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PRÜFER SUBSTITUTIONS ON A COUPLED SYSTEM INVOLVING THE p-LAPLACIAN

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Abstract. In this article, we employ a modified Prüfer substitution acting on a coupled system involving one-dimensional p-Laplacian equations. The basic properties for the initial valued problem and some estimates are obtained. We also derive an analogous Sturmian theory and give a reconstruction formula for the potential function.

1. Introduction

There has been recently a lot of interest in the study of the p-Laplacian eigenvalue problem

$$-\Delta_p y + q|y|^{p-2}y = \lambda|y|^{p-2}y,$$
$$y|_{\partial\Omega} = 0,$$

where $p > 1$ and $q \in C(\Omega)$, $\Omega \subseteq \mathbb{R}^n$. This is a quasilinear partial differential equation when $p \neq 2$. The most cited application is the highly viscid fluid flow (cf. Ladyzhenskaya [4] and Lions [6]). When $p = 2$, q and λ both vanish, it becomes the Laplacian equation. The p-Laplacian operator has the originally physical meaning, and can also be treated as a generalization of the Laplacian operator. For the one-dimensional case, the p-Laplacian eigenvalue problem becomes, after scaling,

$$-(y'^{(p-1)})' = (p-1)(\lambda - q(x))y^{(p-1)}, \tag{1.1}$$
$$y(0) = y(1) = 0, \tag{1.2}$$

where $p > 1$, $f^{(p-1)} \equiv |f|^{p-1}\operatorname{sgn} f = |f|^{p-2}f$, and q is a continuous function defined on $[0, 1]$. The following Sturm-Liouville property for the one-dimensional p-Laplacian operator is well-known now (cf. Binding & Drabek [2], Reichel & Walter [8], Walter [10], etc.).

Theorem 1.1. *For* (1.1)-(1.2)*, there exists a sequence of eigenvalues* $\{\lambda_n\}_{n=1}^{\infty}$ *such that*

$$-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots \to \infty\,,$$

and the eigenfunction corresponding to λ_n *has exactly* $n-1$ *zeros in* $(0, 1)$.

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In this article, we consider the coupled one-dimensional p-Laplacian problem

$$\begin{aligned} (u'(x)^{(p-1)})' + (p-1)\lambda u(x)^{(p-1)} - (p-1)q(x)v(x)^{(p-1)} &= 0, \\ (v'(x)^{(p-1)})' + (p-1)\lambda v(x)^{(p-1)} + (p-1)q(x)u(x)^{(p-1)} &= 0, \end{aligned} \tag{1.3}$$

with the initial conditions

$$u(0) = v(0) = 0, \ u'(0) = v'(0) = \lambda^{1/p}, \tag{1.4}$$

where λ is some positive parameter, $p > 1$, and q is a continuous function defined in $\mathbb{R}$. When $p = 2$, (1.3) reduces to

$$\begin{aligned} u''(x) + \lambda u(x) - q(x)v(x) &= 0, \\ v''(x) + \lambda v(x) + q(x)u(x) &= 0, \end{aligned} \tag{1.5}$$

which is a linear coupled system. One can treat (1.5) as a steady state reaction diffusion model. Define $H(u, v) = \frac{\lambda}{2}u^2 - \frac{\lambda}{2}v^2 - q(x)uv$. Then

$$\frac{\partial H}{\partial u} = \lambda u - q(x)v, \quad -\frac{\partial H}{\partial v} = \lambda v + q(x)u.$$

Equation (1.5) can be viewed as a simplest model of diffusion systems with skew-gradient structure (cf. [11, 12]).

Here we intend to study the existence of sign-changing solutions (or nodal solutions) of (1.3)-(1.4) and try to derive an analog of Theorem 1.1. Employing the information of solutions, a reconstruction formula for $q(x)$ is given. Such a procedure is called an inverse nodal problem. An inverse problem of this type was designated by McLaughlin [7] in 1988. When one studies the inverse nodal problem of (1.3)-(1.4), an interesting observation arises. The asymptotic formula given in Theorem 1.3 (see the following) coincides with the one of the classical Sturm-Liouville eigenvalue problem

$$\begin{gathered} -y'' + w_0(x)y = \mu y, \\ y(0) = y(1) = 0 \end{gathered}$$

(cf. [7, 9, 5]). Besides, the Prüfer substitution is an efficient method in showing the oscillation property for solutions (cf. [1]). In this article we utilize a modified Prüfer substitution to treat this problem. Fortunately we can tackle the effect of the two coupled functions in (1.3)-(1.4), and obtain the detailed estimates of parameters λ_m and nodal points. The following are our main results.

