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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR HÉNON EQUATIONS IN HYPERBOLIC SPACES

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Abstract. In this article, we consider the existence and asymptotic behavior of solutions for the Hénon equation

$$\begin{gathered}-\Delta_{\mathbb{B}^N} u = (d(x))^\alpha |u|^{p-2}u, \quad x \in \Omega \\ u = 0 \quad x \in \partial\Omega,\end{gathered}$$

where $\Delta_{\mathbb{B}^N}$ denotes the Laplace Beltrami operator on the disc model of the Hyperbolic space $\mathbb{B}^N$, $d(x) = d_{\mathbb{B}^N}(0,x)$, $\Omega \subset \mathbb{B}^N$ is geodesic ball with radius 1, $\alpha > 0, N \geq 3$. We study the existence of hyperbolic symmetric solutions when $2 < p < \frac{2N+2\alpha}{N-2}$. We also investigate asymptotic behavior of the ground state solution when p tends to the critical exponent $2^* = \frac{2N}{N-2}$ with $N \geq 3$.

1. Introduction and main result

In this paper, we consider the problem

$$\begin{gathered}-\Delta_{\mathbb{B}^N} u = (d(x))^\alpha |u|^{p-2}u, \quad x \in \Omega \\ u = 0 \quad x \in \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Delta_{\mathbb{B}^N}$ denotes the Laplace Beltrami operator on the disc model of the Hyperbolic space $\mathbb{B}^N$, $d(x) = d_{\mathbb{B}^N}(0,x)$, $\Omega \subset \mathbb{B}^N$ is geodesic ball with radius 1, $\alpha > 0, N \geq 3$.

When posed in Euclidean space $\mathbb{R}^N$, problem (1.1) becomes

$$\begin{gathered}-\Delta u = |x|^\alpha |u|^{p-2}u, \quad x \in \Omega \\ u = 0 \quad x \in \partial\Omega,\end{gathered} \tag{1.2}$$

where Ω is the unit ball in $\mathbb{R}^N$ with $N \geq 3$, $\alpha > 0$ and $p > 2$, which stems from the study of rotating stellar structures and is called Hénon equation [5]. Such a problem has been extensively studied, see for instance [1, 7, 9] etc. Interesting phenomenon concerning with problem (1.2) was revealed recently that the exponent α affects the critical exponent for the existence of solutions. Precisely, it was shown in [7] that for $p \in (2, \frac{2N+2\alpha}{N-2})$, problem (1.2) admits at least one radial solution. One also notices that the moving plane method in [2] can not be applied to (1.2) since the weight function $r \mapsto r^\alpha$ is increasing. So it can be expected that problem (1.2) possesses non-radial solutions. Such solutions were found in [10] for $2 < p < \frac{2N}{N-2}$

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and in [9] for $p = \frac{2N}{N-2}$. Furthermore, in [1], the limiting behavior of the ground state solutions of (1.2) was considered as $p \to 2^* = \frac{2N}{N-2}$. The authors showed that the maximum point of ground state solutions of (1.2) concentrate on a boundary point of the domain as $p \to 2^*$. In their arguments, one of the key ingredients is to show that the ground state solutions $\{u_p\}$, $2 < p < \frac{2N}{N-2}$, of problem (1.2) is actually a minimizing sequence of the problem

$$S = \inf \Big\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |u|^{2^*} \, dx)^{2/2^*}} : u \in H_0^1(\Omega), \, u \not\equiv 0 \Big\}$$

as $p \to 2^*$, and use the fact that S is attained in $\mathbb{R}^N$ by the instanton $U = 1/(1-|x|^2)^{(N-2)/2}$.

In the hyperbolic space, the existence of Brezis-Nirenberg problem for the critical equation

$$-\Delta_{\mathbb{B}^N} u = |u|^{2^*-2} u + \lambda u, \quad u \geq 0, u \in H_0^1(\Omega) \tag{1.3}$$

has been studied in [11] and the results are very similar to the results in the Euclidean case. However, for problem (1.1), there are some difference from Euclidean space. Firstly, the weight function $d(x)$ depends on the Riemannian hyperbolic distance r from a pole o. Secondly, the main purpose in this paper is to study the profile of ground state solution u_p of problem (1.1) as $p \to 2^*$, in particular, the asymptotic behavior of u_p and the limit location of the maximum point of u_p as $p \to 2^*$. In generally, in order to prove the ground state solution is a minimizing sequence of the problem

$$S_{\mathbb{B}^N}(\Omega) = \inf \Big\{ \frac{\int_\Omega |\nabla_{\mathbb{B}^N} u|^2 \, dx}{(\int_\Omega |u|^{2^*} \, dx)^{2/2^*}} : u \in H_0^1(\Omega), \, u \not\equiv 0 \Big\}, \tag{1.4}$$

one will use the unique positive solution of the problem

$$-\Delta_{\mathbb{B}^N} u = u^{2^*-1} \quad \text{in } \mathbb{B}^N. \tag{1.5}$$

However, in [6], Mancini and Sandeep proved that problem (1.5) did not have any positive solutions.

Motivated by above mentioned works, we study problem (1.1) in this paper. Our main results are as follows.

