1^+)))] \\ +\lambda g(t, x(t-\rho_1(t))) + \lambda\int_{t_1}^t \mathcal{S}_\alpha(t-s)h(s, x(s-\rho_2(s)))ds \\ +\lambda\int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s, x(s-\rho_3(s)))dw(s), & t \in (t_1, t_2], \\ \dots \\ \lambda\mathcal{R}_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-)) - g(t_m, x(t_m^+ - \rho_1(t_m^+)))] \\ +\lambda g(t, x(t-\rho_1(t))) + \lambda\int_{t_m}^t \mathcal{S}_\alpha(t-s)h(s, x(s-\rho_2(s)))ds \\ +\lambda\int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s, x(s-\rho_3(s)))dw(s), & t \in (t_m, T]. \end{cases} \tag{3.11}$$

By using Hölder's inequality and Lemma 2.11, we have for $t \in [0, t_1]$

$$\begin{aligned} &E\|x(t)\|_H^p \\ &\leq 4^{p-1}E\|\mathcal{R}_\alpha(t)[\varphi(0) - g(0, \varphi(-\rho_1(0)))]\|_H^p + 4^{p-1}E\|g(t, x(t-\rho_1(t)))\|_H^p \\ &\quad + 4^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)h(s, x(s-\rho_2(s)))ds\Big\|_H^p \\ &\quad + 4^{p-1}E\Big\|\int_0^t \mathcal{S}_\alpha(t-s)f(s, x(s-\rho_3(s)))dw(s)\Big\|_H^p \\ &\leq 8^{p-1}M^p[E\|\varphi(0)\|_H^p + E\|g(0, \varphi(-\rho_1(0)))\|_H^p] + 4^{p-1}E\|g(t, x(t-\rho_1(t)))\|_H^p \\ &\quad + 4^{p-1}M^pE\Big[\int_0^t e^{-\delta(t-s)}\|h(s, x(s-\rho_2(s)))\|_H ds\Big]^p \\ &\quad + 4^{p-1}\Big[\int_0^t [e^{-p\delta(t-s)}(E\|f(s, x(s-\rho_3(s)))\|_H^p)]^{2/p}ds\Big]^{p/2} \\ &\leq 8^{p-1}M^p[E\|\varphi(0)\|_H^p + L_gE\|\varphi(-\rho_1(0))\|_H^p] + 4^{p-1}L_gE\|x(t-\rho_1(t)))\|_H^p \\ &\quad + 4^{p-1}M^p\Big[\int_0^t e^{-(p\delta/p-1)(t-s)}ds\Big]^{p-1}\int_0^t m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds \\ &\quad + 4^{p-1}C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p}ds\Big]^{p/2}. \end{aligned}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, we have

$$\begin{aligned}
&E\|x(t)\|_H^p \\
&\le 4^{p-1}E\|\mathcal{R}_\alpha(t-t_k)[x(t_k^-) + I_k(x(t_k^-)) - g(t_k, x(t_k^+ - \rho_1(t_k^+)))]\|_H^p \\
&\quad + 4^{p-1}E\|g(t, x(t-\rho_1(t)))\|_H^p \\
&\quad + 4^{p-1}E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)h(s, x(s-\rho_2(s)))ds\Big\|_H^p \\
&\quad + 4^{p-1}E\Big\|\int_{t_k}^t \mathcal{S}_\alpha(t-s)f(s, x(s-\rho_3(s)))dw(s)\Big\|_H^p \\
&\le 12^{p-1}M^p[E\|x(t_k^-)\|_H^p + E\|I_k(x(t_k^-))\|_H^p + E\|g(t_k, x(t_k^+ - \rho_1(t_k^+)))\|_H^p] \\
&\quad + 4^{p-1}E\|g(t, x(t-\rho_1(t)))\|_H^p \\
&\quad + 4^{p-1}M^p\Big[\int_{t_k}^t e^{-\delta(t-s)}\|h(s, x(s-\rho_2(s)))\|_H ds\Big]^p \\
&\quad + 4^{p-1}\Big[\int_{t_k}^t [e^{-p\delta(t-s)}(E\|f(s, x(s-\rho_3(s)))\|_H^p)]^{2/p}ds\Big]^{p/2} \\
&\le 12^{p-1}M^p[E\|x(t_k^-)\|_H^p + d_k^1E\|x(t_k^-)\|_H^p + d_k^2 + L_gE\|x(t_k^+ - \rho_1(t_k^+))\|_H^p] \\
&\quad + 4^{p-1}L_gE\|x(t-\rho_1(t))\|_H^p \\
&\quad + 4^{p-1}M^p\Big[\int_{t_k}^t e^{-(p\delta/p-1)(t-s)}ds\Big]^{p-1}\int_{t_k}^t m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds \\
&\quad + 4^{p-1}C_pM^p\Big[\int_{t_k}^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s)))\|_H^p)]^{2/p}ds\Big]^{p/2}.
\end{aligned}$$