Theorem 1.2. *There exists a sequence of real parameters $\{\lambda_k\}_{k=m}^{\infty}$ of the one-dimensional coupled system (1.3)-(1.4), where $m \in \mathbb{N}$ such that*

$$0 < \lambda_m < \lambda_{m+1} < \lambda_{m+2} < \lambda_{m+3} < \cdots \to \infty,$$

and the corresponding solution $u(x, \lambda_k)$ has exactly $k-1$ zeros in $(0,1)$ for $k \geq m$. In particular, the solution pair $\{u(x, \lambda_k), v(x, \lambda_k)\}$ satisfies the following boundary condition

$$u(0, \lambda_k) = v(0, \lambda_k) = 0, \ u(1, \lambda_k) = 0$$

for every $k \geq m$.

Define the zero set (or nodal set) $\{x_i^{(k)}\}_{i=1}^{k-1}$ of the solution $u(x, \lambda_k)$ to (1.3)-(1.4) and the index $i_k(x) = \max\{i : x_i^{(k)} \leq x\}$. Let $\ell_i^{(k)} = x_{i+1}^{(k)} - x_i^{(k)}$ for $0 \leq i \leq k-1$, where $x_0^{(k)} = 0$ and $x_k^{(k)} = 1$. We obtain an asymptotic formula for the function $q(x)$.

Theorem 1.3. *Suppose that the above assumptions hold. Then an asymptotic formula for $q(x)$ in (1.3) is*

$$q(x)-\int_0^1 q(t)dt=\lim_{m\to\infty}[p(m\pi_p)^p(m\ell_{i_m(x)}^{(m)}-1)], \tag{1.6}$$

for all $x\in[0,1]$.

We remark that in Theorem 1.2, the right endpoint conditions $v(1,\lambda_k)$ also vanish when λ_k tends to the infinity. Simultaneously, one can show an analogous Sturmian theory for $v(x,\lambda)$. Then the data coming from $v(x,\lambda_k)$ also make the asymptotic formula (1.6) valid.

This article is organized as follows. After the introduction, we employ a modified Prüfer substitution to show the local solution of the initial value problem (1.3)-(1.4) is unique and can be extended to the whole interval $[0,1]$. In section 3, we derive several lemmas to complete the proof of Theorem 1.2. In section 4, some detailed estimates and the proof of Theorem 1.3 are given.

2. Preliminaries - A modified Prüfer substitution

To discuss the existence and uniqueness of the local solution of (1.3)-(1.4). We need the following lemma.

Lemma 2.1 ([10, p. 180]). *Let $W\in C^1(I)$, $x_0\in I$ and $W(x_0)=0$, where I is a compact interval containing x_0. Denote by $\|W\|_x$ the maximum of W in the interval from x_0 to x. Then $|W'(x)|\le K\|W\|_x$ in I implies*

$$W=0 \quad \text{for } |x-x_0|\le\frac{1}{K},\quad x\in I. \tag{2.1}$$

Proposition 2.2. *For any fixed $\lambda\in\mathbb{R}^+$, the problem (1.3)-(1.4) has a unique local solution which exists on an open interval I containing zero.*

Proof. System (1.3) can be written as

$$\begin{aligned} u'&=U^{(p^*-1)},\\ U'&=(p-1)[qv^{(p-1)}-\lambda u^{(p-1)}],\\ v'&=V^{(p^*-1)},\\ V'&=-(p-1)[qu^{(p-1)}-\lambda v^{(p-1)}], \end{aligned} \tag{2.2}$$

with $u(0)=v(0)=0$ and $U(0)=V(0)=\lambda^{1/p^*}$, where $p^*=p/(p-1)$ is the conjugate exponent of p. Then the local existence of a solution is valid by the Cauchy-Peano theorem. Now it suffices to prove the uniqueness. By (1.4), we may assume that

$$\frac{\lambda^{1/p}}{2}|x-0|<|u(x)|,\ |v(x)|<2\lambda^{1/p}|x-0| \quad \text{for } x\in I. \tag{2.3}$$