Theorem 1.1. *For $\alpha > 0$, problem (1.1) possesses a ground solution u_p which belongs to $H_0^1(\mathbb{B}^N)$ when $p \in (2, \frac{2N}{N-2})$. Moreover, there is a hyperbolic symmetry positive solution u_p^{rad} for problem (1.1) provided that $p \in (2, \frac{2N+2\alpha}{N-2})$.*

Theorem 1.2. *Suppose $p \in (2, 2^*), \alpha > 0$, then the ground state solution u_p satisfies (after passing to subsequence) for some $x_0 \in \partial\Omega$,*

(i) *$|\nabla_{\mathbb{B}^N} u_p|^2 \to \mu\delta_{x_0}$ as $p \to 2^*$ in the sense of measure.*
(ii) *$|u_p|^{2^*} \to \nu\delta_{x_0}$ as $p \to 2^*$ in the sense of measure,*

where $\mu > 0$, $\nu > 0$ satisfy $\mu \geq S\nu^{2/2^*}$, δ_x is the Dirac mass at x.

Theorem 1.3. *Let u_p be as in Theorem 1.2 and $x_p \in \bar{\Omega}$ be such $M_p' = u_p(x_p) = max_{x \in \bar{\Omega}} u_p(x)$, $\lambda_p' = {M_p'}^{-2/(N-2)}$. Then, as $p \to 2^*, M_p' \to +\infty$ and*

(i) *x_p is unique when p close to 2^*. Moreover, as $p \to 2^*, dist_{\mathbb{B}^N}(x_p, \partial\Omega) \to 0$, $\mathrm{dist}_{\mathbb{B}^N}(x_p, \partial\Omega)/\lambda_p' \to \infty$,*
(ii) *$\lim_{p \to 2^*} \int_\Omega |\nabla_{\mathbb{B}^N}(u_p - (\frac{1-|x|^2}{2})^{\frac{N-2}{2}} U_{\lambda_p, x_p})|^2 \, dV_{\mathbb{B}^N} = 0$, where λ_p is defined in Section 4.*

This paper is organized as follows. In section 2, we give some basic facts about hyperbolic space and the proof of Theorem 1.1. In section 3, we show that u_p is a minimizing sequence of S as $p \to 2^*$, and then prove Theorem 1.2 by the concentration compactness principle. In section 4, we prove Theorem 1.3 mainly by a blow-up technique.

2. Preliminaries

Hyperbolic space $\mathbb{H}^N$ is a complete simple connected Riemannian manifold which has constant sectional curvature equal to -1. There are several models for $\mathbb{H}^N$ and we will use the Poincaré ball model $\mathbb{B}^N$ in this article.

The Poincaré ball model for the hyperbolic space is

$$\mathbb{B}^N = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^N \mid |x| < 1\}$$

endowed with Riemannian metric g given by $g_{ij} = (p(x))^2 \delta_{ij}$ where $p(x) = \frac{2}{1-|x|^2}$. We denote the hyperbolic volume by $dV_{\mathbb{B}^N}$ and is given by $dV_{\mathbb{B}^N} = (p(x))^N\,dx$. The hyperbolic gradient and the Laplace Beltrami operator are:

$$\Delta_{\mathbb{B}^N} = (p(x))^{-N}\operatorname{div}((p(x))^{N-2}\nabla u)), \quad \nabla_{\mathbb{B}^N} u = \frac{\nabla u}{p(x)}$$

where ∇ and div denote the Euclidean gradient and divergence in $\mathbb{R}^N$, respectively.

The hyperbolic distance $d_{\mathbb{B}^N}(x, y)$ between $x, y \in \mathbb{B}^N$ in the Poincaré ball model is given by the formula:

$$d_{\mathbb{B}^N}(x, y) = \operatorname{Arccosh}(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}). \tag{2.1}$$

From this we immediately obtain for $x \in \mathbb{B}^N$,

$$d(x) = d_{\mathbb{B}^N}(0, x) = \log(\frac{1+|x|}{1-|x|}).$$

Let us denote the energy functional corresponding to (1.1) by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla_{\mathbb{B}^N} u|^2 dV_{\mathbb{B}^N} - \frac{1}{p}\int_\Omega |d(x)|^\alpha |u|^p dV_{\mathbb{B}^N} \tag{2.2}$$

defined on $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the Sobolev space on $\mathbb{B}^N$ with the above metric g. We see that $u \in H_0^1(\Omega)$ is a solution of problem (1.1) if and only if $v = (\frac{2}{1-|x|^2})^{\frac{N-2}{2}} u$ solves the following equation

$$-\Delta v + \frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2 v = (\ln\frac{1+|x|}{1-|x|})^\alpha (\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}} |v|^{p-2} v, \tag{2.3}$$

for $v \in H_0^1(\Omega')$, where Ω' is a ball in $\mathbb{R}^N$ centered at origin with radius $r = (e-1)/(e+1)$, $\alpha > 0, p > 2$.

Let us define the energy functional corresponding to (2.3) by

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\Omega'} |\nabla v|^2 + \frac{N(N-2)}{4})(\frac{2}{1-|x|^2})^2 v^2\,dx \\ &\quad - \frac{1}{p}\int_{\Omega'} (\ln\frac{1+|x|}{1-|x|})^\alpha (\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}} |v|^p\,dx. \end{aligned} \tag{2.4}$$

Thus for any $u \in H_0^1(\Omega)$ if $\tilde{u}$ is defined as $\tilde{u} = (\frac{2}{1-|x|^2})^{\frac{N-2}{2}} u$, then $I(u) = J(\tilde{u})$. Moreover $\langle I'(u), v\rangle = \langle J'(\tilde{u}), \tilde{v}\rangle$ where $\tilde{v}$ is defined in the same way.