Then, for all $t \in [0, T]$, we have

$$\begin{aligned}
&E\|x(t)\|_H^p \\
&\le \widetilde{M} + 12^{p-1}M^p[E\|x(t_k^-)\|_H^p + d_k^1E\|x(t_k^-)\|_H^p \\
&\quad + L_gE\|x(t_k^+ - \rho_1(t_k^+))\|_H^p] + 4^{p-1}L_gE\|x(s-\rho_1(s))\|_H^p \\
&\quad + 4^{p-1}M^p\Big[\int_0^t e^{-(p\delta/p-1)(t-s)}ds\Big]^{p-1}\int_0^t m_h(s)\Theta_h(E\|x(s-\rho_2(s))\|_H^p)ds \\
&\quad + 4^{p-1}C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(E\|x(s-\rho_3(s))\|_H^p)]^{2/p}ds\Big]^{p/2},
\end{aligned}$$

where

$$\widetilde{M} = \max\{8^{p-1}M^p[E\|\varphi(0)\|_H^p + L_gE\|\varphi(-\rho_1(0))\|_H^p], 12^{p-1}M^p\tilde{d}\},$$

$\tilde{d} = \max_{1\le k\le m} d_k^2$. By the definition of $\mathbb{Y}$, it follows that

$$E\|x(s-\rho_i(s))\|_H^p \le 2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\sup_{s\in[0,t]}\|x(s)\|_H^p, i = 1, 2, 3.$$

If $\mu(t) = 2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\sup_{s\in[0,t]}E\|x(s)\|_H^p$, we obtain that

$$\begin{aligned}
\mu(t) &\le 2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\widetilde{M} + 12^{p-1}M^p[\mu(t) + d_k^1\mu(t) + 2^{p-1}L_g\mu(t)] + 8^{p-1}L_g\mu(t) \\
&\quad + 8^{p-1}M^p\Big[\int_0^t e^{-(p\delta/p-1)(t-s)}ds\Big]^{p-1}\int_0^t m_h(s)\Theta_h(\mu(s))ds
\end{aligned}$$

$$+ 8^{p-1}C_pM^p\Big[\int_0^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(\mu(s))]^{2/p}ds\Big]^{p/2}.$$

Note that

$$\begin{aligned}
&\Big[\int_0^t [e^{-p\delta(t-s)}m_f(s)\Theta_f(\mu(s))]^{2/p}ds\Big]^{p/2}\\
&\leq \Big[\int_0^t e^{-[\frac{2p}{p-2}]\delta(t-s)}ds\Big]^{p/2-1}\int_0^t m_f(s)\Theta_f(\mu(s))ds\\
&\leq [\frac{2\delta p}{p-2}]^{1-p/2}\int_0^t m_f(s)\Theta_f(\mu(s))ds.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\mu(t) &\leq 2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\widetilde{M} + 12^{p-1}M^p[\mu(t) + d_k^1\mu(t) + 2^{p-1}L_g\mu(t)]\\
&\quad + 8^{p-1}L_g\mu(t) + 8^{p-1}M^p[\frac{p\delta}{p-1}]^{1-p}\int_0^t m_h(s)\Theta_h(\mu(s))ds\\
&\quad + 8^{p-1}C_pM^p[\frac{2\delta p}{p-2}]^{1-p/2}\int_0^t m_f(s)\Theta_f(\mu(s))ds.
\end{aligned}$$