Suppose that $\{u_1(x),v_1(x)\}$ and $\{u_2(x),v_2(x)\}$ are two distinct local solutions of (1.3)-(1.4). Without loss of generality, we assume that $u_1(x)\ge u_2(x)$ and $v_1(x)\ge v_2(x)$ in some small interval I which contains zero. By (2.2), for $x\in I$ we have

$$\begin{aligned} &u_1'(x)^{(p-1)}-u_2'(x)^{(p-1)}\\ &=(p-1)\Big\{\int_0^x q(t)[v_1(t)^{(p-1)}-v_2(t)^{(p-1)}]dt-\lambda\int_0^x[u_1(t)^{(p-1)}-u_2(t)^{(p-1)}]dt\Big\}, \end{aligned}$$

$$
\begin{aligned}
&v_1'(x)^{(p-1)} - v_2'(x)^{(p-1)} \\
&= (1-p)\Big\{ \int_0^x q(t)[u_1(t)^{(p-1)} - u_2(t)^{(p-1)}]dt - \lambda \int_0^x [v_1(t)^{(p-1)} - v_2(t)^{(p-1)}]dt \Big\};
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&|u_1'(x)^{(p-1)} - u_2'(x)^{(p-1)} + v_1'(x)^{(p-1)} - v_2'(x)^{(p-1)}| \\
&= (p-1)|\int_0^x (q(t)+\lambda)[v_1(t)^{(p-1)} - v_2(t)^{(p-1)} - u_1(t)^{(p-1)} + u_2(t)^{(p-1)}]dt|.
\end{aligned} \tag{2.4}
$$

It follows from the mean value theorem, that for a_1 and a_2 of the same sign,

$$
a_1^{(p-1)} - a_2^{(p-1)} = (p-1)(a_1 - a_2)|\bar{a}|^{p-2}\,, \tag{2.5}
$$

where $\bar{a}$ lies between a_1, a_2. Note that there exists some c_1 such that the left hand side of (2.4) is greater than or equal to $c_1|u_1'(x) + v_1'(x) - u_2'(x) - v_2'(x)|$. On the other hand, by (2.3) the right hand side of (2.4) is less than or equal to $(p+1)(\|q\|_x + \lambda)\int_0^x |u_1(t) + v_1(t) - u_2(t) - v_2(t)| \cdot 2\lambda^{1/p} t^{p-2} dt$, where the notation $\|\cdot\|_x$ is defined as in Lemma 2.1. Now set $W(x) = u_1(x) + v_1(x) - u_2(x) - v_2(x)$. By Lemma 2.1, we can obtain that $W(x) = 0$ in I. This proves the uniqueness of the local solution. $\square$

Now we introduce a modified Prüfer substitution for the local solution $\{u(x), v(x)\}$ using the generalized sine function $S_p(x)$. The $S_p(x)$ function is well known now (cf. [2, 3, 8]), and satisfies

$$
|S_p(x)|^p + |S_p'(x)|^p = 1, \tag{2.6}
$$

and

$$
(S_p)'' = \frac{-S_p^{(p-1)} S_p'}{(S_p')^{(p-1)}} = \frac{-S_p^{(p-1)}}{|S_p'|^{p-2}}. \tag{2.7}
$$

Thus one has $S_p(\pi_p/2) = 1$, and by (2.6), $S_p'(0) = 1$, $S_p'(\pi_p/2) = 0$. Define

$$
u(x,\lambda) = R(x,\lambda) S_p(\lambda^{1/p}\theta(x,\lambda)), \quad u'(x,\lambda) = \lambda^{1/p} R(x,\lambda) S_p'(\lambda^{1/p}\theta(x,\lambda)), \tag{2.8}
$$

$$
v(x,\lambda) = r(x,\lambda) S_p(\lambda^{1/p}\phi(x,\lambda)), \quad v'(x,\lambda) = \lambda^{1/p} r(x,\lambda) S_p'(\lambda^{1/p}\phi(x,\lambda)). \tag{2.9}
$$

Then, we obtain

$$
\lambda R(x,\lambda)^p = \lambda|u(x,\lambda)|^p + |u'(x,\lambda)|^p, \quad \lambda r(x,\lambda)^p = \lambda|v(x,\lambda)|^p + |v'(x,\lambda)|^p, \tag{2.10}
$$

where $R(x,\lambda)$ and $r(x,\lambda)$ are the Prüfer amplitude functions; and $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are the Prüfer phase angles of $\{u(x), v(x)\}$, respectively. By a direct computation, we have the following lemma.