Proof Theorem 1.1. As $\ln\frac{1+|x|}{1-|x|}\le\frac{2|x|}{1-|x|^2}$, $|x|\le\frac{e-1}{e+1}$ and $\frac{2e}{(e+1)^2}\le\frac{1-|x|^2}{2}\le\frac{1}{2}$, firstly, for $\alpha\ge 0, 2<p<2^*$, we have the variational problem

$$S_{\alpha,p}:=\inf_{0\not\equiv v\in H_0^1(\Omega')}\frac{\int_{\Omega'}|\nabla v|^2+\frac{N(N-4)}{2}(\frac{2}{1-|x|^2})^2v^2dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v|^pdx)^{2/p}} \tag{2.5}$$

which is solved by a v_p. Thus $u_p=(\frac{2}{1-|x|^2})^{-\frac{N-2}{2}}v_p$ is a ground state solution of (1.1).

Secondly, by [7], we have $u\mapsto|x|^{\frac{\alpha}{p}}u$ from $H_r^1(\Omega')$ to $L^p(\Omega')$ is compact for $p\in(2,\frac{2N}{N-2-\frac{2\alpha}{p}})(2<p<\frac{2N+2\alpha}{N-2})$. Then the problem

$$S^R_{\alpha,p}:=\inf_{0\not\equiv v\in H_{0,rad}^1(\Omega')}\frac{\int_{\Omega'}|\nabla v|^2+\frac{N(N-4)}{2}(\frac{2}{1-|x|^2})^2v^2dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v|^pdx)^{2/p}} \tag{2.6}$$

is also attained by a v_p^{rad}, where $H^1_{0,rad}(\Omega')$ denotes the subspace of radial functions in $H_0^1(\Omega')$. Then $u_p^{rad}=(\frac{2}{1-|x|^2})^{-\frac{N-2}{2}}v_p^{rad}$ is a hyperbolic symmetry solution of (1.1). $\square$

3. Proof of Theorem 1.2

Let us consider the problem

$$\begin{gathered}-\Delta v+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2v=(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v|^{p-2}v,\quad x\in\Omega'\\ v=0,\quad x\in\partial\Omega'\end{gathered} \tag{3.1}$$

where Ω' is a ball in $\mathbb{R}^N$ centered at origin with radius $r=\frac{e-1}{e+1}$, $\alpha>0,p>2$.

Lemma 3.1. *The solution of* (3.1) *satisfies*

$$\begin{aligned}&\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}|v_p|^pdx)^{2/p}}\\ &\ge\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}|v_p|^{2^*}dx)^{2/2^*}}+O_{(2^*-p)}(1)\end{aligned} \tag{3.2}$$

for p near 2^*.

Proof. By Hölder inequality

$$\Big(\int_{\Omega'}|v_p|^pdx\Big)^{1/p}\le(\int_{\Omega'}|v_p|^{2^*}dx)^{1/2^*}\Big(\operatorname{meas}\Omega'\Big)^{\frac{2^*-p}{2^*p}},$$

then Lemma 3.1 follows immediately. $\square$

For $\varepsilon>0$ small enough, let $x_0=(\frac{e-1}{e+1}-\frac{1}{|\ln\varepsilon|},0,\dots,0)\in R^N$,

$$U_\varepsilon(x)=\frac{1}{(\varepsilon+|x-x_0|^2)^{\frac{N-2}{2}}},$$

$\varphi\in C_0^\infty(\Omega)$ be a cut-off function satisfying

$$\varphi(x)=\begin{cases}1, & x\in B(x_0,\frac{1}{2|\ln\varepsilon|})\\ 0, & x\in R^n\backslash B(x_0,\frac{1}{|\ln\varepsilon|})\end{cases}$$

and $0 \le \varphi(x) \le 1$, $|\nabla\varphi(x)| \le C|\ln\varepsilon|$ for all $x \in \mathbb{R}^N$, where C is independent of ε, $B(x,r)$ denotes a ball centered x with radius r.

Set $v_\varepsilon = \varphi U_\varepsilon$, then $v_\varepsilon \in H_0^1(\Omega')$.

Lemma 3.2. *Let v_ε be defined as above, then*

$$\lim_{\varepsilon\to 0}\lim_{p\to 2^*}\frac{\int_{\Omega'}\big(|\nabla v_\varepsilon|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_\varepsilon|^2\big)dx}{\big(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v_\varepsilon|^p dx\big)^{2/p}} = S\,.$$

Proof. On the one hand, from [1][11], we have

$$|v_\varepsilon|_p^2 = |U|_p^2\varepsilon^{\frac{N}{p}-(N-2)} + CK_1(\varepsilon)|U|_p^{2-p}\varepsilon^{\frac{(N-2)p}{2}-\frac{N}{2}+\frac{N}{p}-(N-2)}, \tag{3.3}$$

$$|\nabla v_\varepsilon|_2^2 = |\nabla U|_2^2\varepsilon^{-\frac{(N-2)}{2}} + \begin{cases} C|\ln\varepsilon|^{N-2}+o(|\ln\varepsilon|^{N-2}), & N\ge 5,\\ C|\ln\varepsilon|^2(\ln(2|2\ln\varepsilon|))+O(|\ln\varepsilon|^2), & N=4,\\ C|\ln\varepsilon|^2, & N=3,\end{cases} \tag{3.4}$$

and

$$|v_\varepsilon|_2^2 = O(\frac{1}{|\ln\varepsilon|^2}). \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v_\varepsilon|^p dx\\
&\ge (\ln\frac{e-\frac{e+1}{|\ln\varepsilon|}}{1+\frac{e+1}{|\ln\varepsilon|}})^\alpha(\frac{1}{2})^{\frac{(N-2)p-2N}{2}}\int_{\Omega'}|v_\varepsilon|^p dx,
\end{aligned}$$

and

$$\int_{\Omega'}\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_\varepsilon|^2\,dx \le \frac{N(N-2)}{4}\frac{(e+1)^2}{2e}\int_{\Omega'}|v_\varepsilon|^2\,dx.$$