Since $\widetilde{L} = \max_{1\leq k\leq m}\{12^{p-1}M^p(1 + d_k^1 + 2^{p-1}L_g) + 8^{p-1}L_g\} < 1$, we obtain

$$\begin{aligned}
\mu(t) \leq \frac{1}{1-\widetilde{L}}\Big[&2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\widetilde{M} + 8^{p-1}M^p[\frac{p\delta}{p-1}]^{1-p}\int_0^t m_h(s)\Theta_h(\mu(s))ds\\
&+ 8^{p-1}C_pM^p[\frac{2\delta p}{p-2}]^{1-p/2}\int_0^t m_f(s)\Theta_f(\mu(s))ds\Big].
\end{aligned}$$

Denoting by $\zeta(t)$ the right-hand side of the above inequality, we have

$$\begin{gathered}
\mu(t) \leq \zeta(t) \quad \text{for all } t \in [0,T],\\
\zeta(0) = \frac{1}{1-\widetilde{L}}[2^{p-1}\|\varphi\|_{\mathfrak{B}}^p + 2^{p-1}\widetilde{M}],
\end{gathered}$$

$$\begin{aligned}
\zeta'(t) &= \frac{1}{1-\widetilde{L}}\Big[8^{p-1}M^p[\frac{p\delta}{p-1}]^{1-p}m_h(t)\Theta_h(\mu(t))\\
&\quad + 8^{p-1}C_pM^p[\frac{2\delta p}{p-2}]^{1-p/2}m_f(t)\Theta_f(\mu(t))\Big]\\
&\leq \frac{1}{1-\widetilde{L}}\Big[8^{p-1}M^p[\frac{p\delta}{p-1}]^{1-p}m_h(t)\Theta_h(\zeta(t))\\
&\quad + 8^{p-1}C_pM^p[\frac{2\delta p}{p-2}]^{1-p/2}m_f(t)\Theta_f(\zeta(t))\Big]\\
&\leq m^*(t)[\Theta_h(\zeta(t)) + \Theta_f(\zeta(t))],
\end{aligned}$$

where

$$\begin{aligned}
m^*(t) = \max\Big\{&\frac{1}{1-\widetilde{L}}8^{p-1}M^p[\frac{p\delta}{p-1}]^{1-p}m_h(t),\\
&\frac{1}{1-\widetilde{L}}8^{p-1}C_pM^p[\frac{2\delta p}{p-2}]^{1-p/2}m_f(t)\Big\}.
\end{aligned}$$

This implies that

$$\int_{\zeta(0)}^{\zeta(t)}\frac{du}{\Theta_h(u)+\Theta_f(u)} \leq \int_0^T m^*(s)ds < \infty.$$

This inequality shows that there is a constant $\widetilde{K}$ such that $\xi(t) \le \widetilde{K}, t \in [0,T]$, and hence $\|x\|_{\mathbb{Y}} \le \zeta(t) \le \widetilde{K}$, where $\widetilde{K}$ depends only on M, δ, p, C_p and on the functions $m_f(\cdot), \Theta_f(\cdot)$. This indicates that Υ is bounded on $[0,T]$. Consequently, by Lemma 2.12, we deduce that $\Psi_1 + \Psi_2$ has a fixed point $x(\cdot) \in \mathbb{Y}$, which is a mild solution of the system (1.1)-(1.3) with $x(s) = \varphi(s)$ on $[\tilde{m}(0), 0]$ and $E\|x(t)\|_H^p \to 0$ as $t \to \infty$. This shows that the asymptotic stability of the mild solution of (1.1)-(1.3). In fact, let $\varepsilon > 0$ be given and choose $\tilde{\delta} > 0$ such that $\tilde{\delta} < \varepsilon$ and satisfies

$$[16^{p-1}M^p + 8^{p-1}M^p[\delta^{1-p}L_h + C_p(\frac{2\delta(p-1)}{p-2})^{1-2/p}L_f]]\tilde{\delta} + \widetilde{L}\varepsilon < \varepsilon.$$