Lemma 2.3. *For the modified Prüfer substitution* (2.8)-(2.9), *one has*

$$
\theta'(x,\lambda) = 1 - \frac{q(x)}{\lambda}\Big(\frac{r(x,\lambda)}{R(x,\lambda)}\Big)^{p-1} S_p(\lambda^{1/p}\theta(x,\lambda)) S_p(\lambda^{1/p}\phi(x,\lambda))^{(p-1)}, \tag{2.11}
$$

$$
R'(x,\lambda) = \frac{q(x)}{\lambda^{\frac{p-1}{p}}} \frac{r(x,\lambda)^{p-1}}{R(x,\lambda)^{p-2}} S_p(\lambda^{1/p}\phi(x,\lambda))^{(p-1)} S_p'(\lambda^{1/p}\theta(x,\lambda)), \tag{2.12}
$$

$$
\phi'(x,\lambda) = 1 + \frac{q(x)}{\lambda}\Big(\frac{R(x,\lambda)}{r(x,\lambda)}\Big)^{p-1} S_p(\lambda^{1/p}\phi(x,\lambda)) S_p(\lambda^{1/p}\theta(x,\lambda))^{(p-1)}, \tag{2.13}
$$

$$
r'(x,\lambda) = \frac{-q(x)}{\lambda^{\frac{p-1}{p}}} \frac{R(x,\lambda)^{p-1}}{r(x,\lambda)^{p-2}} S_p(\lambda^{1/p}\theta(x,\lambda))^{(p-1)} S_p'(\lambda^{1/p}\phi(x,\lambda)), \tag{2.14}
$$

where $' = \frac{d}{dx}$.

Proof. Here we prove the first two equations, and the rest is similar. For the sake of simplicity, we drop the function variable λ in the proof. By (2.8),

$$\frac{u'(x)^{(p-1)}}{u(x)^{(p-1)}} = \frac{\lambda^{\frac{p-1}{p}} S_p'(\lambda^{1/p}\theta(x))^{(p-1)}}{S_p(\lambda^{1/p}\theta(x))^{(p-1)}}.$$

Differentiating the above equation on both sides and applying (1.3) and (2.7), we obtain

$$[\lambda + |\frac{u'(x)}{u(x)}|^p - q(x)(\frac{v(x)}{u(x)})^{(p-1)}] = \lambda\theta'(x)[1 + |\frac{S_p'(\lambda^{1/p}\theta(x))}{S_p(\lambda^{1/p}\theta(x))}|^p].$$

Multiplying by $|S_p(\lambda^{1/p}\theta(x))|^p$, from (2.6), it follows that (2.11) holds.

Next, differentiate $u(x) = R(x)S_p(\lambda^{1/p}\theta(x))$ with respect to x and employ (2.11), to obtain (2.12). □

Applying Lemma 2.3, we find that $\{u(x), v(x); \lambda\}$ is a solution of (1.3)-(1.4) if and only if $\{\theta(x), R(x), \phi(x), r(x); \lambda\}$ is a solution of (2.11)-(2.14) coupled with the following conditions

$$\theta(0,\lambda) = \phi(0,\lambda) = 0, \ \textit{and}\ R(0,\lambda) = r(0,\lambda) = 1. \tag{2.15}$$

Next we derive some properties for the phase and amplitude functions.

Lemma 2.4. *(i) For $x > 0$, the amplitude functions satisfy that*

$$2\exp[-c_1\lambda^{\frac{1-p}{p}}x] \le R(x,\lambda)^{p-1} + r(x,\lambda)^{p-1} \le 2\exp[c_2\lambda^{\frac{1-p}{p}}x], \tag{2.16}$$

where c_1, c_2 are some positive constants.