By(3.3)–(3.5), for $N\ge 5$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{p\to 2^*}\frac{\int_{\Omega'}(\nabla v_\varepsilon|^2+\frac{N(N-2)}{4}\frac{2}{1-|x|^2}|v_\varepsilon|^2)dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v_\varepsilon|^p dx)^{2/p}}\\
&\le \lim_{\varepsilon\to 0}\lim_{p\to 2^*}\frac{1}{(\ln\frac{e-\frac{e+1}{|\ln\varepsilon|}}{1+\frac{e+1}{|\ln\varepsilon|}})^{\frac{2\alpha}{p}}}\times\frac{1}{(\frac{1}{2})^{\frac{(N-2)p-2N}{2}}}\\
&\quad\times\frac{|\nabla U|_2^2\varepsilon^{-\frac{N-2}{2}}+C|\ln\varepsilon|^{N-2}+o(|\ln\varepsilon|^{N-2})+O(\frac{1}{|\ln\varepsilon|^2})}{|U|_p^2\varepsilon^{\frac{N}{p}-(N-2)}+CK_1(\varepsilon)|U|_p^{2-p}\varepsilon^{\frac{(N-2)p}{2}-\frac{N}{2}+\frac{N}{p}-(N-2)}}\\
&= \lim_{\varepsilon\to 0}(\ln\frac{e-\frac{e+1}{|\ln\varepsilon|}}{1+\frac{e+1}{|\ln\varepsilon|}})^{\frac{2\alpha}{2^*}}\times\frac{|\nabla U|_2^2\varepsilon^{-\frac{N-2}{2}}+C|\ln\varepsilon|^{N-2}+o(|\ln\varepsilon|^{N-2})+O(\frac{1}{|\ln\varepsilon|^2})}{|U|_2^{2^*}\varepsilon^{-\frac{N-2}{2}}+C|\ln\varepsilon|^N\varepsilon}\\
&= \lim_{\varepsilon\to 0}(\ln\frac{e-\frac{e+1}{|\ln\varepsilon|}}{1+\frac{e+1}{|\ln\varepsilon|}})^{\frac{2\alpha}{2^*}}\times\frac{|\nabla U|_2^2+C(\varepsilon^{\frac12}|\ln\varepsilon|)^{N-2}}{|U|_{2^*}^2+C(\varepsilon^{\frac12}|\ln\varepsilon|)^{N-2}} = \frac{|\nabla U|_2^2}{|U|_{2^*}^2}.
\end{aligned} \tag{3.6}$$

Moreover,

$$\frac{\int_{\Omega'}\big(|\nabla v_\varepsilon|^2+\frac{N(N-2)}{4}\frac{2}{1-|x|^2}|v_\varepsilon|^2\big)dx}{\big(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v_\varepsilon|^p dx\big)^{2/p}} \ge \frac{\int_{\Omega'}\big(|\nabla v_\varepsilon|^2+\frac{N(N-2)}{4}\frac{2}{1-|x|^2}|v_\varepsilon|^2\big)dx}{(\frac{e}{(1+e)^2})^{\frac{(N-1)p-2N}{p}}\big(\int_{\Omega'}|v_\varepsilon|^p dx\big)^{2/p}}.$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\lim_{p\to 2^*}\frac{\int_\Omega(|\nabla v_\varepsilon|^2+\frac{N(N-2)}{4}\frac{2}{1-|x|^2}|v_\varepsilon|^2)dx}{(\int_\Omega(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2}dx)^{\frac{(N-2)p-2N}{2}}|v_\varepsilon|^p)^{2/p}}\geq\frac{|\nabla U|_2^2}{|U|_{2^*}^2}. \tag{3.7}$$

Combining (3.6) and (3.7), we can complete the proof for $N\geq 5$. The case $N=3,4$ can been proved similarly. □

Lemma 3.3. *There holds*

$$\lim_{p\to 2^*}\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)(p-1)-N-2}{2}}|v_p|^p dx)^{2/p}}=S, \tag{3.8}$$

$$\lim_{p\to 2^*}\frac{\int_{\Omega'}|\nabla v_p|^2dx}{(\int_{\Omega'}|v_p|^p dx)^{2/p}}=\lim_{p\to 2^*}\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}|v_p|^p dx)^{2/p}}=S. \tag{3.9}$$

Proof. By the definition of $\{v_p\}$ and Lemma 3.2, noting $\ln\frac{1+|x|}{1-|x|}\leq 1$, $\frac{2e}{(e+1)^2}\leq\frac{1-|x|^2}{2}\leq\frac{1}{2}$, and $(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}\to 1$ as $p\to 2^*$, we have

$$\begin{aligned}
&\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}|v_p|^p dx)^{2/p}}\\
&\leq\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v_p|^p dx)^{2/p}}\\
&\leq\frac{\int_{\Omega'}(|\nabla v_\varepsilon|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_\varepsilon|^2)dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha(\frac{1-|x|^2}{2})^{\frac{(N-2)p-2N}{2}}|v_\varepsilon|^p dx)^{2/p}}=S+o(\varepsilon).
\end{aligned} \tag{3.10}$$

In addition, for any p, $2<p<2^*$,

$$S\leq\frac{\int_{\Omega'}|\nabla v_p|^2dx}{(\int_{\Omega'}|v_p|^p dx)^{2/p}}\leq\lim_{p\to 2^*}\frac{\int_{\Omega'}(|\nabla v_p|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_p|^2)dx}{(\int_{\Omega'}|v_p|^p dx)^{2/p}} \tag{3.11}$$

which combined with Lemma 3.1, gives (3.8) and (3.9). □

Lemma 3.3 implies that $\{v_p\}$ is actually a minimizing sequence of S. In fact, we have