If $x(t) = x(t,\varphi)$ is mild solution of (1.1)-(1.3), with $\|\varphi\|_{\mathcal{B}}^p + L_g E\|\varphi(-\rho_1(0))\|_H^p < \tilde{\delta}$, then $(\Psi x)(t) = x(t)$ and satisfies $E\|x(t)\|_H^p < \varepsilon$ for every $t \ge 0$. Notice that $E\|x(t)\|_H^p < \varepsilon$ on $t \in [\tilde{m}(0), 0]$. If there exists $\tilde{t}$ such that $E\|x(\tilde{t})\|_H^p = \varepsilon$ and $E\|x(s)\|_H^p < \varepsilon$ for $s \in [\tilde{m}(0), \tilde{t}]$. Then (3.4) show that

$$\begin{aligned} &E\|x(t)\|_H^p \\ &\le \Big[16^{p-1}M^p e^{-p\delta\tilde{t}} + 8^{p-1}M^p\big[\delta^{1-p}L_h + C_p\big(\frac{2\delta(p-1)}{p-2}\big)^{1-2/p}L_f\big]\Big]\tilde{\delta} + \widetilde{L}\varepsilon < \varepsilon, \end{aligned}$$

which contradicts the definition of $\tilde{t}$. Therefore, the mild solution of (1.1)-(1.3) is asymptotically stable in p-th moment. The proof is complete. □

Remark 3.4. It is well known that the study on nonlocal problems are motivated by physical problems. For example, it is used to determine the unknown physical parameters in some inverse heat conduction problems [10]. Due to the importance of nonlocal conditions in different fields, there has been an increasing interest in study of the fractional impulsive stochastic differential equations involving nonlocal conditions (see [27]). In this remark, we will try to make some simulations about the above results and study the asymptotical stability in p-th moment of mild solutions to a class of fractional impulsive partial neutral stochastic integro-differential equations with nonlocal conditions in Hilbert spaces

$$\begin{gathered} {}^cD^\alpha N(x(t)) = AN(x(t)) + \int_0^t R(t-s)N(x(s))ds + h(t,x(t)) + f(t,x(t))\frac{dw(t)}{dt}, \\ t \ge 0, t \ne t_k, \end{gathered} \tag{3.12}$$

$$x_0 + G(x) = x_0, \quad x'(0) = 0, \tag{3.13}$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \quad t = t_k,\ k = 1, \dots, m, \tag{3.14}$$

where ${}^cD^\alpha, A, Q, w$ are defined as in (1.1)-(1.3). Here $N(x) = x(0) + g(t,x), x \in H$, and $g : [0,\infty) \times H \to H, f : [0,\infty) \times H \to L(K,H)$, are all Borel measurable; $I_k : H \to H(k = 1, \dots, m), G : \mathbb{Y} \to H$ are given functions, where $\mathbb{Y}$ be the space of all $\mathcal{F}_0$-adapted process $\psi(t,\tilde{w}) : [0,\infty) \times \Omega \to \mathbb{R}$ which is almost certainly continuous in t for fixed $\tilde{w} \in \Omega$. Moreover $\psi(0,\tilde{w}) = x_0$ and $E\|\psi(t,\tilde{w})\|_H^p \to 0$ as $t \to \infty$. Also $\mathbb{Y}$ is a Banach space when it is equipped with a norm defined by

$$\|\psi\|_{\mathbb{Y}} = \sup_{t \ge 0} E\|\psi(t)\|_H^p.$$

To prove the Asymptotic stability result, we assume that the following condition holds.

(H6) The functions $G : \mathbb{Y} \to H$ are completely continuous and that there is a constant c such that $E\|G(x)\|_H^p \le c$ for every $x \in \mathbb{Y}$.