(ii) For fixed $x > 0$ and sufficiently large λ, we have

$$\frac{r(x,\lambda)}{R(x,\lambda)} = 1 + o(1). \tag{2.17}$$

Moreover, $\frac{R(x,\lambda)}{r(x,\lambda)}$ has the same asymptotic estimate as in (2.17).

Proof. (i) By assumption and (2.12) and (2.14), there exist some positive constants c_1 and c_2 such that

$$\begin{aligned} &-c_1\lambda^{\frac{1-p}{p}}[R(x)^{p-1} + r(x)^{p-1}] \\ &\le R(x)^{p-2}R'(x) + r(x)^{p-2}r'(x) \le c_2\lambda^{\frac{1-p}{p}}[R(x)^{p-1} + r(x)^{p-1}]. \end{aligned}$$

Solving the above differential inequality and applying the initial condition (2.15), we obtain the inequality (2.16).

(ii) As in (i), there exists some positive constant c_3 such that

$$\frac{R(x)r'(x) - r(x)R'(x)}{R(x)^2} \le c_3\lambda^{\frac{1-p}{p}}[\frac{R(x)^{p-2}}{r(x)^{p-2}} + \frac{r(x)^p}{R(x)^p}].$$

Letting $y(x) = \frac{r(x)}{R(x)}$, we have

$$y'(x) \le c_3\lambda^{\frac{1-p}{p}}[y(x)^{2-p} + y(x)^p].$$

Note that

$$\frac{dy}{dx} \le c_3\lambda^{\frac{1-p}{p}}(\frac{1+y^{2p-2}}{y^{p-2}}); \quad \text{i.e.,}\ \frac{y^{p-2}dy}{1+y^{2p-2}} \le c_3\lambda^{\frac{1-p}{p}}dx.$$

Let $z = y^{p-1}$ and integrate the above inequality; we obtain

$$\tan^{-1}(y(x)^{p-1}) - \tan^{-1}(y(0)^{p-1}) \le (p-1)c_3\lambda^{\frac{1-p}{p}}x;$$

i.e.,

$$0 < \tan^{-1}(y(x)^{p-1}) \le \tan^{-1}(1) + (p-1)c_3\lambda^{\frac{1-p}{p}}x.$$

So

$$y(x)^{p-1} \le 1 + o(1) \tag{2.18}$$

as λ is sufficiently large. This completes the proof. □

From Proposition 2.2 and (2.16), we have the following property.

Proposition 2.5. *For any fixed $\lambda \in \mathbb{R}^+$, problem* (1.3)-(1.4) *has a unique solution which exists over the whole interval* $[0, 1]$.

3. The Sturmian property

In this section, we first derive the following lemma for the proof of Theorem 1.2.

Lemma 3.1. *For $\lambda > 0$, the phase angle function $\theta(x, \lambda)$ satisfies the following properties.*

- (i) *$\theta(\cdot, \lambda)$ is continuous in λ and satisfies $\theta(0, \lambda) = 0$.*
- (ii) *If $\lambda^{1/p}\theta(x_n, \lambda) = n\pi_p$ for some $x_n \in (0, 1)$, then $\lambda^{1/p}\theta(x, \lambda) > n\pi_p$ for every $x > x_n$.*
- (iii)
$$\lim_{\lambda\to\infty} \lambda^{1/p}\theta(1, \lambda) = \infty. \tag{3.1}$$

Proof. For (i), $\theta(\cdot, \lambda)$ is continuous in λ followed by the continuous dependence on parameters. And $\theta(0, \lambda) = 0$ is valid by (2.15). Also if $\lambda^{1/p}\theta(x_n, \lambda) = n\pi_p$ for some $x_n \in (0, 1)$, then by (2.11) and Lemma 2.4, we have

$$\theta'(x_n, \lambda) = 1 > 0. \tag{3.2}$$

This proves (ii). For (iii), integrating (2.11) over $[0, 1]$ and applying (i), one obtains

$$\lambda^{1/p}\theta(1, \lambda) = \lambda^{1/p} - \lambda^{\frac{1-p}{p}} \int_0^1 q(t)(\frac{r(t, \lambda)}{R(t, \lambda)})^{p-1} S_p(\lambda^{1/p}\theta(t, \lambda))S_p(\lambda^{1/p}\phi(t, \lambda))^{(p-1)}dt. \tag{3.3}$$

By (2.17), one has

$$\lambda^{1/p}\theta(1, \lambda) = \lambda^{1/p} + O(\frac{1}{\lambda^{1-\frac{1}{p}}})$$

for sufficiently large λ. This completes the proof. □

We remark that using (2.13) and Lemma 2.4, one can apply the similar arguments as in the above proof to obtain the conclusions in Lemma 3.1 for the phase function $\phi(x, \lambda)$.