Corollary 3.4.

$$\lim_{p\to 2^*}\int_{\Omega'}|\nabla v_p|^2\,dx=S^{\frac{N}{2}}.$$

Corollary 3.5. *When $p=2^*$, Equation (2.3) does not possess any ground state solutions.*

Proof. Assume to the contrary that $S_{\alpha,2^*}$ can be achieved by $v_{2^*}\in H_0^1(\Omega')$, by Lemma 3.3, $S_{\alpha,2^*}=S_{0,2^*}=S$, so

$$\begin{aligned}
S_{\alpha,2^*}&=\frac{\int_{\Omega'}(|\nabla v_{2^*}|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_{2^*}|^2)dx}{(\int_{\Omega'}(\ln\frac{1+|x|}{1-|x|})^\alpha|v_{2^*}|^{2^*}dx)^{2/2^*}}\\
&\geq\frac{\int_{\Omega'}(|\nabla v_{2^*}|^2+\frac{N(N-2)}{4}(\frac{2}{1-|x|^2})^2|v_{2^*}|^2)\,dx}{(\int_{\Omega'}|v_{2^*}|^{2^*}dx)^{2/2^*}}
\end{aligned}$$

$$\geq \frac{\int_{\Omega'} |\nabla v_{2^*}|^2 dx}{(\int_{\Omega'} |v_{2^*}|^{2^*}\, dx)^{2/2^*}} \geq S.$$

Hence $\frac{\int_{\Omega'} |\nabla v_{2^*}|^2\, dx}{(\int_{\Omega'} |v_{2^*}|^{2^*}\, dx)^{2/2^*}} = S$, which is impossible since S cannot be achieved in a bounded domain. $\square$

Proof of Theorem 1.2. Suppose $p \in (2, 2^*), \alpha > 0$. As [1], using the concentration-compactness principle, we can prove that the ground state solution v_p of problem (3.1) satisfies (after passing to subsequence) for some $x_0 \in \partial\Omega'$,

(i) $|\nabla v_p|^2 \rightharpoonup \mu\delta_{x_0}$ as $p \to 2^*$ in the sense of measure.
(ii) $|v_p|^{2^*} \rightharpoonup \nu\delta_{x_0}$ as $p \to 2^*$ in the sense of measure,

where $\mu > 0$, $\nu > 0$ satisfy $\mu \geq S\nu^{2/2^*}$, δ_x is the Dirac mass at x.

Given that $\frac{2e}{(e+1)^2} \leq \frac{1-|x|^2}{2} \leq \frac{1}{2}$, $u_p = (\frac{1-|x|^2}{2})^{\frac{N-2}{2}} v_p$ and $v_p \rightharpoonup 0$ in $H_0^1(\Omega')$, we have

(i) $|\nabla_{\mathbb{B}^N} u_p|^2 \rightharpoonup \mu\delta_{x_0}$ as $p \to 2^*$ in the sense of measure.
(ii) $|u_p|^{2^*} \rightharpoonup \nu\delta_{x_0}$ as $p \to 2^*$ in the sense of measure,

where $\mu > 0$, $\nu > 0$ satisfy $\mu \geq S\nu^{2/2^*}$, δ_x is the Dirac mass at x. $\square$

4. Proof of Theorem 1.3

In this section, we shall study the asymptotic of the ground state solution and prove Theorem 1.3. Set

$$M_p = \sup_{x \in \bar{\Omega}'} v_p(x) = v_p(x_p), x_p \in \bar{\Omega}'.$$

Proposition 4.1. *$M_p \to +\infty$ as $p \to 2^*$.*

Proof. We need only to prove this proposition for any subsequence p_k, such that $p_k \to 2^*$ as $k \to +\infty$. Assume by contradiction that there exists a positive constant c such that $M_{p_k} \leq c$ for all k. For Theorem 1.2, $v_{p_k} \to 0$ a.e. Ω'. By Fatou's Lemma, Egoroff Theorem and the fact that $\int_{\Omega'} |v_{p_k}|^{2^*} = 1$, we have $u_{p_k} \to 0$ weakly in $L^{2^*}(\Omega')$. So, for $\sigma > 0$ small, due to the compactness of $L^{2^*}(\Omega') \hookrightarrow L^{2^*-\sigma}(\Omega')$, we have a subsequence(still denoted by $\{v_{p_k}\}$) such that

$$1 = \int_{\Omega'} |v_{p_k}|^{2^*} dx \leq |v_{p_k}|^{\sigma}_{L^\infty(\Omega')} \int_{\Omega'} |v_{p_k}|^{2^*-\sigma} dx \leq c^\sigma \int_{\Omega'} |v_{p_k}|^{2^*-\sigma} dx \to 0$$

as $k \to \infty$, which is impossible. $\square$

Proof of Theorem 1.3 We follow the blow up technique used by Gidas and Spruck in[3]. Suppose that for a subsequence of p as $p \to 2^*, x_p \to x_0 \in \bar{\Omega}'$. Let λ_p be a sequence of positive numbers defined by $\lambda_p^{\frac{N-2}{2}} M_p = 1$ and $y = \frac{x-x_p}{\lambda_p}$. Define the scaled function

$$w_p(y) = \lambda_p^{\frac{N-2}{2}} v_p(x) \tag{4.1}$$

and the domain

$$\Omega'_p = \{y \in R^N : \lambda_p y + x_p \in \Omega'\}. \tag{4.2}$$

Since $M_p \to +\infty$, we have $\lambda_p \to 0$ as $p \to 2^*$. It is easy to see that $w_p(y)$ satisfies

$$\begin{gathered}
\begin{aligned}
&-\Delta w_p + \frac{N(N-2)}{4}\Big(\frac{2\lambda_p}{1-|\lambda_p y + x_p|^2}\Big)^2 w_p \\
&= \Big(\ln\frac{1+|y\lambda_p + x_p|}{1-|y_p + x_p|}\Big)^\alpha \lambda_p^{\frac{(N-2)(2^*-p)}{2}} \Big(\frac{1-|\lambda_p y + x_p|^2}{2}\Big)^{\frac{(N-2)p-2N}{2}} w_p^{p-1}, \quad y \in \Omega'_p,
\end{aligned} \\
w_p = 0, \quad y \in \partial\Omega'_p, \\
0 < w_p \le 1, \quad w_p = 1.
\end{gathered} \tag{4.3}$$