Further, the mild solution of the Fractional impulsive stochastic system (3.12)-(3.14) can be written as

$$x(t) = \begin{cases} \mathcal{R}_\alpha(t)[x_0 - G(x) - g(0, x(0))] + g(t, x(t)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)h(s, x(s))ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s, x(s))dw(s), & t \in [0, t_1], \\ \mathcal{R}_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-)) - g(t_1, x(t_1^+))] \\ +g(t, x(t)) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)h(s, x(s))ds \\ + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s, x(s))dw(s), & t \in (t_1, t_2], \\ \dots \\ \mathcal{R}_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-)) - g(t_m, x(t_m^+))] \\ +g(t, x(t)) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)h(s, x(s))ds \\ + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s, x(s))dw(s), & t \in (t_m, T]. \end{cases}$$

One can easily prove that by adopting and employing the method used in Theorem 3.3, the fractional impulsive stochastic differential equations (3.12)-(3.14) is asymptotically stable in p-th moment.

4. Example

Consider the fractional impulsive partial stochastic neutral integro-differential equation

$$\begin{aligned} \frac{\partial^\alpha N(z_t)(x)}{\partial t^\alpha} &= \frac{\partial^2 N(z_t)(x)}{\partial x^2} + \int_0^t (t-s)^\sigma e^{-\mu(t-s)} \frac{\partial^2 N(z_t)(x)}{\partial x^2} ds \\ &\quad + \varsigma(t, x, z(t-\rho_2(t), x)) + \varpi(t, x, z(t-\rho_3(t), x)) \frac{dw(t)}{dt}, \\ &\quad t \ge 0, \quad 0 \le x \le \pi, \quad t \ne t_k, \end{aligned} \tag{4.1}$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \ge 0, \tag{4.2}$$

$$z_t(0, x) = 0, \quad 0 \le x \le \pi, \tag{4.3}$$

$$z(\tau, x) = \varphi(\tau, x), \quad \tau \le 0,\ 0 \le x \le \pi, \tag{4.4}$$

$$\triangle z(t_k, x) = z(t_k^+, x) - z(t_k^-, x) = \int_0^\pi \eta_k(s, z(t_k, x))ds, \quad k = 1, 2, \dots, m, \tag{4.5}$$

where $(t_k)_k \in \mathbb{N}$ is a strictly increasing sequence of positive numbers, $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is a Caputo fractional partial derivative of order $\alpha \in (1, 2)$, σ, and μ are positive numbers and $w(t)$ denotes a standard cylindrical Wiener process in H defined on a stochastic space $(\Omega, \mathcal{F}, P)$. In this system, $\rho_i(t) \in C(\mathbb{R}^+, \mathbb{R}^+), i = 1, 2, 3$, and

$$N(z_t)(x) = z(t, x) - \vartheta(t, x, z(t-\rho_1(t), x)).$$

Let $H = L^2([0, \pi])$ with the norm $\|\cdot\|_H$ and define the operators $A : D(A) \subseteq H \to H$ by $A\omega = \omega''$ with the domain

$$D(A) := \{\omega \in H : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in H, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \dots$ is the orthogonal set of eigenvectors of A. It is well known that A generates a strongly continuous semigroup $T(t), t \geq 0$ which is compact, analytic and self-adjoint in H and A is sectorial of type and (P1) is satisfied. The operator $R(t) : D(A) \subseteq H \to H, t \geq 0, R(t)x = t^{\sigma} e^{-\omega t} x''$ for $x \in D(A)$. Moreover, it is easy to see that conditions (P2) and (P3) in Section 2 are satisfied with $b(t) = t^{\sigma} e^{-\mu t}$ and $D = C_0^{\infty}([0, \pi])$, where $C_0^{\infty}([0, \pi])$ is the space of infinitely differentiable functions that vanish at $x = 0$ and $x = \pi$.