Proof of Theorem 1.2. By Lemma 3.1, for every sufficiently large $k \in \mathbb{N}$, there exists $\lambda_k > 0$ satisfies $\lambda_k^{1/p}\theta(1, \lambda_k) = k\pi_p$. This implies that there exists $m \in \mathbb{N}$ such that $\lambda_k^{1/p}\theta(1, \lambda_k) = k\pi_p$ for every $k \ge m$. In this case, $\lambda_m < \lambda_{m+1} < \dots < \lambda_{k+1} < \dots \to \infty$, and $\{\theta(x, \lambda_k), \phi(x, \lambda_k)\}_{k\ge m}$ satisfy (2.11)-(2.15). Hence, $\{u(x, \lambda_k), v(x, \lambda_k)\}_{k\ge m}$ are solutions of (1.3)-(1.4) and satisfy

$$u(0, \lambda_k) = v(0, \lambda_k) = 0,\ u(1, \lambda_k) = 0 \quad \text{for every } k \ge m.$$

This completes the proof. □

4. Some detailed estimates - Proof of Theorem 1.3

Theorem 4.1. *The parameter λ_m of* (1.3)-(1.4) *satisfies*

$$\lambda_m^{1/p} = m\pi_p + \frac{1}{p(m\pi_p)^{p-1}} \int_0^1 q(t)dt + o(\frac{1}{m^{p-1}}) \tag{4.1}$$

as $m \to \infty$.

Proof. First, integrating (2.11) over $[0, x]$, with the associated λ_m,

$$\begin{aligned} &\theta(x, \lambda_m) - \theta(0, \lambda_m) \\ &= x - \frac{1}{\lambda_m} \int_0^x q(t)(\frac{r(t, \lambda)}{R(t, \lambda)})^{p-1} S_p(\lambda^{1/p}\theta(t, \lambda)) S_p(\lambda^{1/p}\phi(t, \lambda))^{(p-1)} dt. \end{aligned} \tag{4.2}$$

Letting $x = 1$, by Theorem 1.2, Lemma 2.4 and the initial condition (1.4), one obtains

$$\frac{m\pi_p}{\lambda_m^{1/p}} = 1 + O(\frac{1}{\lambda_m^{\frac{p-1}{p}}}); \tag{4.3}$$

i.e.,

$$\lambda_m^{1/p} = m\pi_p + O(\frac{1}{m^{p-1}}). \tag{4.4}$$

Again, integrating (2.11) over $[0, x]$, with the associated λ_m, and applying (4.4), one gets

$$\lambda_m^{1/p}\theta(x, \lambda_m) = m\pi_p x + O(\frac{1}{m^{p-1}}). \tag{4.5}$$

Similarly, from (2.13), we obtain

$$\lambda_m^{1/p}\phi(x, \lambda_m) = m\pi_p x + O(\frac{1}{m^{p-1}}). \tag{4.6}$$

Hence, by (4.5),

$$S_p(\lambda_m^{1/p}\theta(x, \lambda_m)) = S_p(m\pi_p x) + S_p'(m\pi_p x)O(\frac{1}{m^{p-1}}) + o(\frac{1}{m^{p-1}}); \tag{4.7}$$

i.e.,

$$S_p(\lambda_m^{1/p}\theta(x, \lambda_m)) = S_p(m\pi_p x) + o(1). \tag{4.8}$$

And the same asymptotic formula is true for $S_p(\lambda_m^{1/p}\phi(x, \lambda_m))$. Now substituting (4.8) into (4.2) and taking $x = 1$, one obtains

$$\begin{aligned} \frac{m\pi_p}{\lambda_m^{1/p}} &= 1 - \frac{1}{\lambda_m} \int_0^1 q(t)|S_p(m\pi_p t)|^p dt + o(\frac{1}{\lambda_m}) \\ &= 1 - \frac{1}{p\lambda_m} \int_0^1 q(t)dt - \frac{1}{\lambda_m} \int_0^1 q(t)[|S_p(m\pi_p t)|^p - \frac{1}{p}]dt + o(\frac{1}{\lambda_m}), \end{aligned} \tag{4.9}$$

for sufficiently large m. By a generalized Riemann-Lebesgue lemma, the asymptotic estimate (4.1) is valid. □

Hence, the asymptotic formula for λ_m is

$$\lambda_m = (m\pi_p)^p + \int_0^1 q(t)dt + o(1). \tag{4.10}$$

Next we derive the asymptotic formula of the nodal length.