By Proposition 4.1, we can have $M_p \ge 1$ for p close to 2^*. Therefore $0 \le \lambda_p \le 1$. Setting $L(p) = \lambda_p^{\frac{(N-2)(2^*-p)}{2}}$, $L(2^*) = \lim_{p\to 2^*} L(p)$, by choosing subsequence if necessary, we have one of the three cases:

(i) $L(2^*) = 0$;
(ii) $L(2^*) = \beta \in (0,1)$;
(iii) $L(2^*) = 1$.

For the location of $x_0 \in \bar{\Omega}'$, we also have one of the two cases: (1) $x_0 \in \Omega'$, and (2) $x_0 \in \partial\Omega'$.

(1) Assume $x_0 \in \Omega'$, let $2d$ denote the distance of x_0 to $\partial\Omega'$. For p close to 2^*, $w_p(y)$ is well defined in the ball $B(0, \frac{d}{\lambda_p})$ and

$$\begin{gathered}
\sup_{y\in B(0,\frac{d}{\lambda_p})} w_p(y) = w_p(0) = 1, \\
\Omega'_p \to \Omega'_{2^*} = R^N, as\ p \to 2^*, \\
B(0, \frac{d}{\lambda_p}) \to R^N, as\ p \to 2^*, \\
\frac{N(N-2)}{4}\Big(\frac{2\lambda_p}{1-|y\lambda_p + x_p|^2}\Big)^2 v_p \to 0, \quad \text{as } p \to 2^*, \\
\Big(\frac{1-|\lambda_p y + x_p|^2}{2}\Big)^{\frac{(N-2)p-2N}{2}} \to 1, \quad \text{as } p \to 2^*.
\end{gathered}$$

Therefore, given any radius l, we have $B(0, 2l) \subset B(0, \frac{d}{\lambda_p})$ for p close to 2^*. By the L^r-estimates in the theory of elliptic equation (see [4], for example), we can find uniform bounds for $\|w_p\|_{W^{2,r}(B(0,2l))}(r > n)$. Choosing p sufficiently close to 2^*, we obtain by Morrey's theorem that $\|w_p\|_{C^{1,\theta}}(B(0,l))(0 < \theta < 1)$ is also uniformly bounded. It follows that for any sequence $p \to 2^*$, there exists a subsequence $p_k \to 2^*$ such that $w_{p_k} \to w$ in $W^{2,r} \cap C^{1,\theta}(r > N)$ on $B(0,l)$. By Hölder continuity $v(0) = 1$. Furthermore, since for $y \in B(0,l)$,

$$\lambda_{p_k} y + x_{p_k} \to x_0 \quad \text{as } k \to +\infty,$$

as in [3] we can also prove that w is well defined in all R^N and $w_{p_k} \to w$ in $W^{2,r} \cap C^{1,\theta}(r > N)$ on any compact subset. Therefore $w(y)$ is a solution of

$$-\Delta w = \Big(\ln\frac{1+|x_0|}{1-|x_0|}\Big)^\alpha L(2^*) w^{2^*-1}. \tag{4.4}$$

If $L(2^*) = 0$ or $x_0 = 0$, then $-\Delta w = 0$ in R^N. Thus $w \equiv 0$, which is impossible since $w(0) = 1$.

If $L(2^*) \in (0,1]$, then by (4.4), Equation (4.3) is

$$\begin{gathered} -\Delta w = cw^{2^*-1}, \quad y \in R^N \\ w \to 0 \quad \text{as } |y| \to \infty \\ 0 < w \leq 1, \quad w(0) = 1 \end{gathered} \tag{4.5}$$

where $0 < c = (\ln \frac{1+|x_0|}{1-|x_0|})^\alpha L(2^*) < 1$, since $0 < |x_0| < \frac{e-1}{e+1}$. Let $z = c^{\frac{1}{2^*-2}} w$, then

$$\begin{gathered} -\Delta z = z^{2^*-1}, \quad y \in R^N \\ z \to 0 \quad \text{as } |y| \to \infty \\ 0 < z \leq c^{\frac{1}{2^*-2}}, \quad z(0) = c^{\frac{1}{2^*-2}}. \end{gathered} \tag{4.6}$$

Hence $z(y) = \varepsilon^{\frac{2-N}{2}} U(\frac{x}{\varepsilon})$, where ε is determined by c.