Additionally, we will assume that

(i) The function $\vartheta : [0, \infty) \times [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant L_{ϑ} such that

$$|\vartheta(t, x, u) - \vartheta(t, x, v)| \leq L_{\vartheta} |u - v|, \quad t \geq 0, x \in [0, \pi], u, v \in \mathbb{R}.$$

(ii) The function $\varsigma : [0, \infty) \times [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive continuous function $m_{\varsigma}(\cdot) : \mathbb{R} \times [0, \pi] \to \mathbb{R}$ such that

$$|\varsigma(t, x, u)| \leq m_{\varsigma}(t, x)|u|, \quad t \geq 0, x \in [0, \pi], u \in \mathbb{R}.$$

(iii) The function $\vartheta : [0, \infty) \times [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive continuous function $m_{\varpi}(\cdot) : \mathbb{R} \times [0, \pi] \to \mathbb{R}$ such that

$$|\varpi(t, x, u)| \leq m_{\varpi}(t, x)|u|, \quad t \geq 0, x \in [0, \pi], u \in \mathbb{R}.$$

(iv) The functions $\eta_k : \mathbb{R}^2 \to \mathbb{R}, k \in \mathbb{N}$, are completely continuous and there are positive continuous functions $L_k : [0, \pi] \to \mathbb{R} (k = 1, 2, \dots, m)$ such that $|\eta_k(s, u)| \leq L_k(s)|u|$, $s \in [0, \pi]$, $u \in \mathbb{R}$.

We can define $N : H \to H$, $g, h : [0, \infty) \times H \to H, f : [0, \infty) \times H \to L(K, H)$ and $I_k : H \to H$ respectively by

$$\begin{gathered} N(z_t)(x) = \varphi(0, x) + g(t, z)(x), \\ g(t, z)(x) = \vartheta(t, x, z(t - \rho_1(t), x)), \\ h(t, z)(x) = \varsigma(t, x, z(t - \rho_2(t), x)), \\ f(t, z)(x) = \varpi(t, x, z(t - \rho_3(t), x)), \\ I_k(z)(x) = \int_0^{\pi} \eta_k(s, z(x)) ds. \end{gathered}$$

Then the problem (4.1)–(4.5) can be written as (1.1)–(1.3). Moreover, it is easy to see that

$$\begin{gathered} E\|g(t, z_1) - g(t, z_2)\|_H^p \leq L_g \|z_1 - z_2\|_H^p, \ z_1, z_2 \in H, \\ E\|h(t, z)\|_H^p \leq m_h(t) \|z\|_H^p, z \in H, \\ E\|f(t, z)\|_H^p \leq m_f(t) \|z\|_H^p, z \in H, \\ E\|I_k(z)\|_H^p \leq d_k \|z\|_H^p, z \in H, k = 1, 2, \dots, m, \end{gathered}$$

where $L_g = L_{\vartheta}^p, m_h(t) = \sup_{x \in [0, \pi]} m_{\tau}^p(t, x)$, $m_f(t) = \sup_{x \in [0, \pi]} m_{\varpi}^p(t, x)$, $d_k = [\int_0^{\pi} L_k(s) ds]^p$, $k = 1, 2, \dots, m$. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 3.3, we can conclude that system (4.1)-(4.5) has at least one mild solution, then the mild solutions is asymptotically stable in the p-th mean.

Conclusions. In this article, we are focused on the theory study on the asymptotical stability in the p-th moment of mild solutions to a class of fractional impulsive partial neutral stochastic integro-differential equations with state-dependent delay. We derive some interesting sufficient conditions to guarantee the asymptotical stability results for fractional impulsive stochastic evolution systems in infinite dimensional spaces. Our techniques rely on the fractional calculus, properties of the α-resolvent operator, and Krasnoselskii-Schaefer type fixed point theorem. Our methods not only present a new way to study such problems under the Lipschitz condition not required in the paper, but also provide new theory results appeared in paper previously are generalized to the fractional stochastic systems settings and the case of state-dependent delay with impulsive conditions. An application is provided to illustrate the applicability of the new result.

Our future work will try to make some the above results and study the exponential stability in p-th moment of mild solutions to fractional impulsive partial neutral stochastic integro-differential equations with state-dependent delay.

Acknowledgments. The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions to improve the the original manuscript.

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