Lemma 4.2. *For $m \to \infty$, the nodal length of the solution $u(x, \lambda_m)$ satisfies*

$$\ell_i^{(m)} = \frac{1}{m} - \frac{1}{pm^{p+1}\pi_p^p}\int_0^1 q(t)dt + \frac{1}{(m\pi_p)^p}\int_{x_i^{(m)}}^{x_{i+1}^{(m)}} q(t)|S_p(m\pi_p t)|^p dt + o(\frac{1}{m^{p+1}}). \tag{4.11}$$

Proof. Letting $\lambda = \lambda_m$ and integrating (2.11) from $x_i^{(m)}$ to $x_{i+1}^{(m)}$, we obtain

$$\frac{\pi_p}{\lambda_m^{1/p}} = \ell_i^{(m)} - \int_{x_i^{(m)}}^{x_{i+1}^{(m)}} \frac{q(t)}{\lambda_m}(\frac{r(t,\lambda_m)}{R(t,\lambda_m)})^{p-1} S_p(\lambda_m^{1/p}\theta(t,\lambda_m))S_p(\lambda_m^{1/p}\phi(t,\lambda_m))^{(p-1)}dt. \tag{4.12}$$

By the parameter estimates (4.1), we have

$$\frac{1}{\lambda_m^{1/p}} = \frac{1}{m\pi_p} - \frac{1}{p(m\pi_p)^{p+1}}\int_0^1 q(t)dt + o(\frac{1}{m^{p+1}}). \tag{4.13}$$

Substituting (2.17), (4.5)-(4.6), (4.10) and (4.13) into (4.12), one can obtain (4.11). □

As in the proof of Lemma 4.2, one can obtain the asymptotic estimate, for the nodal points $x_i^{(m)}$,

$$x_i^{(m)} = \frac{i}{m} - \frac{i}{pm^{p+1}\pi_p^p}\int_0^1 q(t)dt + \frac{1}{(m\pi_p)^p}\int_0^{x_i^{(m)}} q(t)|S_p(m\pi_p t)|^p dt + o(\frac{1}{m^p}), \tag{4.14}$$

which show the existence of a dense subset of nodal points in $[0, 1]$.

Proof of Theorem 1.3. For any $x \in (0, 1)$, write $i_m(x) = i_m$ for the sake of simplicity. Recall an easy identity,

$$(S_p(t)S_p'(t)^{(p-1)})' = 1 - p|S_p(t)|^p.$$

It follows from the mean value theorem for integrals, (4.14), the above identity and a change of variables. Then,

$$\begin{aligned}
\int_{x_{i_m}^{(m)}}^{x_{i_m+1}^{(m)}} q(t)|S_p(m\pi_p t)|^p dt &= \frac{q(x)}{m\pi_p}\int_{m\pi_p x_{i_m}^{(m)}}^{m\pi_p x_{i_m+1}^{(m)}} |S_p(\sigma)|^p d\sigma \\
&= \frac{q(x)}{m\pi_p}\int_0^{\pi_p} |S_p(\sigma)|^p d\sigma (1 + o(1)) \\
&= \frac{q(x)}{m\pi_p}\int_0^{\pi_p} [\frac{1}{p} - \frac{1}{p}(S_p(\sigma)S_p'(\sigma)^{(p-1)})']d\sigma(1 + o(1)) \\
&= \frac{q(x)}{pm}(1 + o(1)).
\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.11), one obtains

$$p(m\pi_p)^p(m\ell_{i_m}^{(m)} - 1) = q(x) - \int_0^1 q(t)dt + o(1). \tag{4.16}$$

Therefore, the asymptotic formula is valid. □

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