By Corollary 3.4 and Fatou's lemma, we have

$$\begin{aligned} S^{N/2} &= \int_{R^N} |\nabla z|^2 dx = c^{\frac{2}{2^*-2}} \int_{R^N} |\nabla w|^2 dx \\ &\leq c^{\frac{2}{2^*-2}} \lim_{p \to 2^*} \int_{\Omega'_p} |\nabla w_p|^2 dx \\ &= c^{\frac{2}{2^*-2}} \lim_{p \to 2^*} \int_{\Omega'} |\nabla v_p|^2 dx \\ &= c^{\frac{2}{2^*-2}} S^{N/2} < S^{N/2} \quad \text{as } p \to 2^*, \end{aligned} \tag{4.7}$$

which is impossible. Thus case (1) cannot occur and x_0 must be on $\partial\Omega'$. Now we straighten $\partial\Omega'$ in a neighorhood of x_0 by a non-singular C^1 change of coordinates as in [3]:

Let $x_n = \psi(x')$ $(x' = (x_1, \dots, x_{N-1}))$. $\psi \in C^1$ be the equation of $\partial\Omega'$. Define a new coordinate system

$$y_i = x_i \ (i = 1, \dots, N-1), \quad y_N = x_N - \psi(x') \tag{4.8}$$

Then v_p is again a solution of an equation of type (1), and $\partial\Omega'$ is contained in the hyperplane $x_N = 0$. Let d_p be the distance from x_p to $\partial\Omega'$ (i.e. $d_p = x_p \cdot e_N$). Note that for p close to 2^*, w_p is well-defined in $B(0, \frac{\delta}{\lambda_p}) \cap \{y_n > -\frac{d_p}{\lambda_p}\}$ for some small $\delta > 0$ and satisfies (4.3). Moreover, $\sup w_p(y) = w_p(0) = 1$.

We assert that

(I) $\frac{d_p}{\lambda_p} \to +\infty$ as $p \to 2^*$;
(II) $L(2^*) = 1$.

Proof of (I). Assume to the contrary that $\frac{d_p}{\lambda_p}$ is uniformly bounded from above, and (by going to a subsequence if necessary) $\frac{d_p}{\lambda_p} \to s$ with $s \geq 0$. Repeating the compactness argument as in the case (1), noting that $|x_0| = \frac{e-1}{e+1}$, we get a subsequence of w_p converging to $w(y)$ satisfying

$$\begin{gathered} -\Delta w = L(2^*) w^{2^*-1}, \quad y \in R^N_s = \{y = (y_1, \dots, y_{n-1}, y_N) : y_N \geq -s\} \\ w = 0, \quad y \in \partial R^N_s, \\ 0 < w \leq 1, \quad w(0) = 1, \quad y \in R^N_s. \end{gathered} \tag{4.9}$$

By a translation, noting the fact that equation

$$\begin{gathered}-\Delta w=cw^{2^*-1},\quad y\in R_+^N=\{y=(y_1,\dots,y_{N-1},y_N)|y_N>0)\},\\ w(y)=0,\quad y\in\partial R_+^N\end{gathered}\tag{4.10}$$

has a unique solution $w=0$, we conclude that (4.9) possesses a unique trivial solution 0 for any case of $L(2^*)$, which contradicts $w(0)=1$. So we can have only $\frac{d_p}{\lambda_p}\to+\infty$ as $p\to 2^*$.

Proof of (II). Assertion(I) implies $\Omega_p'\to\Omega_{2^*}'=R^N$. Similarly by the above regularity theorems in the theory of elliptic equation and $|x_0|=\frac{e-1}{e+1}$, we obtain a subsequence of w_p converging to some function $w(y)$ satisfying

$$\begin{gathered}-\Delta w=L(2^*)w^{2^*-1},\quad y\in R^N,\\ w(y)\to 0,\quad |y|\to\infty,\\ 0<w\le 1,\quad w(0)=1.\end{gathered}\tag{4.11}$$

If $L(2^*)=0$ or $L(2^*)=\beta$, $0<\beta<1$, just as done in case (1) we get the contradiction $w\equiv 0$ or (4.7) respectively. So $L(2^*)=1$, which implies that w solves the equation

$$\begin{gathered}-\Delta w=w^{2^*-1},\quad y\in R^N\\ w(y)\to 0,\quad |y|\to\infty\\ 0<w\le 1,\quad w(0)=1.\end{gathered}\tag{4.12}$$

Hence $w=\varepsilon^{\frac{2-N}{2}}U(\frac{y-y_0}{\varepsilon})$ for some $\varepsilon>0$, $y_0\in R^N$. Since v attains its maximum 1 at $y=0$, we have $\varepsilon=1$ and $y_0=0$. Therefore $w=U$. Note that the limit of $\{w_p\}$ does not depend on the choice of subsequence by the uniqueness of U. Hence the whole sequence $\{w_p\}$ must converge to U.

Let $z_p=w_p-U$. Then $z_p\rightharpoonup 0$ weakly in $H^1(\Sigma)$ for any bounded subset $\Sigma\subset R^N$, and

$$\begin{gathered}-\Delta z_p+\frac{N(N-2)}{4}(\frac{2\lambda_p}{1-|\lambda_p y+x_p|^2})^2w_p=Q_p(y)w_p^{p-1}-U^{2^*-1},\quad y\in\Omega_p'\\ z_p=-U,\quad y\in\partial\Omega_p'\end{gathered}\tag{4.13}$$

where

$$Q_p(y)=(\ln\frac{1+|\lambda_p y+x_p|}{1-|\lambda_p y+x_p|})^\alpha(\frac{1-|\lambda_p y+x_p|^2}{2})^{\frac{(N-2)p-2N}{2}}\lambda_p^{\frac{(N-2)(2^*-p)}{2}}.$$

Multiplying (4.13) by z_p and integrating by parts, we obtain, as $p\to 2^*$,

$$\begin{aligned}\int_{\Omega_p'}|\nabla z_p|^2dx&=\int_{\Omega_p'}[Q_p(y)w_p^{p-1}-U^{2^*-1}]z_p\\ &\quad-\int_{\Omega_p'}\frac{N(N-2)}{4}(\frac{2}{1-|\lambda_p y+x_p|^2})^2w_pz_p+\int_{\partial\Omega_p'}\frac{\partial z_p}{\partial\nu}Uds\\ &=\int_{\Omega_p'}Q_p(y)|z_p|^p+o_{(2^*-p)}(1).\end{aligned}\tag{4.14}$$

The last equality follows from the weak convergence of w_p in $H^1(\Sigma)$ and the decay of U at infinity.

As $p \to 2^*$,

$$\int_{\Omega_p'} |\nabla z_p|^2 \geq S\Big(\int_{\Omega_p'} Q_p(y)|z_p|^p\Big)^{2/p} + o_{2^*-p}(1) \tag{4.15}$$

If $\int_{\Omega_p'} |\nabla z_p|^2 dx \to \rho > 0$, by (4.15), we see easily that

$$\int_{\Omega_p'} |\nabla z_p|^2 = \int_{\Omega_p'} Q_p(y)|z_p|^p dx + o_{2^*-p}(1) \geq S^{N/2} + o_{2^*-p}(1) \quad \text{as } p \to 2^*.$$

Then by (2.3) and Corollary 3.4, we have

$$J(v_p) = \frac{1}{N} S^{N/2} + o_{(2^*-p)}(1) \ as\ p \to 2^*. \tag{4.16}$$

On the other hand, as we done in obtaining (4.14),

$$\begin{aligned}
J(v_p) &= \frac{1}{2}\int_{\Omega_p'} |\nabla U|^2 - \frac{1}{p}\int_{\Omega_p'} (\frac{2}{1-|\lambda_p y + x_p|^2})^{\frac{(N-2)p-2N}{2}} U^p \\
&\quad + \frac{1}{2}\int_{\Omega_p'} |\nabla w_p|^2 - \frac{1}{p}\int_{\Omega_p'} Q_p(y)(\frac{2}{1-|\lambda_p y + x_p|^2})^{\frac{(N-2)p-2N}{2}} w_p^p \\
&\quad + \frac{N(N-2)}{4}\int_{\Omega_p'} (\frac{2\lambda_p}{1-|\lambda_p y + x_p|^2})^2 w_p^2 \\
&\quad + \frac{N(N-2)}{4}\int_{\Omega_p'} (\frac{2\lambda_p}{1-|\lambda_p y + x_p|^2})^2 U^2 + o_{(2^*-p)}(1) \\
&= \frac{1}{2}\int_{R^N} |\nabla U|^2 - \frac{1}{2^*}\int_{\Omega_p'} U^{2^*} + \frac{1}{2}\int_{\Omega_p'} |\nabla w_p|^2 - \frac{1}{p}\int_{\Omega_p'} Q_p(y) w_p^p + o_{2^*-p}(1) \\
&\geq \frac{2}{N} S^{N/2} + o_{(2^*-p)}(1)
\end{aligned}$$

which contradicts (4.16). Thus $\rho = 0$, and we obtain

$$\lim_{p \to 2^*} \int_{\Omega'} |\nabla(v_p - U_{\lambda_p, x_p})|^2 = 0. \tag{4.17}$$

Since

$$\frac{2e}{(e+1)^2} \leq \frac{1-|x|^2}{2} \leq \frac{1}{2}, \quad u_p = (\frac{1-|x|^2}{2})^{\frac{N-2}{2}} v_p,$$

part (ii) of Theorem 1.3 is proved.

To complete our proof of Theorem 1.3, we need only to show that x_p is unique for p close to 2^*. Suppose that this is not true, then exist x_p^i, $i = 1, 2$, such that $M_p = v_p(x_p^i)$ for $i = 1, 2$. For x_p^i by choosing subsequence as $p \to 2^*$, we have either

$$\frac{|x_p^1 - x_p^2|}{\lambda_p} \to +\infty \tag{4.18}$$

or

$$\frac{|x_p^1 - x_p^2|}{\lambda_p} \leq c < +\infty \tag{4.19}$$

where c is some positive constant independent of p.

Suppose that (4.19) holds, then the scaled function w_p would have two local maximum points in $B(0, l)$ for l large enough and p close to 2^*. On the other hand, by [8, Lemma 4.2] and by using the similar arguments to [8], we can also verify that w_p has only one local maximum point. So we get a contradiction.

Assume that (4.18) holds, then from (4.17) we obtain

$$\lim_{p\to 2^*}\int_{\Omega'}|\nabla(U_{\lambda_p,x_p^1}-U_{\lambda_p,x_p^2})|^2=0. \tag{4.20}$$

Setting $(\Omega')_p^1=\{y|\lambda_p y+x_p^1\in\Omega\}$ and $m_p=\frac{x_p^1-x_p^2}{\lambda_p}$, we have

$$0=2S^{N/2}-2\lim_{p\to 2^*}\int_{(\Omega')_p^1}\nabla U\nabla U_{1,z_p}. \tag{4.21}$$

Since $|m_p|\to+\infty$, we obtain $\lim_{p\to 2^*}\int_{(\Omega')_p^1}\nabla U\nabla U_{1,z_p}=0$, this contradicts (4.20) and hence (4.18) does not hold, either.

Since

$$u_p=(\frac{1-|x|^2}{2})^{\frac{N-2}{2}}v_p,\quad \frac{2e}{(e+1)^2}\le\frac{1-|x|^2}{2}\le\frac{1}{2},$$
$$M_p'=u_p(x_p)=\max_{x\in\bar\Omega}u_p(x)\,,$$

it follows that $M_p'\to+\infty$ as $p\to 2^*$. Thus part (i) of Theorem 1.3 is proved. □

From Theorem 1.3, we can obtain easily the following result.

Corollary 4.2. *For p close to* 2^**, the ground state solution of* (1.1) *is not radially symmetric.*

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